Chapter 5

Error Estimates for Wavelet-Galerkin Solution of BVP

5.1 Introduction

Tn this chapter, we have developed an error estimates for Wavelet-Galerkin solution of elliptic BVP in one dimension. We have made an attempt to solve elliptic PDE numerically using wavelet based Green's function approach. At the end, we have compared analytical solution, wavelet solution, and finite difference solution in terms of relative error and computational time.

5.2 Error Estimates

5.2.1 Introduction

Wavelets have caught the eye of so many researchers in the field of differential equations. The general characteristic that has brought wavelet research to so many disciplines, including partial differential equations, is the ability to explicitly consider scale and location, simultaneously. When one is trying to solve partial differential equations over large dynamic ranges, there is an increased richness and flexibility that stems from including scale and location on more or less equal footing.

In many applications of partial differential equations, it is essential that a high level of ac-

curacy should be achievable. For instance, in solving the Schrodinger wave equation, we need to be able to evaluate integrals involving the scaling functions and wavelet functions that constitute a wavelet family. A reasonable answer to this problem is the use of quadrature integration, now implemented for orthogonal wavelets, biorthogonal wavelets, and multi-wavelets, coupled with refinement in scale according to the needs of the problem. Such an adaptive wavelet quadrature can control the accuracy of integral evaluation.

This general strategy for using wavelets is similar to the use of finite elements and related numerical techniques. Finite elements can be multiscale, but the availability of orthogonality, biorthogonality, or other variations offers guaranteed avoidance of linear-dependence problems for many classes of wavelets. For finite elements, it is generally required to set up sophisticated structured meshes, while wavelet calculations can start from a simply-structured hierarchy and can be made to adapt resolution automatically. For dynamics calculations, finite elements require re-gridding as time progresses, while orthogonal wavelets can, in principle, be added or subtracted individually as their coefficients grow or shrink without disturbing the other basis members.

The methods like variational, wavelet-Galerkin, or other partial differential equation methods are available but still one needs to develop better methods for handling boundaries. The various types of finite difference methods for boundary value problems are known. The approximation of wavelet-Galerkin method for dealing with boundary value problem is known recently. For a certain class of equations, using wavelets in conjunction with the Galerkin method give the two primary desired features for the associated linear system: sparseness and low condition number.

In finite difference method, solution of differential equation is obtained by the discretization of derivatives. Finally, differential equation reduces to the matrix equation $\mathbf{A}\chi = \mathbf{y}$. The condition number of a matrix plays an important role for the stability of solutions of differential equations. A system is more stable if the condition number of matrix of the system is very low. The condition number of $\mathbf{A}$ measures how unstable the linear system $\mathbf{A}\chi = \mathbf{y}$ is under the perturbation of $\mathbf{y}$. So, for application point of view, we are looking for a small condition number. If the condition number of $\mathbf{A}$ is high, we would like to replace the linear system whose matrix has a low condition number. For example, multiplying the system by a preconditioning matrix $\mathbf{B}$, we get the equivalent system $\mathbf{BA}\chi = \mathbf{By}$, where the condition number of $\mathbf{BA}$ is smaller than the condition number of $\mathbf{A}$.

We shall start with the following definitions:

Definition 5.2.1 *Let* $\mathbf{A}$ *be* $n \times n$ *matrix. Define* $\| A \|$, *called the operator norm or just the*

norm, of $\mathbf{A}$ *by*

$$\| \mathbf{A} \| = sup \frac{\| \mathbf{Az} \|}{\| \mathbf{z} \|} \tag{5.1}$$

where the supremum is taken over all non-zero vectors $\mathbf{z}$ *in* C^n.

Definition 5.2.2 *Let* $\mathbf{A}$ *be an invertible* $n \times n$ *matrix, then the condition number of* $\mathbf{A}$, *denoted by* $C_\sharp(\mathbf{A})$, *is given by*

$$C_\sharp(\mathbf{A}) = \| \mathbf{A} \| \| \mathbf{A}^{-1} \| \tag{5.2}$$

We quote some specific lemmas about condition number which will be used in our later discussion.

Lemma 5.2.1 *Suppose that* $\mathbf{A}$ *is an* $n \times n$ *normal invertible matrix. Let*

$$|\lambda|_{max} = max\{|\lambda| : \lambda \text{ is an eigen value of } \mathbf{A}\} \tag{5.3}$$

and

$$|\lambda|_{min} = min\{|\lambda| : \lambda \text{ is an eigen value of} \mathbf{A}\}. \tag{5.4}$$

Then

$$C_\sharp(\mathbf{A}) = \frac{|\lambda|_{max}}{|\lambda|_{min}} \tag{5.5}$$

Proof: See([Fra99])

Lemma 5.2.2 *Let* $\{\psi_{j,k}\}_{(j,k)\in\tau}$ *be a complete orthonormal system, where* τ *is a certain subset of* $\mathbf{Z} \times \mathbf{Z}$. *Then wavelet system* $\{\psi_{j,k}\}_{(j,k)\in\tau}$ *satisfies the following estimate:*
There exists constants C_4, $C_5 > 0$ *such that for all functions* g *of the form*

$$g = \sum_{j,k} C_{j,k}\psi_{j,k},$$

where the sum is finite, we have

$$C_4 \sum_{j,k} 2^{2j}|C_{j,k}|^2 \leq \int_0^1 |g'(t)|^2 dt \leq C_5 \sum_{j,k} 2^{2j}|C_{j,k}|^2. \tag{5.6}$$

An estimate of this form is called norm equivalence.

Proof: See([Fra99]).

5.2.2 Finite Difference Method

Let L be a uniformly elliptic Sturm-Liouville operator defined as

$$L = -\frac{d}{dt}\left(a(t)\frac{d}{dt}\right) + b(t) \tag{5.7}$$

Our goal in this section is to obtain a C^2 function u that is a solution to equation

$$-\frac{d}{dt}\left(a(t)\frac{du}{dt}\right) + b(t)u(t) = f(t) \quad for\ 0 \leq t \leq 1. \tag{5.8}$$

i.e

$$Lu = f(t) \quad for\ 0 \leq t \leq 1. \tag{5.9}$$

with Dirichlet boundary conditions

$$u(0) = 0 \ \ and \ \ u(1) = 0. \tag{5.10}$$

We assume that f and b continuous and a has a continuous derivatives on $[0, 1]$. We also assume that this operator is uniformly elliptic, i.e. there exists finite constants C_1, C_2 and C_3 such that

$$0 < C_1 \leq a(t) < C_2 \ ; \quad 0 \leq b(t) \leq C_3 \quad for\ 0 \leq t \leq 1. \tag{5.11}$$

We shall assume $a(t) = 1$ and $b(t) = 0$ and use finite difference method to solve (5.8) and (5.10). It is based on approximating the derivatives in equation (5.8) by differences evaluated on a finite set of points in the interval $[0, 1]$. By the definition of the derivative

$$u'(t) \approx \frac{u(t + \Delta t) - u(t)}{\Delta t}$$

for small Δt. We may also write

$$u'(t) \approx \frac{u(t + \frac{h}{2}) - u(t - \frac{h}{2})}{h}. \tag{5.12}$$

where

$$h = 2\triangle t$$

Now,

$$u''(t) \approx \frac{u(t + h) - 2u(t) + u(t - h)}{h^2}. \tag{5.13}$$

We consider the points $t_j = \frac{j}{N}$, $j = 0, 1, 2, \cdots, N$ and we assume that $h = \frac{1}{N}$. We set

$$\chi(j) = u\left(\frac{j}{N}\right) \ ; \ y(j) = \frac{1}{N^2} f\left(\frac{j}{N}\right) \ \ for \ \ j = 0,\ 1,\ 2,\ \cdots,\ N. \tag{5.14}$$

So, when we have to solve $-u''(t_j) = f(t_j)$, we approximate $u''(t_j)$ using above approximation and consider the system of equations

$$-u\left(\frac{j+1}{N}\right) + 2u\left(\frac{j}{N}\right) - u\left(\frac{j-1}{N}\right) = \frac{1}{N^2} f\left(\frac{j}{N}\right)$$

with boundary conditions $u(0) = u(1) = 0$. When $j = 0$ and $j = N$, this equation does not make sense because $u\left(-\frac{1}{N}\right)$ and $u\left(1+\frac{1}{N}\right)$ are undefined. So, we restrict ourselves to $1 \leq j \leq N-1$. Thus, we consider

$$-\chi(j+1) + 2\chi(j) - \chi(j-1) = y(j); \quad j = 1, 2, 3 \cdots N-1. \tag{5.15}$$

with boundary conditions $\chi(0) = 0$ and $\chi(N) = 0$. Now for $j = 1$, equation (5.15) reduces to $-\chi(2) + 2\chi(1) = y(1)$ because $\chi(0) = 0$ and when $j = N-1$ equation (5.15) reduces to $2\chi(N-1) - \chi(N-2) = 0$ because $\chi(N) = 0$. Thus, equation (5.15) is a linear system of $(N-1)$ equations in $(N-1)$ unknowns $\chi(1), \chi(2), \cdots \chi(N-1)$ represented by

$$\begin{bmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & -1 & 2. \end{bmatrix} \begin{bmatrix} \chi(1) \\ \chi(2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \chi(N-1) \end{bmatrix} = \begin{bmatrix} y(1) \\ y(2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y(N-1) \end{bmatrix} \tag{5.16}$$

which we denote by

$$\chi = \mathbf{A}_N{}^{-1} y. \tag{5.17}$$

Since determinant of $\mathbf{A}_N = N$, it is easy to see that $\mathbf{A}_N$ is invertible and there is a unique solution χ to equation (5.17) for each vector $\mathbf{y}$. As we let $h \to 0$, i.e $N \to \infty$, we expect our solution χ to approximate the true values of u with greater accuracy. However, it is important numerically for a linear system to be well conditioned. By Lemma 5.2.1, the condition number of $\mathbf{A}_N$ is given by (see [Fra99])

$$C_{\natural}(\mathbf{A}_N) = \frac{|\lambda|_{max}}{|\lambda|_{\min}} = \frac{4\sin^2(\frac{\pi(N-1)}{2N})}{4\sin^2(\frac{\pi}{2N})}.$$

Now, as $N \to \infty$,

$$4\sin^2(\frac{\pi(N-1)}{2N}) \to 1.$$

Also, $4\sin^2(\frac{\pi}{2N})$ behaves like $\frac{\pi^2}{4N^2}$, which gives

$$C_{\natural}(\mathbf{A}_N) \approx \frac{4N^2}{\pi^2}. \tag{5.18}$$

Thus, the condition number of $\mathbf{A}_N$ goes to ∞ proportional to N^2. So, although increasing N should increase the accuracy of the approximation to the solution u of equations (5.8) and (5.10), the linear system $\mathbf{A}_N\chi = \mathbf{y}$ becomes increasing unstable and the solution becomes more and more unreliable.

Since the operator $-\frac{d^2}{dt^2}$ is translation-invariant, we were able to explicitly diagonalize the matrix $\mathbf{A}_N$ arising in the finite difference approximation (see [Fra99]). With general operator L given by equation (5.7), the matrix $\mathbf{A}$, obtained after finite difference approximation, is not close to circulant matrix and as a result, there is a little hope for explicit diagonalization of matrix $\mathbf{A}$. The condition number of $\mathbf{A}$ is expected to be large (see [Fra99]), because the same thing was happened in a very simple case. The alternate approach to remove this difficulty is wavelets.

5.2.3 Wavelet Galerkin Method

We shall introduce this approach for general operator L. This approach leads to linear systems with bounded condition number.

Consider the general boundary value problem (5.8) and (5.10). There is a unique function u satisfying equation (5.8) and (5.10) under certain conditions. We know that $L^2[0,1]$ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt. \tag{5.19}$$

For the Galerkin method, we assume that $\{v_j\}_j$ is a complete orthonormal system for $L^2[0,1]$ and that every v_j is C^2 on $[0,1]$ and satisfies

$$v_j(0) = v_j(1) = 0. \tag{5.20}$$

We select some finite set Λ of indices j and consider the subspace

$$S = span\{v_j : j \in \Lambda\}.$$

We look for an approximation to the solution u of equation (5.8) of the form

$$u_s = \sum_{k\in\Lambda} \chi_k v_k \in S, \tag{5.21}$$

where each χ_k is a scalar. Our criterion for determining the coefficients χ_k is that u_s should behave like the true solution u on the subspace S, i.e

$$\langle Lu_s, v_j\rangle = \langle f, v_j\rangle \quad \forall\, j \in \Lambda. \tag{5.22}$$

The approximate solution u_s satisfies the boundary conditions $u_s(0) = u_s(1) = 0$ because of equation (5.20). If we substitute equation (5.21) in equation (5.22), we get

$$\left\langle L\left(\sum_{k\in\Lambda}\chi_k v_k\right), v_j\right\rangle = \langle f, v_j\rangle \quad \forall\, j \in \Lambda,$$

or

$$\sum_{k\in\Lambda}\langle Lv_k, v_j\rangle\chi_k = \langle f, v_j\rangle \quad \forall\, j \in \Lambda. \tag{5.23}$$

Let χ denote the vector $(\chi_k)_{k\in\Lambda}$ and $\mathbf{y}$ be the vector $(y_k)_{k\in\Lambda}$, where $y_k = \langle f, v_k\rangle$. Let $\mathbf{A} = [a_{j,k}]_{j,k\in\Lambda}$ where

$$a_{j,k} = \langle Lv_k, v_j\rangle. \tag{5.24}$$

Thus, equation (5.23) is the linear system of equations

$$\sum_{k\in\Lambda} a_{j,k}\chi_k = y_j, \quad \forall\, j \in \Lambda,$$

i.e.

$$\mathbf{A}\chi = \mathbf{y}. \tag{5.25}$$

In the Galerkin's method, for each subset Λ we obtain an approximation $u_s \in S$ to u by solving the linear system (5.25) for χ and using these components to determine u_s by equation (5.21). As we increase our set Λ in some symmetric way, our approximation u_s should converge to the actual solution u.

We are concerned with the nature of the linear system (5.25) that results from choosing a wavelet as a basis for the Galerkin method as opposed to some other basis, say Fourier basis. There are two properties that we would like to have for the matrix $\mathbf{A}$ in the linear system (5.25) for our numerical problem. First, as to have a small condition number to obtain stability of the solution under small perturbations in the data. Second, for performing calculations with $\mathbf{A}$ quickly, we would like $\mathbf{A}$ to be sparse, which means that $\mathbf{A}$ should have a high proportion of entries that are 0. The best case is when $\mathbf{A}$ is diagonal but the next best case is when $\mathbf{A}$ is sparse. There is a way of modifying the wavelet system for $L^2(R)$ so as to obtain a complete orthonormal system

$$\{\psi_{j,k}\}_{(j,k)\in\tau} \quad for\ L^2[0,1].$$

The set τ is a certain subset of $\mathbf{Z}\times\mathbf{Z}$. The function $\psi_{j,k}$ are not exactly the same functions as in a wavelet basis for $L^2(R)$ but they are similar. In particular, $\psi_{j,k}$ has a scale of about 2^{-j}, $\psi_{j,k}$ is concentrated near the point $2^{-j}k$ and $\psi_{j,k}$ is 0 outside an interval centered at $2^{-j}k$ of length proportional to 2^{-j}. Wavelets concentrated well into the interior of $[0,1]$ are nearly the same

as usual wavelets, but those concentrated near the boundary points are substantially modified. For each $(j,k) \in \tau$, $\psi_{j,k}$ is C^2 and satisfies the boundary conditions.

$$\psi_{j,k}(0) = \psi_{j,k}(1) = 0.$$

For wavelets, we write equations (5.21) as

$$u_s = \sum_{(j,k)\in\tau} \chi_{j,k}\psi_{j,k}$$

and equation (5.23) as

$$\sum_{(j,k)\in\tau} \langle L\psi_{j,k}, \psi_{l,m}\rangle \chi_{j,k} = \langle f, \psi_{l,m}\rangle \quad \forall \ (l,m) \in \tau, \tag{5.26}$$

for some finite set of indices τ. We can still regard this as a matrix equation

$$\mathbf{A}\,\chi = \mathbf{y},$$

where

$$\chi = (\chi_{j,k})_{(j,k)\in\tau}$$

and

$$y = (y_{j,k})_{(j,k)\in\tau}$$

are indexed by the pairs $(j,k) \in \tau$. Let matrix

$$\mathbf{A} = [a_{l,m;j,k}]_{(l,m),(j,k)\in\tau}$$

defined by

$$a_{l,m;j,k} = \langle L\psi_{j,k}, \psi_{l,m}\rangle. \tag{5.27}$$

We would like $\mathbf{A}$ to be sparse and have a low condition number. Actually, $\mathbf{A}$ itself does not have a low condition number, but we can replace the system $\mathbf{A}\,\chi = \mathbf{y}$ by a equivalent system $\mathbf{Mz} = \nu$ for which the new matrix $\mathbf{M}$ has the desired properties. To see this, first define the diagonal matrix

$$\mathbf{D} = [d_{l,m;j,k}]_{(l,m),(j,k)\in\tau}$$

by

$$d_{l,m;j,k} = \begin{cases} 2^j & if \quad (l,m) = (j,k) \\ 0 & otherwise. \end{cases}$$

Define

$$\mathbf{M} = [m_{l,m;j,k}]_{(l,m),(j,k)\in\tau}$$

by

$$\mathbf{M} = \mathbf{D}^{-1}\mathbf{A}\mathbf{D}^{-1}. \tag{5.28}$$

By writing this out, we see that

$$m_{l,m;j,k} = 2^{-j-l}a_{l,m;j,k} = 2^{-j-l}\langle L\psi_{j,k}, \psi_{l,m}\cdot\rangle \tag{5.29}$$

The system $\mathbf{A}\,\chi = \mathbf{y}$ is equivalent to

$$\mathbf{D}^{-1}\mathbf{A}\mathbf{D}^{-1}\mathbf{D}\mathbf{x} = \mathbf{D}^{-1}\mathbf{y}.$$

Now, take $\mathbf{z} = \mathbf{D}\mathbf{x}$ and $\nu = \mathbf{D}^{-1}y$ we get

$$\mathbf{M}\mathbf{z} = \nu. \tag{5.30}$$

The matrix in the preconditioned system (5.30), has a condition number bounded independently of the set τ. So, we increase τ to approximate our solution with more accuracy as the condition number stays bounded. This is much better than the finite difference case, as we have seen earlier, in which the condition number grows as N^2. There are other complete orthonormal systems, for example Fourier system, in which a similar preconditioning can be done to yield a bounded condition number. In our discussion, we see the advantage of the Galerkin method over finite difference, but the advantages of wavelets over the Fourier system is still not clear (see [Fra99]).

We should consider the other features of the matrix $\mathbf{M}$ that is sparseness of matrix $\mathbf{M}$. We can see from equation (5.29) that this is the case, because of the localization of the wavelets. Namely, $\psi_{i,j}$ is 0 outside an interval of length $c2^{-j}$ around the point $2^{-j}k$, for some constant c. Because L involves only differentiation and multiplication by another function, it does not change this localization property. So, $L\psi_{i,j}$ is 0 outside this interval also. Similarly, $\psi_{l,m}$ is 0 outside an interval of length $c2^{-l}$ around the point $2^{-l}m$. As j and l get large, fewer and fewer of these intervals intersect, so more and more of the matrix entries

$$m_{l,m;j,k} = 2^{-j-l}\langle L\psi_{i,j}, \psi_{l,m}\rangle = 2^{-j-l}\int_0^1 L\psi_{i,j}(t)\overline{\psi_{l,m}(t)}dt$$

are 0. So $\mathbf{M}$ is sparse, which makes computation easier. The basic reason for this sparseness is the compact support of the wavelets.

The wavelet system is more crude than the Fourier system in the sense that there are few, if any, naturally occurring operators that are diagonalized by a wavelet basis. But for a very large class of operators, for instance, the variable coefficient operator considered here, the matrices representing these operators in the wavelet system are sparse, which we regard as being nearly

diagonal. Thus, although the wavelet system does not exactly diagonalize much of anything, it nearly diagonalizes a very large class of operators, a much larger class than the translation-invariant operators, which are perfectly diagonalized by the Fourier system.

The fact that a wavelet system nearly diagonalizes a very broad class of operators is one of the key properties of wavelets. We have seen that this property is important in applications to numerical differential equations. Another key property of wavelets is that norm equivalences for wavelets such as relation (5.6) (see Lemma 5.2.2) hold for a much large class of function spaces than for the Fourier system (see [HW96]).

5.2.4 Some Error Estimates

We have made an attempt to extend the error estimates of Neumann kind of boundary value problem (see [BS02]) to Dirichlet kind of boundary value problem for which we have both advantages like sparsity and small condition number. We have also given more generalized result for wavelet-Galerkin solution.

We have the following two point boundary value problem

$$-u'' = f \tag{5.31}$$

with boundary conditions

$$u(0) = u(1) = 0.$$

Let v be any sufficiently regular function such that $v(0) = 0$. Multiplying equation (5.31) with v and integrating from 0 to 1, we get

$$-\int_0^1 u''v dx = \int_0^1 fv dx$$

$$\Longrightarrow \qquad -\{u'(1)v(1) - u'(0)v(0) - \int_0^1 u'v' dx\} = \int_0^1 fv\ dx$$

Now, $v(0) = 0$

$$\Longrightarrow \qquad -u'(1)v(1) + \int_0^1 u'v' dx = \int_0^1 fv dx$$

Define

$$a(u, v) := \int_0^1 u'v' dx, \tag{5.32}$$

and

$$(f, v) := \int_0^1 f\ v\ dx. \tag{5.33}$$

Thus, we get

$$a(u,v) = (f,v) + u'(1)v(1). \tag{5.34}$$

Define

$$V = \{v \in L^2(0,1) : a(v,v) \ < \ \infty \ \textit{and} \ \ v(0) = 0\}.$$

Thus, our problem is to find $u \in V$ such that

$$a(u,v) = (f,v) + u'(1)v(1) \quad \forall v \in V. \tag{5.35}$$

Let $S \subset V$ be any finite dimensional subspace. Let us consider equation (5.34) with V replaced by S. So, we have to find $u_s \in S$ such that

$$a(u_s,v) = (f,v) + u'(1)v(1) \quad \forall v \in S. \tag{5.36}$$

Subtracting (5.36) from (5.34), we get

$$a(u - u_s, w) = 0 \qquad \forall w \in S. \tag{5.37}$$

Now, we shall state our estimates in the form of theorems. Our theorems are as follows:

Theorem 5.2.3 *Suppose*

$$\|v\|_E = \sqrt{a(v,v)}, \ \ \forall \ \ v \in V$$

then

$$\|u - u_s\|_E = inf\{\|u - v\|_E \ \ : \ v \in S\},$$

where $S \subset V$ is any finite dimensional subspace.

This is the basic error estimates for the Ritz-Galerkin method, and it says that the error is optimal in the energy norm. In the next theorem, we will use this error estimate for deriving more concrete error estimate.

Proof of Theorem 5.2.3: We know that

$$\|v\|_E = \sqrt{a(v,v)}, \ \ \forall \ \ v \in V$$

as the energy norm. The energy norm and inner-product is related by Schwarz inequality:

$$|a(v,w)| \leq \|v\|_E \|w\|_E \ \ \forall \ \ v,w \in V$$

Now, we wish to consider the size of the error $u - u_s$ in this norm. So, for any $v \in S$, consider

$$\begin{aligned} \|u - u_s\|_E^2 &= a(u - u_s, u - u_s) \\ &= a(u - u_s, u - v) + a(u - u_s, v - u_s) \end{aligned}$$

Take $w = v - u_s$ and apply equation (5.37), we shall get

$$\|u - u_s\|_E^2 = a(u - u_s, u - v).$$

By applying Schwarz inequality, we have

$$\|u - u_s\|_E^2 \leq \|u - u_s\|_E \|u - v\|_E$$

If $\|u - u_s\|_E \neq 0$, then we can write

$$\|u - u_s\|_E \leq \|u - v\|_E \qquad for\ any\ v \in S.$$

Taking, the infimum over $v \in S$ yields

$$\|u - u_s\|_E \leq inf\{\|u - v\|_E : \quad v \in S\}.$$

Since $u_s \in S$, we have

$$inf\{\|u - v\|_E : \quad v \in S\} \leq \|u - u_s\|_E.$$

Therefore,

$$\|u - u_s\|_E = inf\{\|u - v\|_E : \quad v \in S\}.$$

Hence, the proof.

Theorem 5.2.4 *Let $\epsilon > 0$. Suppose w be solution of*

$$-w'' = u - u_s \quad on \quad [0,1]; \quad with \ \ w(0) = w(1) = 0 \tag{5.38}$$

and, if

$$\inf_{v \in S} \|w - v\|_E \leq \epsilon \|w''\|_E \tag{5.39}$$

then

$$\|u - u_s\|_{L^2} \leq \epsilon \|u - u_s\|_E \leq \epsilon^2 \|u''\| \ = \epsilon^2 \|f\|. \tag{5.40}$$

Proof of Theorem 5.2.4: Define

$$\|v\|_{L^2(0,1)} = (v, v)^{1/2} = \left(\int_0^1 (v(x))^2 \ dx \right)^{1/2}.$$

Now, we shall estimate $\|u - u_s\|$ using the concept of duality.

Given that w is a solution of

$$-w'' = u - u_s \quad on \quad [0,1] \tag{5.41}$$

with

$$w(0) = w(1) = 0.$$

Now,

$$\begin{aligned} \|u - u_s\|_{L^2}^2 &= (u - u_s, u - u_s) \\ &= (u - u_s, -w'') \\ &= a(u - u_s, w) \\ &\quad [\text{since } \ (u - u_s)(0) = (u - u_s)(1) = 0] \\ &= a(u - u_s, w - v). \\ &\quad [\text{from equation (5.37)}] \end{aligned}$$

Therefore,

$$\|u - u_s\|_{L^2}^2 = a(u - u_s, w - v), \quad \forall v \in S.$$

Thus, Schwarz inequality implies that

$$\begin{aligned} \|u - u_s\|_{L^2} &\leq \frac{\|u - u_s\|_E \|w - v\|_E}{\|u - u_s\|_{L^2}} \\ &= \frac{\|u - u_s\|_E \|w - v\|_E}{\|w''\|_{L^2}} \end{aligned}$$

By taking the infimum over $v \in S$, we get

$$\|u - u_s\|_{L^2} \leq \frac{\|u - u_s\|_E \ \inf_{v \in S}\{\|w - v\|_E\}}{\|w''\|_{L^2}} \tag{5.42}$$

So, from equation (5.39) and equation (5.42), we have

$$\|u - u_s\|_{L^2} \leq \epsilon \|u - u_s\|_E.$$

Applying equation (5.39) again, with w replaced by u, and using

$$\|u - u_s\|_E = inf\{\|u - v\|_E \ : v \in S\},$$

we have

$$\|u - u_s\|_E \leq \epsilon \|u''\|.$$

Therefore, using equation (5.31), we have

$$\|u - u_s\|_{L^2} \leq \epsilon \|u - u_s\|_E \leq \epsilon^2 \|u''\| = \epsilon^2 \|f\|.$$

Thus,

$$\|u - u_s\|_{L^2} \leq \epsilon \|u - u_s\|_E \leq \epsilon^2 \|f\|.$$

Hence, the proof.

Theorem 5.2.5 *Let*

$$\Psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Consider

$$\Psi_{j,k}(x) = 2^{\frac{-j}{2}}(2^{-j}x - k), \tag{5.43}$$

where j and k are integers. If u_{ws} be wavelet-Galerkin solution in space S and it is of the form

$$u_{ws} = \sum_{j=1}^{n}\sum_{k=1}^{n} C_{j,k}\Psi_{j,k}(x) \tag{5.44}$$

Then we have

$$\|u - u_{ws}\|_{L^2} = \inf_{v \in S}\{2\|u - v\|_{L^2} + O(1)\}. \tag{5.45}$$

Proof of Theorem 5.2.5: Consider

$$u_{ws} = \sum_{j=1}^{n}\sum_{k=1}^{n} C_{j,k}\Psi_{j,k}(x) \tag{5.46}$$

$$\begin{aligned} \|u - u_{ws}\|_{L^2}^2 &= \|u - \sum_{j=1}^{n}\sum_{k=1}^{n} C_{j,k}\Psi_{j,k}(x)\|_{L^2}^2 \\ &= \int_0^1 |u - \sum_{j=1}^{n}\sum_{k=1}^{n} C_{j,k}\Psi_{j,k}(x)|^2 dx \\ &= \int_0^1 |u - \sum_{j=1}^{n}\sum_{k=1}^{n} C_{j,k}2^{-\frac{j}{2}}\Psi(2^{-j}x - k)|^2 dx \end{aligned}$$

We know that, $|a+b|^p \le 2^p\left(|a|^p + |b|^p\right)$. Applying, we get

$$\begin{aligned} \|u - u_{ws}\|_{L^2}^2 &\le 2^2\left[\int_0^1 |u|^2 dx + \int_0^1 \left|\sum_{j=1}^{n}\sum_{k=1}^{n} 2^{-\frac{j}{2}} C_{j,k}\Psi(2^{-j}x - k)\right|^2 dx\right] \\ &= 4\int_0^1 |u|^2 dx + 4\sum_{j=1}^{n}\sum_{k=1}^{n} 2^{-j}\int_0^1 C_{j,k}^2\Psi^2(2^{-j}x - k)dx \\ &= 4\int_0^1 |u|^2 dx + 4\sum_{j=1}^{n}\sum_{k=1}^{n} 2^{-j}C_{j,k}^2 \\ &\le 4\int_0^1 |u|^2 dx + 2\sum_{j=1}^{n}\sum_{k=1}^{n} C_{j,k}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|u - u_{ws}\|_{L^2} &\leq 2\|u\|_{L^2} + O(1) \\ &= 2\|u - v + v\|_{L^2} + O(1) \\ &= 2\|u - v\| + 2\|v\|_{L^2} + O(1) \end{aligned}$$

Since $v \in S \subset V$ and $V \subset L^2[0,1]$, we get

$$2\|v\|_{L^2} = O(1)$$

By taking the infimum over $v \in S$ yields

$$\|u - u_{ws}\|_{L^2} \leq \inf_{v \in S}\{2\|u - v\|_{L^2} + O(1)\}$$

Since $u_{ws} \in S$

$$\inf_{v \in S}\{2\|u - v\|_{L^2} + O(1)\} \leq \|u - u_{ws}\|_{L^2}$$

So, we finally get

$$\|u - u_{ws}\|_{L^2} = \inf_{v \in S}\{2\|u - v\|_{L^2} + O(1)\}. \tag{5.47}$$

Hence, the proof.

5.3 Wavelet-based Green's Function Approach to Hlemholtz & Modified Helmholtz Problem

5.3.1 Introduction

In this section, We shall describe how wavelets may be used to solve partial differential equations. These problems are currently solved by techniques such as finite differences, finite elements, and multigrid. We know from previous chapters that the wavelet method, however, offers several advantages over traditional methods. Wavelets have good ability to represent functions at different levels of resolution, thereby providing a logical means of hierarchy of solutions. Furthermore, compactly supported wavelets (such as those due to Daubechies [Dau92]) are localized in space, which means that solution can be refined in regions of higher gradient, e.g., stress concentrations, without having to regenerate the mesh for the entire problem. In order to demonstrate the wavelet technique, we consider Helmholtz equation and modified Helmholtz equation in two dimensions. By doing comparison with a simple finite difference solution to this problem with periodic boundary conditions, we show how a wavelet technique may be efficiently developed.

Dirchlet/Neumann/Robin boundary conditions are then imposed, using Capacitance Matrix method described by Proskurowski and Widlund [PW96] and others. The convergence of the wavelet solutions are examined and they are compared favorably to the finite difference solutions.

Wavelets are a family of orthonormal functions which are characterized by the translation and dilation of a single function $\psi(x)$. This family of functions denoted by $\psi_{m,k}(x)$ and given by

$$\psi_{m,k}(x) = 2^{\frac{m}{2}}\psi(2^m x - k); \quad m,\ k \in \mathbf{Z}$$

is a basis for the space of square integrable functions $L^2(\mathbf{R})$, i.e.,

$$f(x) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} d_{m,k}\psi_{m,k}(x) \quad \in L^2(\mathbf{R}).$$

Wavelets are derived from scaling functions, i.e. functions which satisfy recursion

$$\phi(x) = \sum_{k=0}^{\infty} a_k \phi(2x - k)$$

in which a finite number of filter coefficients a_k are non-zero.

Any $L^2(\mathbf{R})$ function $f(x)$ may be approximated at resolution m by

$$P_m(f)(x) = \sum_{k=0}^{\infty} C_{m,k}\phi_{m,k}(x) \quad k \in \mathbf{Z}.$$

where using Daubechies notation [Dau92], $P_m(f)$ represent the projection of the function f onto the space of scaling functions at resolution m. Also

$$\phi_{m,k}(x) = 2^{\frac{m}{2}}\phi(2^m x - k); \quad k \in \mathbf{Z}$$

is a scaling function basis at resolution m approximation of $L^2(\mathbf{R})$. The set of approximations $P_m(f)(x)$ constitute a multiresolution representation of the function $f(x)$.

In two dimensions, the space of square integrable functions is $L^2(\mathbf{R}^2)$ and any function $f(x,y)$ which lies in this space may be expressed in terms of the orthonormal basis

$$\psi_{i,k}(x)\psi_{j,l}(y); \quad i,k,j,l \in \mathbf{Z}.$$

This is simply the tensor product of the one dimensional bases in the two co-ordinate directions x and y. $f(x,y)$ may be represented at resolution m by

$$P_m(f)(x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_{m,k,l}\phi_{m,k}(x)\phi_{m,l}(y); \quad k,l \ \in \mathbf{Z}.$$

For a greater insight into the properties of wavelets and their constructions, we shall see [Dau92]. This work details how a hierarchy of wavelet solutions to two dimensional PDE may be developed using scaling functions bases.

In order to demonstrate the wavelet technique, we consider Helmholtz equation or Modified Helmholtz equation in two dimensions:

$$u_{xx} + u_{yy} \pm u = f \tag{5.48}$$

where

$$u = u(x, y) \quad \text{and} \quad f = f(x, y).$$

5.3.2 Finite Difference Solution of the Periodic Problem

Consider the problem

$$u_{xx} + u_{yy} \pm u = f$$

where u and f are periodic functions in x and y. Let dx and dy be the periods in the x and y directions, respectively, i.e.,

$$u(0, y) = u(dx, y); \quad f(0, y) = f(dx, y);$$

$$u(x, 0) = u(x, dy); \quad f(x, 0) = f(x, dy).$$

Now, suppose that we have a $n_x \times n_y$ mesh descritization of the rectangular region $[0, dx]$, $[0, dy]$ so that

$$u_{j,k} = u(jh, kh); \quad f_{j,k} = f(jh, kh)$$

where $j = 0, 1, 2, \cdots, (n_x - 1)$, $k = 0, 1, 2, \cdots, (n_y - 1)$ and

$$h = \frac{dx}{n_x} = \frac{dy}{n_y}.$$

The finite difference approximation to u_{xx} is

$$\begin{aligned} (u_{xx})_{j,k} &= \frac{1}{h}\left(\frac{u_{j+1,k} - u_{j,k}}{h} - \frac{u_{j,k} - u_{j-1,k}}{h}\right) \\ &= \frac{u_{j+1,k} - 2u_{j,k} + u_{j-1,k}}{h^2} \end{aligned}$$

Similarly,

$$(u_{yy})_{j,k} = \frac{u_{j,k+1} - 2u_{j,k} + u_{j,k-1}}{h^2}$$

So, the discrete form of equation (5.48) is;

$$u_{j+1,k} + u_{j,k+1} - 4u_{i,j} + u_{j-1,k} + u_{j,k-1} \pm h^2 u_{j,k} = h^2 f_{j,k}$$

Noting that

$$u_{-1,k} = u_{n_x-1,k}$$

$$u_{n_x,k} = u_{0,k}$$

$$u_{j,-1} = u_{j,n_y-1}$$

$$u_{j,n_y} = u_{j,0}$$

So,

$$\begin{bmatrix} T & I & 0 & 0 & - & - & 0 & I \\ I & T & I & 0 & - & - & 0 & 0 \\ 0 & I & T & I & - & - & 0 & 0 \\ 0 & 0 & I & T & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & T & I \\ I & 0 & 0 & 0 & - & - & I & T \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \\ - \\ - \\ U_{n_y-2} \\ U_{n_y-1} \end{bmatrix} = h^2 \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ - \\ - \\ f_{n_x-2} \\ f_{n_x-1} \end{bmatrix} \tag{5.49}$$

where $\mathbf{I}$ is the identity matrix,

$$\mathbf{T} = \begin{bmatrix} -4 \pm h^2 & 1 & 0 & 0 & - & - & 0 & 1 \\ 1 & -4 \pm h^2 & 1 & 0 & - & - & 0 & 0 \\ 0 & 1 & -4 \pm h^2 & 1 & - & - & 0 & 0 \\ 0 & 0 & 1 & -4 \pm h^2 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & -4 \pm h^2 & 1 \\ 1 & 0 & 0 & 0 & - & - & 1 & -4 \pm h^2 \end{bmatrix}_{n_x,n_x}$$

$$\mathbf{U}_k = \begin{bmatrix} u_{0,k} \\ u_{1,k} \\ u_{2,k} \\ u_{3,k} \\ - \\ - \\ u_{n_y-2} \\ u_{n_y-1} \end{bmatrix}; \quad f_k = \begin{bmatrix} f_{0,k} \\ f_{1,k} \\ f_{2,k} \\ f_{3,k} \\ - \\ - \\ f_{n_x-2} \\ f_{n_x-1} \end{bmatrix}$$

Now, the matrix $\mathbf{T}$ is a circulant matrix and the matrix on the left hand side of equation (5.49) is a block circulant matrix, i.e., a circulant matrix whose elements are matrices in themselves. Thus, equation (5.49) may be written as

$$U_{k-1} + IU_k + U_{k+1} = h^2 f_k; \quad k = 0, 1, 2, ..., D_{n_y-1}. \tag{5.50}$$

The form of the equations may be efficiently solved using the FFT. Let $\mathbf{F}_n$ be the $n \times n$ Fourier Matrix, i.e.,

$$\mathbf{F}_n = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & 1 & - & - & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & - & - & \omega^{n-2} & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & - & - & \omega^{2(n-2)} & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & - & - & \omega^{3(n-2)} & \omega^{3(n-1)} \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 1 & \omega^{(n-2)} & \omega^{2(n-2)} & \omega^{3(n-2)} & - & - & \omega^{(n-2)^2} & \omega^{(n-2)(n-1)} \\ 1 & \omega^{(n-1)} & \omega^{2(n-1)} & \omega^{3(n-1)} & - & - & \omega^{(n-1)(n-2)} & \omega^{(n-1)^2} \end{bmatrix}$$

where $\omega = e^{\frac{2\pi i}{n}}$ and $i = \sqrt{-1}$. Then DFT of n-dimensional vector $\vec{a}$ is

$$\hat{a} = \mathbf{F}_n^{-1} a = n\bar{\mathbf{F}}_n a.$$

Furthermore, if $\mathbf{A}$ is an $n \times n$ circulant matrix, then $\mathbf{F}_n^{-1}\mathbf{A}\mathbf{F}_n$ is a diagonal matrix containing the eigenvalues of $\mathbf{A}$. Then eigenvalues may also be obtained by taking the discrete Fourier Transform of the first column of $\mathbf{A}$.

Multiplying equation (5.50) by $\mathbf{F}_{n_x}^{-1}$, we get

$$\mathbf{F}_{n_x}^{-1}U_{k-1} + \mathbf{F}_{n_x}^{-1}\mathbf{T}U_k + \mathbf{F}_{n_x}^{-1}U_{k+1} = h^2\mathbf{F}_{n_x}^{-1}f_k$$

$$V_{k-1} + \Lambda V_k + V_{k+1} = h^2 C_k; \;\; 0, 1, 2, ..., n_y - 1 \tag{5.51}$$

where

$$\Lambda = \mathbf{F}_{n_x}^{-1}\mathbf{T}\mathbf{F}_{n_x} = \begin{bmatrix} \lambda_0 & 0 & 0 & 0 & - & - & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & - & - & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & - & - & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & \lambda_{n_x-2} & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 & \lambda_{n_x-2} \end{bmatrix}_{(n_x, n_x)}$$

$$V_k = F_{n_x}^{-1} U_k = \hat{U}_k = \begin{bmatrix} v_{0,k} \\ v_{1,k} \\ v_{2,k} \\ v_{3,k} \\ - \\ - \\ v_{n_x-2} \\ v_{n_x-1} \end{bmatrix}; \quad C_k = F_{n_x}^{-1} f_k = \hat{f}_k = \begin{bmatrix} c_{0,k} \\ c_{1,k} \\ c_{2,k} \\ c_{3,k} \\ - \\ - \\ c_{n_x-2} \\ c_{n_x-1} \end{bmatrix}$$

Since Λ is a diagonal matrix, each set of equations in (5.51) is a decoupled system in itself

$$v_{j,k-1} + \lambda_j v_{j,k} + v_{j,k+1} = h^2 c_{j,k};$$
$$j = 0, 1, 2, ..., n_x - 1;$$
$$k = 0, 1, 2, ..., n_y - 1;$$

or, we can write

$$\begin{bmatrix} \lambda_j & 1 & 0 & 0 & - & - & 0 & 1 \\ 1 & \lambda_j & 1 & 0 & - & - & 0 & 0 \\ 0 & 1 & \lambda_j & 1 & - & - & 0 & 0 \\ 0 & 0 & 1 & \lambda_j & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & \lambda_j & 1 \\ 1 & 0 & 0 & 0 & - & - & 1 & \lambda_j \end{bmatrix} \begin{bmatrix} v_{j,0} \\ v_{j,1} \\ v_{j,2} \\ v_{j,3} \\ - \\ - \\ v_{j,n_y-2} \\ v_{j,n_y-1} \end{bmatrix} = h^2 \begin{bmatrix} c_{j,0} \\ c_{j,1} \\ c_{j,2} \\ c_{j,3} \\ - \\ - \\ c_{j,n_y-2} \\ c_{j,n_y-1} \end{bmatrix} \tag{5.52}$$

where $j = 0, 1, 2, \cdots, n_x - 1$. Once again, a circulant matrix is encountered. In fact, the left hand side of equation (5.52) represents a discrete convolution. It follows from the properties of Fourier matrix that

$$\left(\mathbf{F}_{n_y}^{-1} \begin{bmatrix} \lambda_j \\ 1 \\ 0 \\ 0 \\ - \\ - \\ 0 \\ 1 \end{bmatrix} \right) \cdot \left(F_{n_y}^{-1} \begin{bmatrix} v_{j,0} \\ v_{j,1} \\ v_{j,2} \\ v_{j,3} \\ - \\ - \\ v_{j,n_y-2} \\ v_{j,n_y-1} \end{bmatrix} \right) = h^2 \mathbf{F}_{n_y}^{-1} \begin{bmatrix} c_{j,0} \\ c_{j,1} \\ c_{j,2} \\ c_{j,3} \\ - \\ - \\ c_{j,n_y-2} \\ c_{j,n_y-1} \end{bmatrix}$$

where the dot represents the componentwise multiplication of the parenthesized vectors. Transposing this system of equations and expanding for $j = 0, 1, 2, \cdots, n_x - 1$, we get

$$\left(\mathbf{L}\mathbf{F}_{n_y}^{-1} \right) \cdot \left(U\mathbf{F}_{n_y}^{-1} \right) = h^2 \left(C\mathbf{F}_{n_y}^{-1} \right) \tag{5.53}$$

where

$$\mathbf{L} = \begin{bmatrix} \lambda_0 & 1 & 0 & - & - & 0 & 0 & 1 \\ \lambda_1 & 1 & 0 & - & - & 0 & 0 & 1 \\ \lambda_2 & 1 & 0 & - & - & 0 & 0 & 1 \\ \lambda_3 & 1 & 0 & - & - & 0 & 0 & 1 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ \lambda_{n_x-2} & 1 & 0 & - & - & 0 & 0 & 1 \\ \lambda_{n_x-1} & 1 & 0 & - & - & 0 & 0 & 1 \end{bmatrix}_{(n_x,n_y)}$$

$$\mathbf{V} = \begin{bmatrix} v_{0,0} & v_{0,1} & v_{0,2} & v_{0,3} & - & - & v_{0,n_y-2} & v_{0,n_y-1} \\ v_{1,0} & v_{1,1} & v_{1,2} & v_{1,3} & - & - & v_{1,n_y-2} & v_{1,n_y-1} \\ v_{2,0} & v_{2,1} & v_{2,2} & v_{2,3} & - & - & v_{2,n_y-2} & v_{2,n_y-1} \\ v_{3,0} & v_{3,1} & v_{3,2} & v_{3,3} & - & - & v_{3,n_y-2} & v_{3,n_y-1} \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ v_{n_x-2,0} & v_{n_x-2,1} & v_{n_x-2,2} & v_{n_x-2,3} & - & - & v_{n_x-2,n_y-2} & v_{n_x-2,n_y-1} \\ v_{n_x-1,0} & v_{n_x-1,1} & v_{n_x-1,2} & v_{n_x-1,3} & - & - & v_{n_x-1,n_y-2} & v_{n_x-1,n_y-1} \end{bmatrix}$$

and

$$\mathbf{C} = \begin{bmatrix} c_{0,0} & c_{0,1} & c_{0,2} & c_{0,3} & - & - & c_{0,n_y-2} & c_{0,n_y-1} \\ c_{1,0} & c_{1,1} & c_{1,2} & c_{1,3} & - & - & c_{1,n_y-2} & c_{1,n_y-1} \\ c_{2,0} & c_{2,1} & c_{2,2} & c_{2,3} & - & - & c_{2,n_y-2} & c_{2,n_y-1} \\ c_{3,0} & c_{3,1} & c_{3,2} & c_{3,3} & - & - & c_{3,n_y-2} & c_{3,n_y-1} \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ c_{n_x-2,0} & c_{n_x-2,1} & c_{n_x-2,2} & c_{n_x-2,3} & - & - & c_{n_x-2,n_y-2} & c_{n_x-2,n_y-1} \\ c_{n_x-1,0} & c_{n_x-1,1} & c_{n_x-1,2} & c_{n_x-1,3} & - & - & c_{n_x-1,n_y-2} & c_{n_x-1,n_y-1} \end{bmatrix}$$

From the properties of the Fourier matrix, the first column of $\mathbf{A}$ is Inverse Fourier transform of the first column of L. Hence,

$$\mathbf{F}_{n_x}\mathbf{L} = \begin{bmatrix} -4 \pm h^2 & 1 & 0 & 0 & - & - & 0 & 1 \\ 1 & 0 & 0 & - & - & 0 & 0 & 0 \\ 0 & 0 & 0 & - & - & 0 & 0 & 0 \\ 0 & 0 & 0 & - & - & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & - & - & 0 & 0 & 0 \\ 1 & 0 & 0 & - & - & 0 & 0 & 0 \end{bmatrix} = K(say)$$

K is referred to as the Convolution Kernel. Thus,

$$\mathbf{L} = \mathbf{F}_{n_x}^{-1}\mathbf{K}$$

Also, from the definition of V_k and C_k, we get

$$\mathbf{V} = \mathbf{F}_{n_x}^{-1}\mathbf{U}$$

and

$$\mathbf{C} = \mathbf{F}_{n_x}^{-1}f$$

where

$$\mathbf{U} = \begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} & u_{0,3} & - & - & u_{0,n_y-2} & u_{0,n_y-1} \\ u_{1,0} & u_{1,1} & u_{1,2} & u_{1,3} & - & - & u_{1,n_y-2} & u_{1,n_y-1} \\ u_{2,0} & u_{2,1} & u_{2,2} & u_{2,3} & - & - & u_{2,n_y-2} & u_{2,n_y-1} \\ u_{3,0} & u_{3,1} & u_{3,2} & u_{3,3} & - & - & u_{3,n_y-2} & u_{3,n_y-1} \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ u_{n_x-2,0} & u_{n_x-2,1} & u_{n_x-2,2} & u_{n_x-2,3} & - & - & u_{n_x-2,n_y-2} & u_{n_x-2,n_y-1} \\ u_{n_x-1,0} & u_{n_x-1,1} & u_{n_x-1,2} & u_{n_x-1,3} & - & - & u_{n_x-1,n_y-2} & u_{n_x-1,n_y-1} \end{bmatrix}$$

and

$$f = \begin{bmatrix} f_{0,0} & f_{0,1} & f_{0,2} & f_{0,3} & - & - & f_{0,n_y-2} & f_{0,n_y-1} \\ f_{1,0} & f_{1,1} & f_{1,2} & f_{1,3} & - & - & f_{1,n_y-2} & f_{1,n_y-1} \\ f_{2,0} & f_{2,1} & f_{2,2} & f_{2,3} & - & - & f_{2,n_y-2} & f_{2,n_y-1} \\ f_{3,0} & f_{3,1} & f_{3,2} & f_{3,3} & - & - & f_{3,n_y-2} & f_{3,n_y-1} \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ f_{n_x-2,0} & f_{n_x-2,1} & f_{n_x-2,2} & f_{n_x-2,3} & - & - & f_{n_x-2,n_y-2} & f_{n_x-2,n_y-1} \\ f_{n_x-1,0} & f_{n_x-1,1} & f_{n_x-1,2} & f_{n_x-1,3} & - & - & f_{n_x-1,n_y-2} & f_{n_x-1,n_y-1} \end{bmatrix}$$

Equation (5.53) may be written as

$$\hat{\hat{K}} \cdot \hat{\hat{U}} = h^2 \hat{\hat{f}}$$

$$\Rightarrow \qquad \left(F_{n_x}^{-1} K F_{n_y}^{-1}\right) \cdot \left(F_{n_x}^{-1} U F_{n_y}^{-1}\right) = h^2 \left(F_{n_x}^{-1} f F_{n_y}^{-1}\right)$$

where

$$\hat{\hat{A}} = \left(F_{n_x}^{-1} A F_{n_y}^{-1}\right)$$

denotes the two-dimensional FFT of $n_x \times n_y$ matrix A. Then the solution to equation (5.48) is obtained by taking the Inverse two-dimensional FFT of

$$\hat{\hat{U}} = h^2 \left(\hat{\hat{f}} / \hat{\hat{K}}\right)$$

where / denotes the componentwise division.

In the particular case of Poisson equation, the convolution kernel is singular. As a consequence of this singularity, the first element of $\hat{\tilde{K}}$ is zero, i.e.,

$$\left(\hat{\tilde{K}}\right)_{0,0} = 0.$$

To avoid the division by zero, $\left(\hat{\tilde{K}}\right)_{0,0}$ mat be set to arbitrary non-zero value. If the function f is chosen such that its mean over the period is zero in both the x and y directions, then the value assigned to $\left(\hat{\tilde{K}}\right)_{0,0}$ is immaterial.

5.3.3 Wavelet-Galerkin Solution to the Periodic Problem

The Wavelet-Galerkin method (see [RGT90], [ALT91], [LT90], [Wei]) gives the solution u and the right hand side f as expansions of scaling functions at scale m. For the purpose of our work, it will suffice to say that the scaling function ϕ is defined by a dilation equation of the form

$$\phi(x) = \sum_{k=-\infty}^{\infty} a_k \phi(2x - k)$$

and that the values of the scaling function may be calculated using this recursion. Compactly supported scaling functions, such as those belonging to the Daubechies family of wavelets (see [Dau92]), have a finite number of non-zero filter coefficients a_k. The number of nonzero filter coefficients is denoted by N. Figure-5.1 shows the Daubechies 6 coefficient scaling function. For a more detailed description of scaling functions, their construction and their properties (see [AWQW92], and [Str89]).

The Wavelet-Galerkin solution of the periodic problem is slightly more complicated than the finite difference solution, since the solution procedure consists of solving a set of simultaneous equations in wavelet space and not in physical space. This means that we have to transform the right hand side function into wavelet space, solve the set of simultaneous equations to get the solution in wavelet space, and then transform the solution from wavelet space back into physical space. Consider again the equation

$$u_{xx} + u_{yy} \pm u = f \tag{5.54}$$

The Wavelet-Galerkin approximation to the solution $u(x, y)$ at scale m is

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \tilde{c}_{k,l} 2^{m/2}\phi\left(2^m x - k\right) 2^{m/2}\phi\left(2^m x - l\right); \quad k, l \in \mathbf{Z} \tag{5.55}$$

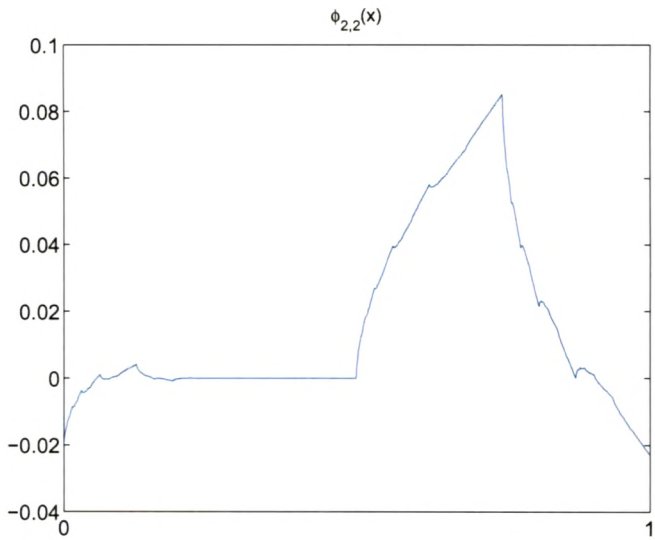

Figure 5.1: **Daubechies 6 coefficient scaling function**

where $\tilde{c}_{k,l}$ are the wavelet coefficients of u, i.e., they define the solution in wavelet space. The transformation from wavelet space to physical space (or vice versa) can be easily accomplished using the FFT if the wavelet expansion is expressed as discrete convolution. To do this, we make the following substitution:

$$X = 2^m x; \quad Y = 2^m y.$$

Then

$$\begin{aligned} U(X,Y) &= u(x,y) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \tilde{C}_{k,l} 2^{m/2} \phi(X-k)\phi(Y-l) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} C_{k,l} \phi(X-k)\phi(Y-l) \end{aligned}$$

where

$$C_{k,l} = 2^{m/2} \tilde{c}_{k,l} \tag{5.56}$$

Now, $u(x,y)$ is periodic in x and y with periods dx and dy, so that $U(X,Y)$ is periodic in X and Y with periods $2^m dx$ and $2^m dy$. Assuming that $dx, dy \in \mathbf{Z}$, so that

$$n_x = 2^m dx \in \mathbf{Z}$$

and

$$n_y = 2^m dy \in \mathbf{Z}$$

then $C_{k,l}$ is periodic in k and l with periods n_x and n_y.

Now, consider an $n_x \times n_y$ discretization of $U(X,Y)$, obtained by letting X and Y taking integer values only. This gives the values of $u(x,y)$ at all the dyadic points $(x,y) = (2^{-m}X, 2^{-m}Y)$, i.e., the discretization of $u(x,y)$ depends on scale that we have chosen (or vice versa). Thus,

$$U_{i,j} = U(i\Delta X, j\Delta Y) = U(i,j);$$
$$i = 0,1,2,3,...,n_x - 1;$$
$$j = 0,1,2,3,...,n_y - 1.$$

Equation (5.56) may then be written as

$$U_{i,j} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} C_{k,l}\phi_{i-k}\phi_{j-l}$$

$$\Rightarrow U_{i,j} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} C_{j-k,j-l}\phi_k\phi_l,$$

where $\phi(k) = \phi_k$. In matrix from, we may write it as

$$\mathbf{U} = \Phi_{n_x}\mathbf{C}\Phi_{n_y}^T \tag{5.57}$$

where

$$\mathbf{U} = \begin{bmatrix} U_{0,0} & U_{0,1} & U_{0,2} & U_{0,3} & - & - & U_{0,n_y-2} & U_{0,n_y-1} \\ U_{1,0} & U_{1,1} & U_{1,2} & U_{1,3} & - & - & U_{1,n_y-2} & U_{1,n_y-1} \\ U_{2,0} & U_{2,1} & U_{2,2} & U_{2,3} & - & - & U_{2,n_y-2} & U_{2,n_y-1} \\ U_{3,0} & U_{3,1} & U_{3,2} & U_{3,3} & - & - & U_{3,n_y-2} & U_{3,n_y-1} \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ U_{n_x-2,0} & U_{n_x-2,1} & U_{n_x-2,2} & U_{n_x-2,3} & - & - & U_{n_x-2,n_y-2} & U_{n_x-2,n_y-1} \\ U_{n_x-1,0} & U_{n_x-1,1} & U_{n_x-1,2} & U_{n_x-1,3} & - & - & U_{n_x-1,n_y-2} & U_{n_x-1,n_y-1} \end{bmatrix};$$

$$\mathbf{C} = \begin{bmatrix} c_{0,0} & c_{0,1} & c_{0,2} & c_{0,3} & - & - & c_{0,n_y-2} & c_{0,n_y-1} \\ c_{1,0} & c_{1,1} & c_{1,2} & c_{1,3} & - & - & c_{1,n_y-2} & c_{1,n_y-1} \\ c_{2,0} & c_{2,1} & c_{2,2} & c_{2,3} & - & - & c_{2,n_y-2} & c_{2,n_y-1} \\ c_{3,0} & c_{3,1} & c_{3,2} & c_{3,3} & - & - & c_{3,n_y-2} & c_{3,n_y-1} \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ c_{n_x-2,0} & c_{n_x-2,1} & c_{n_x-2,2} & c_{n_x-2,3} & - & - & c_{n_x-2,n_y-2} & c_{n_x-2,n_y-1} \\ c_{n_x-1,0} & c_{n_x-1,1} & c_{n_x-1,2} & c_{n_x-1,3} & - & - & c_{n_x-1,n_y-2} & c_{n_x-1,n_y-1} \end{bmatrix};$$

$$\Phi_{n_x} = \begin{bmatrix} 0 & 0 & 0 & - & - & \phi_{N-2} & - & - & \phi_2 & \phi_1 \\ \phi_1 & 0 & 0 & - & - & 0 & - & - & \phi_3 & \phi_2 \\ \phi_2 & \phi_1 & 0 & - & - & 0 & - & - & \phi_4 & \phi_3 \\ - & - & - & - & - & - & - & - & - & - \\ \phi_{N-2} & \phi_{N-3} & \phi_{N-4} & - & - & 0 & - & - & 0 & 0 \\ 0 & \phi_{N-2} & \phi_{N-3} & - & - & 0 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & \phi_{N-3} & - & - & - & - & \phi_1 & 0 \end{bmatrix}_{n\times n};$$

$$\Phi_{n_y}^T = \begin{bmatrix} 0 & 0 & 0 & - & - & \phi_{N-2} & - & - & \phi_2 & \phi_1 \\ \phi_1 & 0 & 0 & - & - & 0 & - & - & \phi_3 & \phi_2 \\ \phi_2 & \phi_1 & 0 & - & - & 0 & - & - & \phi_4 & \phi_3 \\ - & - & - & - & - & - & - & - & - & - \\ \phi_{N-2} & \phi_{N-3} & \phi_{N-4} & - & - & 0 & - & - & 0 & 0 \\ 0 & \phi_{N-2} & \phi_{N-3} & - & - & 0 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & \phi_{N-3} & - & - & - & - & \phi_1 & 0 \end{bmatrix}^T_{n\times n};$$

Taking two dimensional FFT of equation (5.57)

$$\mathbf{F}_{n_x}^{-1}\mathbf{U}\mathbf{F}_{n_y}^{-1} = F_{n_x}^{-1}\Phi_{n_x}C\Phi_{n_y}^T F_{n_y}^{-1}$$

or

$$\hat{\hat{U}} = \left(F_{n_x}^{-1}\Phi_{n_x}F_{n_x}\right)\left(F_{n_x}^{-1}CF_{n_y}^{-1}\right)\left(F_{n_y}\Phi_{n_y}^T F_{n_y}^{-1}\right).$$

But Φ_n, where $n = (n_x, n_y)$, are circulant matrices and hence

$$\left(F_{n_x}^{-1}\Phi_{n_x}F_{n_x}\right) = M_{n_x}$$

$$\left(F_{n_y}\Phi_{n_y}^T F_{n_y}^{-1}\right) = \left(F_{n_y}^{-1}\Phi_{n_y}^T F_{n_y}\right) = M_{n_y}$$

where

$$\mathbf{M}_n = \begin{bmatrix} \mu_0 & 0 & 0 & 0 & - & - & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & - & - & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & - & - & 0 & 0 \\ 0 & 0 & 0 & \mu_3 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & \mu_{n-2} & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 & \mu_{n-1} \end{bmatrix}$$

Thus,

$$\hat{\hat{U}} = \mathbf{M}_{n_x}\hat{\hat{C}}\mathbf{M}_{n_y} \tag{5.58}$$

Equation (5.58) may be written as

$$\hat{\hat{\mathbf{U}}} = \mathbf{B}_x \cdot \mathbf{B}_y \cdot \hat{\hat{\mathbf{C}}} \tag{5.59}$$

where

$$\mathbf{B}_x = \begin{bmatrix} \mu_0 & \mu_0 & \mu_0 & \mu_0 & - & - & \mu_0 & \mu_0 \\ \mu_1 & \mu_1 & \mu_1 & \mu_1 & - & - & \mu_1 & \mu_1 \\ \mu_2 & \mu_2 & \mu_2 & \mu_2 & - & - & \mu_2 & \mu_2 \\ \mu_3 & \mu_3 & \mu_3 & \mu_3 & - & - & \mu_3 & \mu_3 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ \mu_{n_x-2} & \mu_{n_x-2} & \mu_{n_x-2} & \mu_{n_x-2} & - & - & \mu_{n_x-2} & \mu_{n_x-2} \\ \mu_{n_x-1} & \mu_{n_x-1} & \mu_{n_x-1} & \mu_{n_x-1} & - & - & \mu_{n_x-1} & \mu_{n_x-1} \end{bmatrix}_{(n_x,n_y)}$$

$$\mathbf{B}_y = \begin{bmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 & - & - & \mu_{n_x-2} & \mu_{n_x-1} \\ \mu_0 & \mu_1 & \mu_2 & \mu_3 & - & - & \mu_{n_x-2} & \mu_{n_x-1} \\ \mu_0 & \mu_1 & \mu_2 & \mu_3 & - & - & \mu_{n_x-2} & \mu_{n_x-1} \\ \mu_0 & \mu_1 & \mu_2 & \mu_3 & - & - & \mu_{n_x-2} & \mu_{n_x-1} \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ \mu_0 & \mu_1 & \mu_2 & \mu_3 & - & - & \mu_{n_x-2} & \mu_{n_x-1} \\ \mu_0 & \mu_1 & \mu_2 & \mu_3 & - & - & \mu_{n_x-2} & \mu_{n_x-1} \end{bmatrix}_{(n_y,n_x)}$$

But from the properties of Fourier matrix

$$\mathbf{B}_x = \hat{\hat{K}}\phi_x$$

$$\mathbf{B}_y = \hat{\hat{K}}\phi_y$$

where

$$K_{\phi_x} = \begin{bmatrix} 0 & 0 & 0 & 0 & - & - & 0 & 0 \\ \phi_1 & 0 & 0 & 0 & - & - & 0 & 0 \\ \phi_2 & 0 & 0 & 0 & - & - & 0 & 0 \\ \phi_3 & 0 & 0 & 0 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ \phi_{N-2} & 0 & 0 & 0 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & 0 & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 & 0 \end{bmatrix}$$

$$K_{\phi_y} = \begin{bmatrix} 0 & \phi_1 & \phi_2 & - & - & \phi_{N-2} & - & - & 0 & 0 \\ 0 & 0 & 0 & - & - & 0 & - & - & 0 & 0 \\ 0 & 0 & 0 & - & - & 0 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & - & - & 0 & - & - & 0 & 0 \\ 0 & 0 & 0 & - & - & 0 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & - & - & 0 & - & - & 0 & 0 \end{bmatrix}.$$

So, equation (5.59) becomes

$$\hat{\hat{U}} = \hat{\hat{K}}_{\phi_x} \cdot \hat{\hat{K}}_{\phi_y} \cdot \hat{\hat{C}} \tag{5.60}$$

giving a relationship between U and its wavelet coefficients C. Similar relationships to equation (5.56) and (5.60) exist for right hand side function f, i.e.

$$F(X,Y) = f(x,y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g_{k,l}\phi(X-k)\phi(Y-l) \tag{5.61}$$

and

$$\hat{\hat{F}} = \hat{\hat{K}}_{\phi_x} \cdot \hat{\hat{K}}_{\phi_y} \cdot \hat{\hat{g}} \tag{5.62}$$

where $\mathbf{g}$ is the matrix of wavelet coefficients of $F(X,Y)$.

5.3.4 Differential Equation in Wavelet Space

Now, we substitute equation (5.56) and (5.61) into equation (5.62), and we get

$$\frac{\partial^2}{\partial x^2}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} C_{k,l}\phi(X-k)\phi(Y-l)\right] + \frac{\partial^2}{\partial y^2}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} C_{k,l}\phi(X-k)\phi(Y-l)\right]$$

$$\pm\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} C_{k,l}\phi(X-k)\phi(Y-l) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g_{k,l}\phi(X-k)\phi(Y-l)$$

i.e.

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2^{2m}C_{k,l}\phi''(X-k)\phi(Y-l) + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2^{2m}C_{k,l}\phi(X-k)\phi''(Y-l)$$

$$\pm\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} C_{k,l}\phi(X-k)\phi(Y-l) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g_{k,l}\phi(X-k)\phi(Y-l)$$

Taking the inner product of both sides, first with $\phi(X-p)$ and then with $\phi(Y-q)$; $p,q \in Z$, we get

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2^{2m} C_{k,l} \int \phi''(X-k)\phi(X-p)dX \int \phi(Y-l)\phi(Y-q)dY$$
$$+\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2^{2m} C_{k,l} \int \phi(X-k)\phi(X-p)dX \int \phi''(Y-l)\phi(Y-q)dY$$
$$\pm\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} C_{k,l} \int \phi(X-k)\phi(X-p)dX \int \phi(Y-l)\phi(Y-q)dY$$
$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g_{k,l} \int \phi(X-k)\phi(X-p)dX \int \phi(Y-l)\phi(Y-q)dY$$

Since the translation of the scaling functions are mutually orthogonal, this simplifies to

$$\sum_{k=0}^{\infty} C_{k,q} \int \phi''(X-k)\phi(X-p)dX + \sum_{l=0}^{\infty} C_{p,l} \int \phi''(Y-l)\phi(Y-q)dY \pm \frac{C_{p,q}}{2^{2m}} = \frac{1}{2^{2m}} g_{p,q}.$$

So,

$$\sum_{k=0}^{\infty} C_{k,q}\Omega_{p-k} + \sum_{l=0}^{\infty} C_{p,l}\Omega_{q-l} = \frac{1}{2^{2m}}\left[g_{p,q} \pm C_{p,q}\right]$$

where

$$\Omega_{j-k} = \int \phi''(Y-k)\phi(Y-j)dY$$

are the connection coefficients described by Latto et. at [ALT91]. Remembering that $c_{j,k}$ and $g_{j,k}$ have periods n_x and n_y in x and y directions, the matrix form of the differential equation is

$$\mathbf{R}_{n_x}\mathbf{C} + \mathbf{C}\mathbf{R}_{n_y}^{T} = \frac{1}{2^{2m}}\left[\mathbf{g} \mp \mathbf{C}\right] \tag{5.63}$$

where

$$\mathbf{R}_n = \begin{bmatrix}
\Omega_0 & \Omega_{-1} & - & - & \Omega_{2-N} & - & \Omega_{N-2} & - & \Omega_1 \\
\Omega_1 & \Omega_0 & - & - & \Omega_{3-N} & - & 0 & - & \Omega_2 \\
- & - & - & - & - & - & - & - & - \\
\Omega_{N-2} & \Omega_{N-3} & - & - & \Omega_0 & - & 0 & - & 0 \\
0 & \Omega_{N-2} & - & - & \Omega_1 & - & 0 & - & 0 \\
- & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - \\
0 & 0 & - & - & - & - & \Omega_{-1} & - & \Omega_{2-N} \\
\Omega_{2-N} & 0 & - & - & - & - & \Omega_0 & - & \Omega_{3-N} \\
- & - & - & - & - & - & - & - & - \\
\Omega_{-1} & \Omega_{-2} & - & - & 0 & - & - & \Omega_{N-1} & \Omega_0
\end{bmatrix}.$$

The two dimensional FFT of equation (5.63) is

$$\mathbf{F}_{n_x}^{-1}(\mathbf{R}_{n_x}\mathbf{C})\mathbf{F}_{n_y}^{-1} + \mathbf{F}_{n_x}^{-1}(\mathbf{C})\mathbf{R}_{n_y}^{T}\mathbf{F}_{n_y}^{-1} = \mathbf{F}_{n_x}^{-1}\left(\frac{1}{2^{2m}}\left[\mathbf{g} \mp \mathbf{C}\right]\right)\mathbf{F}_{n_y}^{-1}$$

i.e.

$$\left(\mathbf{F}_{n_x}^{-1}\mathbf{R}_{n_x}\mathbf{F}_{n_x}\right)\left(\mathbf{F}_{n_x}^{-1}\mathbf{C}\mathbf{F}_{n_y}^{-1}\right) + \left(\mathbf{F}_{n_x}^{-1}\mathbf{C}\mathbf{F}_{n_y}^{-1}\right)\left(\mathbf{F}_{n_y}\mathbf{R}_{n_y}^{T}\mathbf{F}_{n_y}^{-1}\right) = \frac{1}{2^{2m}}\left(\mathbf{F}_{n_x}^{-1}\left[\mathbf{g} \mp \mathbf{C}\right]\mathbf{F}_{n_y}^{-1}\right)$$

It can be written in the form

$$\Lambda_{n_x}\hat{\hat{C}} + \hat{\hat{C}}\Lambda_{n_y} = \frac{1}{2^{2m}}\left[\hat{\hat{g}} \mp \hat{\hat{C}}\right] \tag{5.64}$$

where

$$F_{n_x}^{-1}R_{n_x}F_{n_x} = \Lambda_{n_x}$$
$$F_{n_y}^{-1}R_{n_y}F_{n_y} = F_{n_y}R_{n_y}^{T}F_{n_y}^{-1} = \Lambda_{n_y}$$

and

$$\Lambda_n = \begin{bmatrix} \lambda_0 & 0 & 0 & 0 & - & - & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & - & - & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & - & - & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & \lambda_{n-2} & 0 \\ 0 & 0 & 0 & 0 & - & - & 0 & \lambda_{n-1} \end{bmatrix}_{(n,n)}.$$

Equation (5.64) may be written as

$$(\mathbf{H}_x + \mathbf{H}_y)\cdot\hat{\hat{C}} = \frac{1}{2^{2m}}\left[\hat{\hat{g}} \mp \hat{\hat{C}}\right] \tag{5.65}$$

where

$$\mathbf{H}_x = \begin{bmatrix} \lambda_0 & \lambda_0 & \lambda_0 & \lambda_0 & - & - & \lambda_0 & \lambda_0 \\ \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & - & - & \lambda_1 & \lambda_1 \\ \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 & - & - & \lambda_2 & \lambda_2 \\ \lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 & - & - & \lambda_3 & \lambda_3 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ \lambda_{n_x-1} & \lambda_{n_x-2} & \lambda_{n_x-2} & \lambda_{n_x-2} & - & - & \lambda_{n_x-2} & \mu_{n_x-2} \\ \lambda_{n_x-1} & \lambda_{n_x-1} & \lambda_{n_x-1} & \lambda_{n_x-1} & - & - & \lambda_{n_x-1} & \lambda_{n_x-1} \end{bmatrix}_{(n_x,n_y)}$$

and

$$\mathbf{H}_y = \begin{bmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & - & - & \lambda_{n_y-2} & \lambda_{n_y-1} \\ \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & - & - & \lambda_{n_y-2} & \lambda_{n_y-1} \\ \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & - & - & \lambda_{n_y-2} & \lambda_{n_y-1} \\ \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & - & - & \lambda_{n_y-2} & \lambda_{n_y-1} \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & - & - & \lambda_{n_y-2} & \lambda_{n_y-1} \\ \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & - & - & \lambda_{n_y-2} & \lambda_{n_y-1} \end{bmatrix}_{(n_x,n_y)}$$

From the properties of Fourier matrix

$$\mathbf{H}_x = \hat{\hat{K}}\Omega_x$$

and

$$\mathbf{H}_y = \hat{\hat{K}}\Omega_y$$

where

$$K_{\Omega_x} = \begin{bmatrix} \Omega_0 & 0 & 0 & 0 & - & - & 0 & 0 \\ \Omega_1 & 0 & 0 & 0 & - & - & 0 & 0 \\ \Omega_2 & 0 & 0 & 0 & - & - & 0 & 0 \\ - & 0 & 0 & 0 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ \Omega_{N-2} & 0 & 0 & 0 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & 0 & 0 \\ \Omega_{2-N} & 0 & 0 & 0 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ \Omega_{-1} & 0 & 0 & 0 & - & - & 0 & 0 \end{bmatrix}$$

$$K_{\Omega_y} = \begin{bmatrix} \Omega_0 & \Omega_1 & \Omega_2 & - & - & \Omega_{N-2} & - & \Omega_{2-N} & - & \Omega_{-1} \\ 0 & 0 & 0 & - & - & 0 & - & - & 0 & 0 \\ 0 & 0 & 0 & - & - & 0 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & - & - & 0 & - & - & 0 & 0 \\ 0 & 0 & 0 & - & - & 0 & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & - & - & 0 & - & - & 0 & 0 \end{bmatrix}$$

Let

$$K_\Omega = K_{\Omega_x} + K_{\Omega_y}$$

This is the convolution kernel of the left hand side of equation (5.63). Thus, equation (5.65) becomes

$$\hat{\hat{K}}_\Omega \cdot \hat{\hat{C}} = \frac{1}{2^{2m}} \hat{\hat{g}} \tag{5.66}$$

Combining equations (5.60), (5.62) and (5.66), we get

$$\hat{\hat{U}} = \left(\hat{\hat{K}}_{\phi_x} \cdot \hat{\hat{K}}_{\phi_y}\right) \cdot \left(\left(\frac{1}{2^{2m}} \hat{\hat{F}} / \left(\hat{\hat{K}}_{\phi_x} \cdot \hat{\hat{K}}_{\phi_y}\right)\right) / \hat{\hat{K}}_\Omega\right)$$

$$\Rightarrow \hat{\hat{U}} = \frac{1}{2^{2m}} \hat{\hat{F}} / \hat{\hat{K}}_\Omega$$

Taking the inverse two dimensional Fourier transform, we get the solution U.

5.3.5 Incorporation of Boundary Conditions

The Capacitance Matrix Method

Boundary conditions may be incorporated using the Capacitance Matrix method (Proskurowski and Widlund [PW96], Qian and Weiss [QW93] and others).

Suppose, we have to solve the problem

$$u_{xx} + u_{yy} \pm u = f \quad in \quad S \tag{5.67}$$

with Dirichlet boundary conditions

$$u = u_\Gamma(x, y) \ \ on \ \ the \ \ boundary \ \ \Gamma.$$

Again, suppose that $u(x, y)$ and $f(x, y)$ are periodic with periods dx and dy in x and y direction, so that the region S lies in the periodic cell $[0, dx]$, $[0, dy]$. f can be made periodic by making it zero outside S. If necessary, the function f may be extended smoothly outside S, so as to make it periodic.

Let the solution to the differential equation with periodic boundary condition be $u(x, y)$. The solution u to the differential equation with Dirichlet boundary condition may be obtained by adding in another function $w(x, y)$ such that

$$u = v + w \tag{5.68}$$

Since,

$$v_{xx} + v_{yy} \pm v = f; \ \ in \ \ S,$$

w must satisfy

$$w_{xx} + w_{yy} \pm w = 0; \ \ in \ \ S.$$

However, on or outside the boundary Γ, $\Delta w = w_{xx} + w_{yy}$ may take such values as to make u satisfy the given boundary condition. The desired effort nay be achieved by placing sources (or delta functions) along a closed curve Γ, which encompasses the region S. Thus, w is given by the solution to the equation

$$w_{xx} + w_{yy} \pm w = X_1; \ \ in \ [0, dx] \times [0, dy]$$

where

$$X_1 \equiv X_1(x, y) = \int_\Gamma X_0(p, q)\delta(x - p, y - q)d\Gamma_1; \ \ (p, q) \in \Gamma_1$$

and $d(x, y)$ is the delta function at $(0, 0)$.

Let $G(x, y)$ be the Green's function of the differential equation, i.e.,

$$G_{xx} + G_{yy} \pm G = \delta(x, y); \ \ in \ [0, dx] \times [0, dy]$$

The periodic Green's function may be easily computed using the periodic solvers. Thus, the solution of BVP is given by convolution

$$W = G(x, y) * X_1(x, y) \tag{5.69}$$

i.e.

$$w = \int_{\Gamma_1} X_0(p, q)G(x - p, y - q)d\Gamma_1; \ \ (p, q) \in \Gamma_1$$

In order to discretize the equation (5.69), consider a finite number of points (p_j, q_j); $j = 1, 2, 3, ..., n_\Gamma$ on the curve Γ_1, where n_Γ is the number of mesh points lying on the boundary Γ. Thus,

$$w(x, y) = \sum_{j=0}^{\infty} X_0(p_j, q_j)G(x - p_j, y - q_j)$$

or simply

$$w(x, y) = \sum_j X_j G(x - p_j, y - q_j) \tag{5.70}$$

Now, the values of $w(x, y)$ on the boundary Γ are known from the BC and the solutions to the periodic problem. Let $(x_i, y_i), i = 1, 2, 3, ..., n_\Gamma$ be the set of mesh points lying on Γ. Then from equation (5.68)

$$w_i \equiv w(x_i, y_i) = u_\Gamma(x_i, y_i) - v_\Gamma(x_i, y_i); \ \ i = 1, 2, 3, ..., n_\Gamma$$

Equation (5.70) may now be written for each point (x_i, y_i) on the boundary as

$$w_i = \sum_{j=0}^{\infty} X_j G(x_i - p_j, y_i - q_j); \quad i = 1, 2, 3, ..., n_\Gamma \tag{5.71}$$

In matrix form this equation becomes

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ - \\ - \\ w_{n_\Gamma - 1} \\ w_{n_\Gamma} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} & - & - & - & - & G_{1n_\Gamma} \\ G_{21} & G_{22} & G_{23} & - & - & - & - & G_{2n_\Gamma} \\ G_{31} & G_{32} & G_{33} & - & - & - & - & G_{3n_\Gamma} \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ G_{n_\Gamma-1,1} & G_{n_\Gamma-1,2} & G_{n_\Gamma-1,3} & - & - & - & - & G_{n_\Gamma-1,n_\Gamma} \\ G_{n_\Gamma,1} & G_{n_\Gamma,2} & G_{n_\Gamma,3} & - & - & - & - & G_{n_\Gamma,n_\Gamma} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ - \\ - \\ X_{n_\Gamma - 1} \\ f_{n_\Gamma} \end{bmatrix}$$

where

$$G_{i,j} = G(x_i - p_i, y_i - q_i).$$

Solving this set of equations yields the values of X_j, which may then be substituted into equation (5.70) to obtained value w. The solution u to the given differential equation is then obtained from (4.8.1).

The choice of the curve Γ, is a matter of critical importance. Qian and Weiss (see [QW93]) have shown that Γ_1 should be offset from the boundary Γ by at least N mesh points (where N is the support of the scaling function) order to control the residual error arising from the finite support of the delta function in wavelet space. The support of delta function in wavelet space may be shown to the same as that of the scaling function by taking the inner product of the expansion

$$\delta(x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} g_{k,l} \phi(X - k) \phi(Y - l)$$

with each of the basis function $\phi(X - i)\phi(Y - j)$. This gives

$$g_{i,j} = 2^{2m} \phi(-i) \phi(-j)$$

from which the support of $g_{i,j}$ is given both coordinate directions.

5.3.6 Convergence and Computation Time for Test Examples

Consider the differential equation

$$u_{xx} + u_{yy} + u = 4 + x^2 + y^2$$

on

$$\Omega = \{(x, y) | -1 \leq x \leq 1; \quad -1 \leq y \leq 1\}$$

with boundary conditions

$$u = x^2 + y^2 \ \ on \ \ \partial\Omega.$$

The exact solution for this problem is

$$u(x, y) = x^2 + y^2.$$

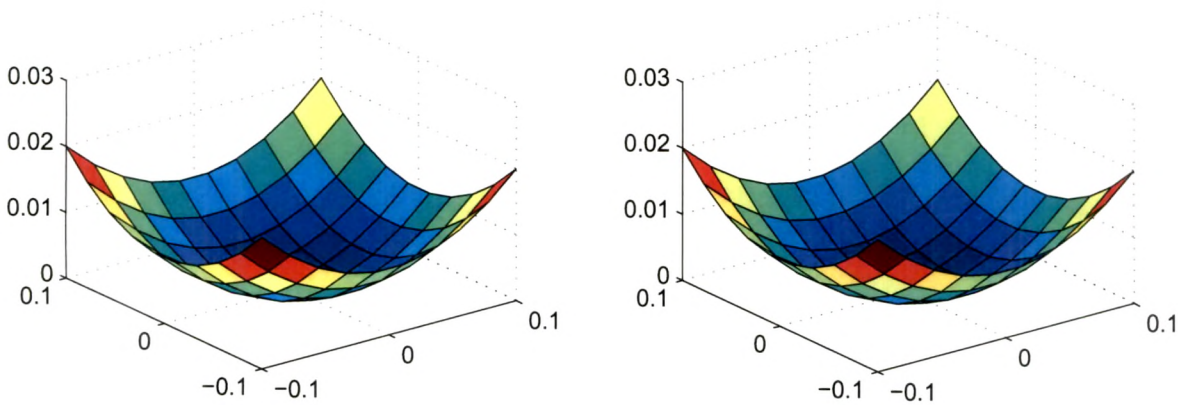

Figure 5.2: **Analytical and Numerical solution**

Figure-5.2 shows the decay of the analytical and numerical solution. The Table 5.1 shows the decay of numerical residual error with increasing sample size n. The figure clearly indicates the high rate of convergence that is obtained with wavelet method. Table-5.2 indicates the variation of computation time, in seconds, with increasing sample size n. The wavelet solution takes slightly longer than the finite difference solution, because we have to transform the sample from physical space into wavelet space and back again. From these results, it can be seen that wavelet solution compares extremely favorably with the finite difference solution.

5.3.7 Conclusion

The wavelet method has been shown to be a powerful numerical tool for the fast and accurate solution of partial differential equations. The procedure described here shows that the solution to the differential equation is related to the equations on right hand side by a sequence of discrete convolution which can be rapidly performed using the FFT. Although the FFT implies that the

n	With Wavelet Method	With Finite Difference
3	2.32 e -2	4.12 e -2
4	2.21 e -2	4.69 e -2
5	2.15 e -2	3.62 e -2
6	1.00 e -2	2.62 e -2

Table 5.1: **Residual error of wavelet method and finite difference method**

n	With Wavelet Method	With Finite Difference
3	10.52	20.51
4	13.89	30.52
5	18.21	39.53
6	19.41	50.63

Table 5.2: **Computational time for wavelet method and finite difference method**

solution is periodic, we may incorporate non-periodic boundary conditions using the periodic Green's function. Solutions obtained using the wavelet method have been compared with those obtained using finite difference method and the wavelet solutions have been found to converge much faster than the finite difference solutions, the gains in accuracy, particularly with higher order wavelets, far outweighs the increase in cost. Furthermore, wavelet have the capacity of representing the solutions at different levels of resolution, which makes them particularly useful for developing hierarchical solutions to the engineering problems.