Chapter 10

Application of Wavelet Methods and FPM to Industrial Problems

In this chapter, we have made an attempt to develop two mathematical models of real life problem. We have solved one model by wavelet technique and another model by FPM technique and then we have compared them with the finite difference solution.

10.1 Numerical Simulation of of Cooling of Coke in a Can

10.1.1 Introduction

People like to drink cold coke. Unfortunately, cold coke is not always available. So, we try to make the coke cool by putting it into the refrigerator. Now, the natural question arises how much time we have to put the can into the refrigerator so that we will get a cooled coke. We have made an attempt to study the cooling process of coke in a refrigerator. We may assume that the can has a temperature of $24°C$ originally, and the refrigerator is at $4°C$ and then we shall try to simulate how the heat diffuses out of the can containing coke. Initially, we have presented a mathematical model under certain assumptions. After doing necessary transformations and scaling of the problem, we have used numerical scheme to solve the problem for heat diffusion to achieve our objective.

10.1.2 Mathematical Model

We shall begin with the geometry of can. A can of coke is usually almost like a cylinder. We shall consider that the 1/2 liter can have radius of $r_c = 32\ mm$ and the height is about $h_c = 160\ mm$, approximately (see Figure-10.1). The process of heat conduction is unsteady or transient if the temperature field varies with time, i.e. the body is being heated or cooled. If a solid body is subjected to sudden heating or cooling, sometimes it must lapse before an equilibrium temperature is reached. It is during this intrim period that the change in internal energy of the body as the time takes place whereupon the temperature profile that the body can be represented by the three dimensional heat conduction partial differential equation which reads as follows:

$$\rho C \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + f(x, y, z, t), \tag{10.1}$$

where $T(x, y, z, t)$, ρ, C, K, and $f(x, y, z, t)$ are respectively, the temperature, the density, the specific heat, the heat conductivity, and the heat source per unit time per unit volume, at a point (x, y, z) at time t.

Here, we assume the following:

- The physical properties of coke are equal to water because most of the coke is water.
- The numbers ρ, C, and K which we are going to mention in our simulation will vary with pressure and temperature. These numbers will be the approximate numbers at the room temperature and with earth pressure.
- There is no generation of heat in coke, i.e. $f(x, y, z, t) = 0$.

Thus, the equation (10.1) can be reduced to

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C} \nabla \cdot (\nabla T). \tag{10.2}$$

So,

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right).$$

Now, assume that there is no variation of heat in z-direction which shows that the temperature is a function of x, y and t only, i.e.

$$T = T(x, y, t).$$

So, equation (10.2) can be represented as

$$\frac{\partial T(x, y, t)}{\partial t} = \frac{k}{\rho C} \nabla \cdot (\nabla T(x, y, t)), \tag{10.3}$$

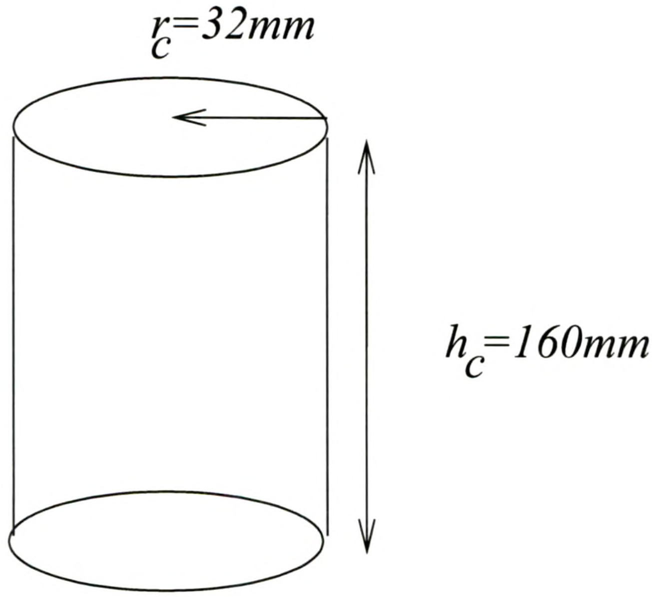

Figure 10.1: **The Can of Coke**

which gives

$$\frac{\partial T(x,y,t)}{\partial t} = \frac{k}{\rho C}\left(\frac{\partial^2 T(x,y,t)}{\partial x^2} + \frac{\partial^2 T(x,y,t)}{\partial y^2}\right).$$

In general, the assumption $\partial T/\partial z = 0$, will not be true. However, if the bottom and the top of the can are insulated, it will be a reasonable assumption. So, it is reasonable to consider that all changes in temperature are in (x, y) directions. Also, if the height of can of coke is much larger than the radius, then the temperature variations in z-direction will be small enough. In this situation also, $\partial T/\partial z = 0$ is a reasonable assumption.

Now, we shall convert equation (10.3) into the polar co-ordinates as the can of coke is almost a cylinder. So, we have a two-dimensional problem with circular geometry. We may take the solution as radially symmetric where the coordinate system is placed with the assumption that the origin is at the center of the base circle of the can. Physically, this assumption is a reasonable one because the temperature is constant initially which is radially symmetric function and the boundary is also symmetric. Therefore, the heat flow will be symmetric.

If $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$, then we have

$$\nabla(\nabla T) = \nabla^2 T = \Delta T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}.$$

But we assumed that

$$\frac{\partial T}{\partial \theta} = 0; \quad \frac{\partial T}{\partial z} = 0.$$

Thus, we have

$$\nabla \cdot (\nabla T(r,t)) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T(r,t)}{\partial r}\right). \tag{10.4}$$

Thus, equation (10.3) can be written as

$$\frac{\partial T(r,t)}{\partial t} = \frac{k}{\rho C}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T(r,t)}{\partial r}\right). \tag{10.5}$$

When we want to solve the equation (10.5), we need certain initial conditions and boundary conditions. We have assumed that the temperature of coke is $24°C$ initially, i.e.

$$T(r,0) = T_0(= 24°C). \tag{10.6}$$

If the refrigerator is large compared to the can of coke, we can assume that the air in the refrigerator is constant at $4°C$, i.e. temperature of the refrigerator is $T_R = 4°C$.

Now, Newton's law of cooling says that the heat flow out of the boundary is proportional to the difference between the temperature in the coke and the temperature in the refrigerator, i.e.

$$-k\frac{\partial T(r_c,t)}{\partial r} = h_{T_c}\left(T(r_c,t) - T_R\right). \tag{10.7}$$

Here, the heat transfer coefficient is usually be approximated by

$$h_{T_c} = 5\frac{W}{m^2K}, \tag{10.8}$$

when the air is not moving. However, if the air in the refrigerator starts to move, the coefficient h_{T_c} will increase. If the coefficient increases very much, we may get the boundary condition:

$$T(r_c,t) = T_R. \tag{10.9}$$

If we assume that

$$\left|\frac{\partial T(r_c,t)}{\partial r}\right| < \infty,$$

and when $h_{T_c} \to \infty$, we say that (10.9) follows from (10.7). The boundary condition at $r = 0$ (see Figure-10.2) is given by

$$\frac{\partial T}{\partial r}(0,t) = 0. \tag{10.10}$$

We assume that (10.9) is valid and we write the full system as

$$\frac{\partial T(r,t)}{\partial t} = \frac{k}{\rho C}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T(r,t)}{\partial r}\right); \quad (r,t) \in (0,r_c) \times (0,t_{end}) \tag{10.11}$$

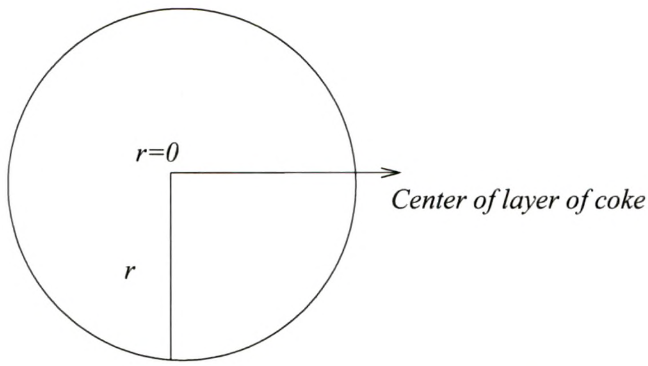

Figure 10.2: **Center of one layer of Coke**

$$\frac{\partial T(0,t)}{\partial r} = 0; \tag{10.12}$$

$$T(r_c, t) = T_R; \tag{10.13}$$

$$T(r, 0) = T_0. \tag{10.14}$$

If Newton's law of cooling is used for the boundary, then (10.13) is replaced by

$$-k\frac{\partial T(r_c,t)}{\partial r} = h_{T_c}\left(T(r_c,t) - T_R\right). \tag{10.15}$$

10.1.3 Dimension Analysis

Now, we would like to scale our problem in the following way:

$$\bar{r} = \frac{r}{r_c}; \quad \bar{T} = \frac{T - T_R}{T_0 - T_R}; \quad and \quad \bar{t} = \frac{t}{t_c} \tag{10.16}$$

where $r_c = 0.032\ m$, $T_R = 24°C$, and t_c is the characteristic time. It can be proved very easily that

$$\frac{\partial T}{\partial t} = \left(\frac{T_0 - T_R}{t_c}\right)\frac{\partial \bar{T}}{\partial \bar{t}}; \tag{10.17}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \left(\frac{T_0 - T_R}{r_c^2}\right)\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \bar{T}}{\partial \bar{r}}\right). \tag{10.18}$$

The Newton's law of cooling on scaled form can be written as

$$-\frac{k}{h_{T_c} r_c}\frac{\partial \bar{T}(1,\bar{t})}{\partial \bar{r}} = \bar{T}(1,\bar{t}). \tag{10.19}$$

So, equation (10.11) can be written as

$$\left(\frac{T_0 - T_R}{t_c}\right)\frac{\partial \bar{T}}{\partial \bar{t}} = \frac{k}{\rho C}\left(\frac{T_0 - T_R}{r_c^2}\right)\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \bar{T}}{\partial \bar{r}}\right).$$

So,

$$\frac{\partial \bar{T}}{\partial \bar{t}} = \frac{t_c k}{\rho C r_c^2}\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \bar{T}}{\partial \bar{r}}\right).$$

Now, put

$$t_c = \frac{\rho C r_c^2}{k}.$$

So,

$$\frac{\partial \bar{T}}{\partial \bar{t}} = \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \bar{T}}{\partial \bar{r}}\right).$$

Thus, we have the following set of equations:

$$\frac{\partial \bar{T}}{\partial \bar{t}} = \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \bar{T}}{\partial \bar{r}}\right); \quad (\bar{r}, \bar{t}) \in (0,1) \times (0, \bar{t}_{end}); \tag{10.20}$$

$$\frac{\partial \bar{T}}{\partial \bar{r}} = 0; \tag{10.21}$$

$$\bar{T}(1, \bar{t}) = 0; \tag{10.22}$$

$$\bar{T}(\bar{r}, 0) = 1; \tag{10.23}$$

Now, we define $\bar{T} = u$ and $\bar{r} = x$, and write the above system as:

$$\frac{\partial u}{\partial \bar{t}} = \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right); \quad (x, \bar{t}) \in (0,1) \times (0, \bar{t}_{end}); \tag{10.24}$$

$$\frac{\partial u}{\partial x}(0, \bar{t}) = 0; \tag{10.25}$$

$$u(1, \bar{t}) = 0; \tag{10.26}$$

$$u(x, 0) = 1; \tag{10.27}$$

where $h_{T_c} = \infty$. When $h_{T_c} < \infty$, (10.25) is replaced with

$$-\frac{k}{h_{T_c} r_c}\frac{\partial u(1, \bar{t})}{\partial x} = u(1, \bar{t}). \tag{10.28}$$

Note that these equations are in dimensionless form.

10.1.4 Finite Difference Method

Now, we shall use stable finite difference method to solve the equations (10.24) - (10.27). Equation (10.24) can be written as follows:

$$\frac{\partial u}{\partial \bar{t}} = \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)$$

i.e.

$$\frac{\partial u}{\partial \bar{t}} = \frac{1}{x}\left[1.\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2}\right].$$

So, we have

$$\frac{\partial u}{\partial \bar{t}} = \left[\frac{1}{x}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right]; \tag{10.29}$$

with the boundary conditions

$$\frac{\partial u}{\partial x}(0, \bar{t}) = 0; \tag{10.30}$$

$$u(1, \bar{t}) = 0; \tag{10.31}$$

$$u(x, 0) = 1. \tag{10.32}$$

We have considered only the case when $h_{T_c} = \infty$, but not the case when $h_{T_c} < \infty$. Now, we shall use explicit finite difference method to solve equations (10.29) to (10.32).

Now, in general we know the following descretizations

$$\left(\frac{\partial u}{\partial t}\right)_i = \frac{u_{i+1} - u_i}{\Delta t} + O(\Delta t);$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} + O\left((\Delta x)^2\right).$$

Using above finite differences, we have the following discretization of equation (10.29):

$$u(i+1, j) = u(i, j) + \frac{\Delta t}{x\Delta x}\left[u(i, j+1) - u(i, j)\right] + \frac{\Delta t}{(\Delta x)^2}\left[u(i, j+1) - 2u(i, j) + u(i, j-1)\right]. \tag{10.33}$$

Our numerical algorithm is as follows:

1. Input time step-size and space step-size.
2. Input initial conditions or boundary conditions, (i.e. equations (10.31) and (10.32))

$$u(i, m+1) = 0; \quad i = 2 : n+1,$$

$$u(1, j) = 1; \quad j = 1 : m+1.$$

3. Evaluate $u(i+1, j)$ by the formula (10.33).
4. Use the condition (10.30), i.e.

$$u(i+1,1) = u(i+1,2); \quad for \ \ i = 1 : n.$$

5. End

10.1.5 Numerical Simulation

We use the following data to do numerical simulations:

1. We assume that the physical properties of coke and water are equal. Therefore,

$$\rho = 1000 \ kg/m^3; \quad C = 4200 \ J/kg \circ K; \quad k = 0.58 \ W/m \circ K.$$

2. Radius and temperature is taken to be

$$r_c = 0.032 \ m; \quad T_R = 4°C; \quad T_0 = 24°C.$$

3. We may calculate t_c according to the formula

$$t_c = \frac{\rho C r_c^2}{k}.$$

We have used Matlab software to run our program.

10.1.6 Results and Discussion

The solution of this model is calculated by Finite difference method and wavelet method using preconditioning concepts. The simulation results shows that:

1. At time $t = t_1$ (say), as displacement increases from the center of the can of coke, the temperature of coke is 1 for certain displacement and then it fall down.
2. As time $t = t_2$, as displacement increases from the center of the can of coke, the temperature of coke is 1 for certain displacement and then it fall down.

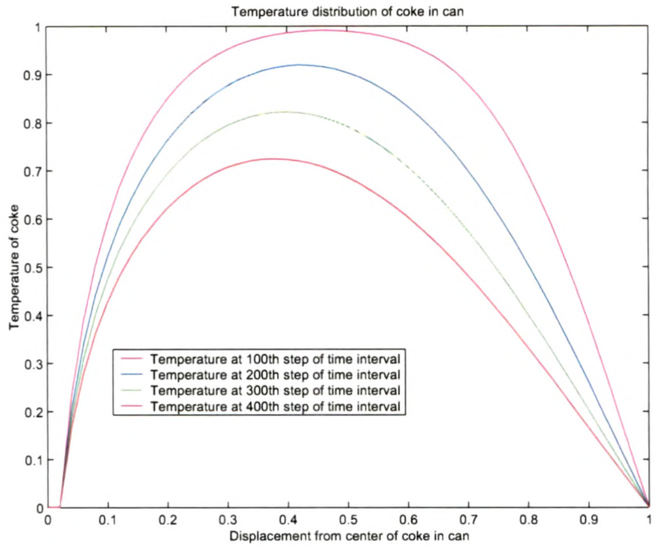

Figure 10.3: **Numerical Simulation of Cooling of Coke in a Can**

Figure-10.3 shows the temperature profile of coke in a can at various time. It is obvious that as we move along the radius of can of a coke towards the boundary, the temperature of the coke decreases, i.e. as time increases the temperature falls down very rapidly. The error FPM solution and wavelet solution is 10^{-2}. As discussed in Chapter 7, 8, and 9, we have used the preconditioners of the type ILU (0) and GMRES method to solve the linear system of equations.

Δt	$\|u_{WG} - u_{EFD\|}$
0.01	0.1450 e - 2
0.02	0.2415 e - 2
0.03	0.8450 e - 2
0.04	0.9450 e - 2
0.05	1.2 e - 2

Table 10.1: **The relative error at different time steps.**

10.2 Mathematical Models for Pressure Distribution in a Slider Bearing Lubricated with Viscous Fluid

10.2.1 Introduction

The term lubricate means: to make smooth, slippery or oily in motion. Lubricants are used to reduce friction. Today's technology demands sophisticated lubricants which can be solids, liquids, or gaseous which perform very well even in bad environment. Lubricants form a layer between two surfaces and thereby prevent there direct contact, which reduces friction between the moving parts. However, there are a number of factors to be taken into account while deciding the nature of lubricant which may be suitable for given application. The properties of lubricants like pressure, viscosity, ignition temperature, and thermal conductivity are important. The pressure distribution in the bearing should be known for proper functioning of machinery. Some mathematical models are presented for simulating various bearing configuration.

10.2.2 Navier-Stokes Lubrication Equations

The study of the motion of fluids is very much important to lubrication: the displacement of two solid surfaces by a thin film of fluid. The forces that allow a thin film of fluid to separate the two even under very heavy loads are connected by the hydrodynamic forces. Hence, hydrodynamics is a basic element of lubrication theory. However, hydrodynamics alone is insufficient to describe the full reality of lubrication, because the lubricant properties are temperature dependent. Moreover, viscosity is a dissipative agent, turning mechanical energy into heat energy which gives itself as a temperature change in the fluid. Thus, in many areas of lubrication, one must consider thermal effects which are linked through the viscosity to the hydrodynamics of the system. Another basic element of lubrication theory is the relationship between hydrodynamics and flow channel. The geometry of the flow channel can change with pressure if the surface are complicated. This couples hydrodynamics with deformation mechanics.

The basic equations of hydrodynamics for a viscous fluid are the Navier-Stokes equations. Although Navier-Stokes equations are rather complicated, but the approximations can be made for study of lubrication because the lubrication involves thin films of fluids. The thin film approximation is the classical approximation for boundary layer theory. However, in addition to the thin film approximations, in most of the situations, the lubricant is so viscous that momentum convected by the fluid is insufficient. In this situation, one can remove inertial terms in the Navier-Stokes equations and develop from them the Reynold's equations, which is the nucleus of the hydrodynamics of lubrication. The Reynold's equations provide a relationship between the fluid pressure, film thickness, fluid velocity, speed of the lubricated surface, and geometry of the lubricant channel.

Now, we shall present steady-state Navier-Stokes lubrication equations.

Steady-State Navier-Stokes Lubrication Equations

Consider the lubricant channel as shown in the Figure-10.4. In two dimensions, the Navier-

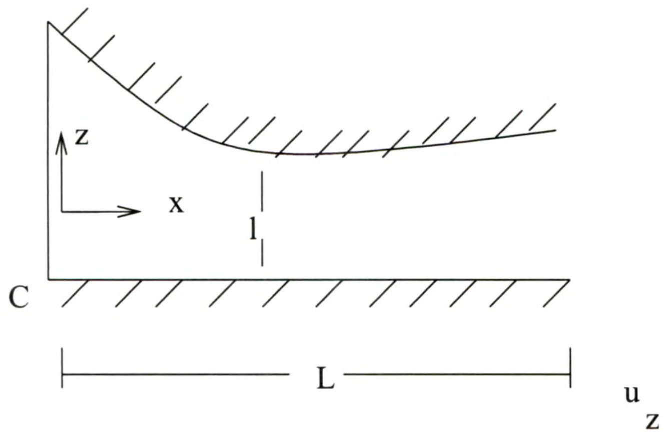

Figure 10.4: Schematic of Lubricant flow channel

Stokes equations can be represented as:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{10.34}$$

$$\rho\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{10.35}$$

$$\rho\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right), \tag{10.36}$$

where ρ, μ, and p are density, kinematic viscosity, and pressure, respectively. u and w are velocity components in x and z directions, respectively. Now, we change to dimensionless variable to scale the equations. Let $u = u_p\bar{u}$, $x = L\bar{x}$, $z = l\bar{z}$, $w = w_0\bar{w}$, and $p = P\bar{p}$. With these new variables, equation (10.34) becomes

$$\frac{\partial \bar{u}}{\partial \bar{x}}\frac{u_p}{L} + \frac{\partial \bar{w}}{\partial \bar{z}}\frac{w_0}{l} = 0,$$

or

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{w}}{\partial \bar{z}} \frac{w_0}{l} \frac{L}{u_p} = 0.$$

Since u will depend on x, neither term of the continuity equation can be removed. So, reasonable value of w_0 is $w_0 = \frac{u_p l}{L}$. So,

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0. \tag{10.37}$$

Now, equation (10.35) becomes

$$\rho \left(u_p \bar{u} \frac{u_p}{L} \frac{\partial \bar{u}}{\partial \bar{x}} + w_0 \bar{w} \frac{u_p}{l} \frac{\partial \bar{u}}{\partial \bar{z}} \right) = -\frac{P}{L} \frac{\partial \bar{p}}{\partial \bar{x}} + \mu \left(\frac{u_p}{L^2} \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{u_p}{l^2} \frac{\partial^2 \bar{u}}{\partial \bar{z}^2} \right).$$

So,

$$\rho \left(\bar{u} \frac{u_p^2}{L} \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{w_0 \bar{w} u_p}{l} \frac{\partial \bar{u}}{\partial \bar{z}} \right) = -\frac{P}{L} \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\mu u_p}{l^2} \left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} \left(\frac{l}{L} \right)^2 + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2} \right).$$

Since $\frac{l}{L} << 1$, the term containing l/L can be removed. Then rewriting the above equation, we have

$$\frac{\rho u_p^2}{L} \left(\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{w} \frac{\partial \bar{u}}{\partial \bar{z}} \right) = -\frac{P}{L} \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\mu u_p}{l^2} \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}.$$

So,

$$\frac{\rho u_p^2}{L} \frac{l^2}{\mu u_p} \left(\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{w} \frac{\partial \bar{u}}{\partial \bar{z}} \right) = -\frac{P}{L} \frac{l^2}{\mu u_p} \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}.$$

Let

$$P = \frac{\mu u_p L}{l^2}.$$

Then, above equation reduces to

$$\frac{\rho u_p L}{\mu} \left(\frac{l}{L} \right)^2 \left(\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{w} \frac{\partial \bar{u}}{\partial \bar{z}} \right) = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}.$$

Hence, we finally get

$$R \left(\frac{l}{L} \right)^2 \left(\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{w} \frac{\partial \bar{u}}{\partial \bar{z}} \right) = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}, \tag{10.38}$$

where R is the Reynold's number and is given by

$$R = \frac{\rho u_p L}{\mu}.$$

In general, for many lubrication problem $R \left(\frac{l}{L} \right)^2 << 1$. So, the left side of equation (10.38), i.e. the inertial term can be removed. Hence, we get

$$\frac{\partial \bar{p}}{\partial \bar{x}} = \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}. \tag{10.39}$$

In a similar fashion, equation (10.36) becomes

$$\rho\left(u_p\bar{u}\frac{w_0}{L}\frac{\partial\bar{w}}{\partial\bar{x}}+w_0\bar{w}\frac{w_0}{l}\frac{\partial\bar{w}}{\partial\bar{z}}\right)=-\frac{P}{l}\frac{\partial\bar{p}}{\partial\bar{z}}+\mu\left(\frac{w_0}{L^2}\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right)+\mu\left(\frac{w_0}{l^2}\frac{\partial^2\bar{w}}{\partial\bar{z}^2}\right).$$

So, we have

$$\rho\left(\bar{u}\frac{u_p^2l}{L^2}\frac{\partial\bar{w}}{\partial\bar{x}}+\frac{\bar{w}u_p^2l}{L^2}\frac{\partial\bar{w}}{\partial\bar{z}}\right)=-\frac{\mu u_pL}{l^3}\frac{\partial\bar{p}}{\partial\bar{z}}+\frac{\mu u_pl}{L^3}\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+\frac{\mu u_p}{lL}\frac{\partial^2\bar{w}}{\partial\bar{z}^2}.$$

Hence

$$\rho\left(\bar{u}\frac{u_p^2l}{L^2}\frac{\partial\bar{w}}{\partial\bar{x}}+\frac{\bar{w}u_p^2l}{L^2}\frac{\partial\bar{w}}{\partial\bar{z}}\right)=-\frac{\mu u_pL}{l^3}\frac{\partial\bar{p}}{\partial\bar{z}}+\frac{\mu u_p}{lL}\left[\left(\frac{l}{L}\right)^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+\frac{\partial^2\bar{w}}{\partial\bar{z}^2}\right].$$

Since $\left(\frac{l}{L}\right)^2 << 1$, we can neglect the first term in the right hand side brackets. Hence,

$$\rho\left(\bar{u}\frac{u_p^2l}{L^2}\frac{\partial\bar{w}}{\partial\bar{x}}+\frac{\bar{w}u_p^2l}{L^2}\frac{\partial\bar{w}}{\partial\bar{z}}\right)=-\frac{\mu u_pL}{l^3}\frac{\partial\bar{p}}{\partial\bar{z}}+\frac{\mu u_p}{lL}\frac{\partial^2\bar{w}}{\partial\bar{z}^2}.$$

So,

$$\frac{\rho u_p^2l}{L^2}\frac{l^3}{\mu u_pL}\bar{u}\frac{\partial\bar{w}}{\partial\bar{x}}+\frac{\rho u_p^2l}{L^2}\frac{l^3}{\mu u_pL}\bar{w}\frac{\partial\bar{w}}{\partial\bar{z}}=-\frac{\partial\bar{p}}{\partial\bar{x}}+\frac{\mu u_p}{lL}\frac{l^3}{\mu u_pL}\frac{\partial^2\bar{w}}{\partial\bar{z}^2}.$$

Finally, we have

$$R\left(\frac{l}{L}\right)^4\left(\bar{u}\frac{\partial\bar{w}}{\partial\bar{x}}+\bar{w}\frac{\partial\bar{u}}{\partial\bar{z}}\right)=-\frac{\partial\bar{p}}{\partial\bar{x}}+\left(\frac{l}{L}\right)^2\frac{\partial^2\bar{w}}{\partial\bar{z}^2}. \tag{10.40}$$

Since $R\left(\frac{l}{L}\right)^4$ is negligible, the left hand side of equation (10.40) is essentially zero. Further, second term on the right hand side is also negligible, since $\left(\frac{l}{L}\right)^2 << 1$. So, from equation (10.40), we have only

$$\frac{\partial\bar{p}}{\partial\bar{z}}=0. \tag{10.41}$$

So, equation (10.37), (10.39), and (10.41) constitute lubrication equations in two dimensions.

In three dimensions, if we take $v=u_p\bar{v}$, and $y=L\bar{y}$, we get an equation similar to (10.38). So, the lubrication equations in three dimensional form of equations are as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{10.42}$$

$$\frac{\partial p}{\partial x}=\mu\frac{\partial^2u}{\partial z^2}, \tag{10.43}$$

$$\frac{\partial p}{\partial y}=\mu\frac{\partial^2v}{\partial z^2}, \tag{10.44}$$

$$\frac{\partial p}{\partial z}=0. \tag{10.45}$$

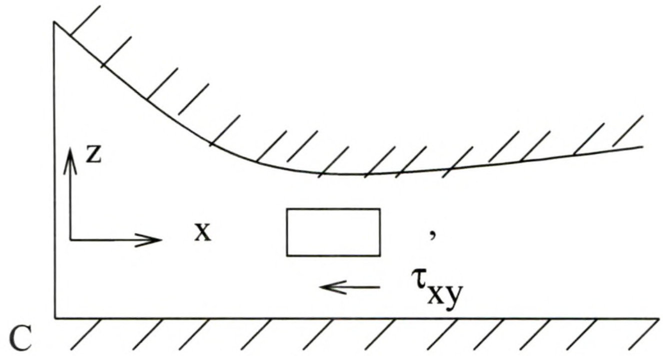

Figure 10.5: Shear stress on element of fluid

We have derived the Navier-Stokes equations with constant viscosity. Now, we shall remove this restriction. Define shear stresses τ'_{xz}, τ'_{yz} on an element of fluid as shown in the Figure-10.5. A force balance on the element of the fluid gives

$$\frac{\partial p}{\partial x} = \frac{\partial \tau'_{xz}}{\partial z}; \quad \frac{\partial p}{\partial y} = \frac{\partial \tau'_{yz}}{\partial z}. \tag{10.46}$$

Now, if the fluid is Newtonian, then we have

$$\tau'_{xz} = \mu \frac{\partial u}{\partial z}; \quad \tau'_{yz} = \mu \frac{\partial v}{\partial z}. \tag{10.47}$$

From equation (10.46) and equation (10.47), we have

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right); \quad \frac{\partial p}{\partial y} = \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right); \quad \frac{\partial p}{\partial z} = 0. \tag{10.48}$$

We can add more general form of the continuity equation, which is as follows:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0. \tag{10.49}$$

Reynold's Equation for Viscosity Constant Across Thin Film

Now, we assume that $\mu \neq \mu(z)$, but $\mu = \mu(p)$. From first two equations of (10.48) we have

$$\frac{1}{\mu}\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial z^2}; \quad \frac{1}{\mu}\frac{\partial p}{\partial y} = \frac{\partial^2 v}{\partial z^2}. \tag{10.50}$$

Now, integrating equation (10.50) with respect to z with conditions:

$$u(0) = u_0; \quad v(0) = v_0,$$

and

$$u(h) = v(h) = 0,$$

we have

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}z(z-h) + u_0\left(\frac{h-z}{h}\right); \quad v = \frac{1}{2\mu}\frac{\partial p}{\partial y}z(z-h) + v_0\left(\frac{h-z}{h}\right). \tag{10.51}$$

Now, assuming $\rho \neq \rho(z)$ and integrating the continuity equation (10.49), we shall get

$$\int_0^h \frac{\partial}{\partial x}(\rho u)dz + \int_0^h \frac{\partial}{\partial y}(\rho v)dz + \int_0^h \frac{\partial}{\partial z}(\rho w)dz + \int_0^h \frac{\partial \rho}{\partial t}dz = 0,$$

$$\int_0^h \frac{\partial}{\partial x}(\rho u)dz + \int_0^h \frac{\partial}{\partial y}(\rho v)dz + \rho w(h) + \frac{\partial \rho}{\partial t}h = 0, \tag{10.52}$$

where $w(h) = \frac{\partial h}{\partial t}$. But, Leibnitz's rule says that

$$\frac{\partial}{\partial t}\left(\int_{a(t)}^{b(t)}\right) = f\left[b(t), t\right]\frac{db}{dt} - f\left[a(t), t\right]\frac{da}{dt} + \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}dx.$$

So, equation (10.52) becomes

$$\frac{\partial}{\partial x}\int_0^h (\rho u)dz + \frac{\partial}{\partial y}\int_0^h (\rho v)dz + \frac{\partial \rho h}{\partial t} = 0. \tag{10.53}$$

Since $\rho \neq \rho(z)$ is assumed, it can be reduced from integral, from equation (10.51), we have

$$\int_0^h udz = \frac{-1}{12\mu}\frac{\partial p}{\partial x}h^3 + \frac{u_0 h}{2}; \quad \int_0^h vdz = \frac{-1}{12\mu}\frac{\partial p}{\partial y}h^3 + \frac{v_0 h}{2}. \tag{10.54}$$

Note that u_0 and v_0 could be dependent on x, y, and t. Substituting equation (10.54) into equation (10.53), we have

$$\frac{\partial}{\partial x}\rho\left(\frac{-1}{12\mu}\frac{\partial p}{\partial x}h^3 + \frac{u_0 h}{2}\right) + \frac{\partial}{\partial y}\rho\left(\frac{-1}{12\mu}\frac{\partial p}{\partial y}h^3 + \frac{v_0 h}{2}\right) + \frac{\partial \rho h}{\partial t} = 0.$$

So,

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\rho h^3}{\mu}\frac{\partial p}{\partial y}\right) - 12\frac{\partial \rho h}{\partial t} - 6\left[\frac{\partial}{\partial x}(\rho u_0 h) + \frac{\partial}{\partial y}(\rho v_0 h)\right] = 0.$$

Finally, we have

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\rho h^3}{\mu}\frac{\partial p}{\partial y}\right) = 12\frac{\partial \rho h}{\partial t} + 6\left[\frac{\partial}{\partial x}(\rho u_0 h) + \frac{\partial}{\partial y}(\rho v_0 h)\right], \tag{10.55}$$

which is the Reynold's equation for unsteady, compressible flow with restriction that $\mu \neq \mu(z)$ and $\rho \neq \rho(z)$.

10.2.3 Slider Bearing

Model 1: Mathematical Model using Reynold's Equation

Consider the slider bearing geometry as shown in Figure-10.6. A simple slider bearing has two

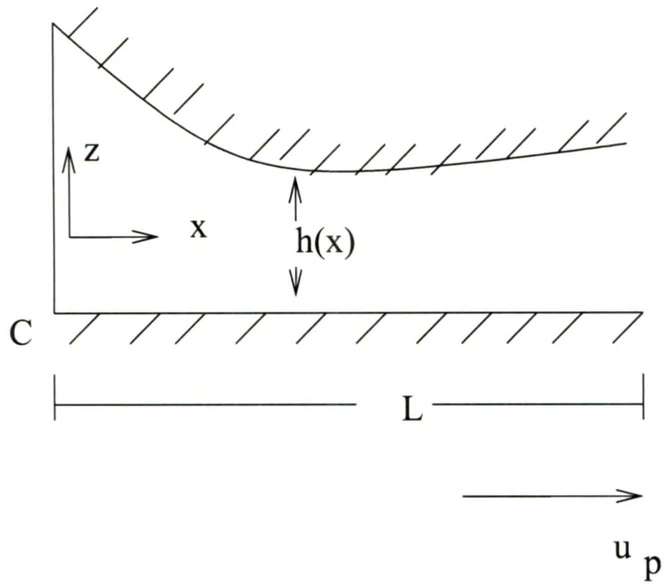

Figure 10.6: Lubricant flow channel with varying height

surfaces of given profile separated by a gap filled with the lubricant. One surface moves with velocity u_p related to other surface. From equation (10.55), we get

$$\frac{d}{dx}\left(h^3\frac{dp}{dx}\right) = 6\mu u_p\frac{dh}{dx}.$$

Integrating above equation, we have

$$\frac{dp}{dx} = 6\mu u_p\left(\frac{h-c}{h^3}\right). \tag{10.56}$$

With $p(0) = p(L)$, integration of above equation from 0 to L gives the constant of integration. So, we have

$$c = \int_0^L \frac{dx}{h^2} \div \int_0^L \frac{dx}{h^3}.$$

Define

$$I_{mn} = \int_0^L \frac{x^m dx}{h^n}.$$

Then

$$c = \frac{I_{02}}{I_{03}}.$$

Now, integrating equation equation (10.56) from 0 to x, we obtain the pressure as follows:

$$p(x) = 6\mu u_p \left[\int_0^x \frac{dx'}{h^2(x')} - \frac{I_{02}}{I_{03}} \int_0^x \frac{dx'}{h^3(x')} \right].$$

For design purpose of bearing, it is better to calculate load. Load is defined as follows:

$$w = \int_0^L p(x)dx.$$

After some calculation, we have

$$w = 6\mu u_p \left(\frac{I_{02} I_{13}}{I_{03}} - I_{12} \right).$$

Model 2: Sophisticated Model

In [PPU02], we have proved that bearing with parallel plates can support no load. Now, we assume that bottom surface is flat and the profile of top surface is given by $z = h(x)$. If we scale $h(x)$ by $l\bar{h}(\bar{x})$, we again have the same equations as equations (10.37), (10.39), and (10.41). Let $\bar{p}(\bar{x})$ be the pressure inside the fluid. From equation (10.39), we obtain $\bar{u}$ by integrating twice. We have

$$\bar{u} = \frac{d\bar{p}}{d\bar{x}} \frac{\bar{z}^2}{2} + C_2 \bar{z} + C_3.$$

Now, impose boundary conditions

$$\bar{u} = 0 \ \ at \ \ \bar{z} = 0,$$

and

$$\bar{u} = 1 \ \ at \ \ \bar{z} = \bar{h}(\bar{x}),$$

we have

$$\bar{u} = \frac{\bar{z}^2 - \bar{z}\bar{h}}{2} \frac{d\bar{p}}{d\bar{x}} + \frac{\bar{z}}{\bar{h}}.$$

Now, from continuity equation (10.37) and $\bar{w} = 0$ at $\bar{z} = \bar{h}(\bar{x})$, we get

$$\frac{\bar{h}^3(\bar{x})}{12} p''(\bar{x}) + \frac{\bar{h}^2(\bar{x})}{4} \frac{d\bar{h}}{d\bar{x}} p'(\bar{x}) + \frac{1}{2} \frac{d\bar{h}}{d\bar{z}} = 0.$$

We can assume that $\bar{h}(\bar{x}) = k_1 \bar{x} + k_2 \bar{x}$ and if we let $k_1 = k_2 = 1$, we have

$$p(\bar{x}) = \frac{6k_1}{k_1\bar{x} + k_2} + \frac{k_1 \epsilon_1}{(k_1\bar{x} + k_2)^2} - \epsilon_2.$$

where ϵ_1 and ϵ_2 are determined by putting $\bar{p} = 0$ at $\bar{z} = 0$ and $\bar{z} = \bar{h}(\bar{x})$. So,

$$p(\bar{x}) = \frac{2\bar{x}(1 - \bar{x})}{(1 + \bar{x})^2},$$

and load is given by

$$w = \int_0^1 \bar{p}(\bar{x}) d\bar{x}.$$

Model 3: Mathematical Model with a Pressure Difference at the Boundaries

In earlier models, it was assumed that the pressure was zero at $x = 0$ and $z = L$. The problem will now be considered in which the boundary conditions are $p(0) = p_1$ and $p(L) = p_2$. We may simplified the analysis by recognizing that the solution for $u_p = 0$ with a pressure gradient simply superimposes on the solution with a finite u_p. Thus, the Reynold's equation (10.55) is reduced to

$$\frac{d}{dx}\left(h^3 \frac{dp}{dx}\right) = 0.$$

Integrating, we have

$$\frac{dp}{dx} = \frac{C}{h^3}.$$

So,

$$p(x) = C \int_0^x \frac{dx'}{h^3} + C_1. \tag{10.57}$$

But $p(0) = p_1$, so

$$p_1 = C_1. \tag{10.58}$$

Now, $p(L) = p_2$ gives

$$p_2 = C \int_0^L \frac{dx'}{h^3} + p_1.$$

Hence,

$$\frac{p_2 - p_1}{I_{03}} = C. \tag{10.59}$$

So, equation (10.57), (10.58), and (10.59) gives

$$p(x) = \frac{p_2 - p_1}{I_{03}} \int_0^x \frac{dx'}{h^3} + p_1.$$

and the load is given by

$$w = \int_0^L p(x) dx = \frac{p_2 - p_1}{I_{03}}(L I_{03} - I_{13}) + p_2 L.$$

10.2.4 Results and Discussion

The analytical and numerical study of a slider bearing with some viscous lubricant is considered. The pressure distribution and load on bearing is calculated using perturbation techniques. Two

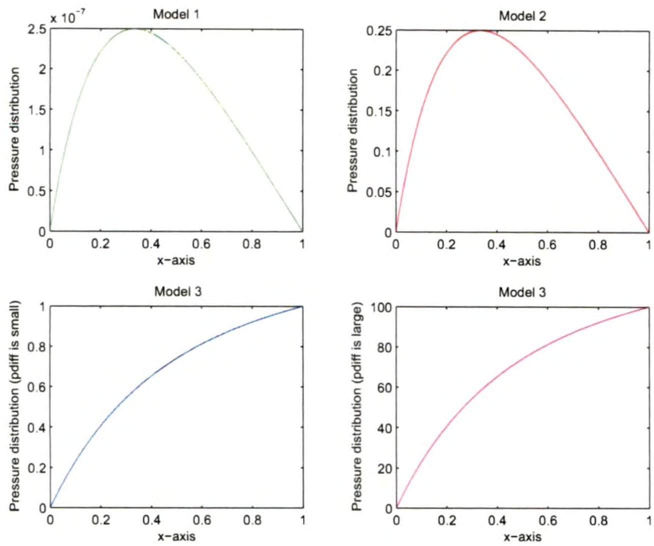

Figure 10.7: **Pressure Distribution in Slider Bearing**

upper parts of Figure-10.7 shows the pressure distribution in a slider bearing with pressure zero at end using model 1 and model 2. Two lower parts of Figure-10.7 shows the pressure distribution in a slider bearing with some pressure difference between the ends. We have also calculated load. The load according to the first and second model is given by $9.1589\ e-7$ and $6\log 2\ e-4$, respectively. According to the third model, the load is 3.6667. This paper deals with the modelling part of lubrication theory and it may be useful to an engineer to search for the validity of our predicted theoretical results and investigate the new design of bearing to extract better benefits of lubricants. The error between analytical solution and FDM solution is discussed in this chapter. We have used GMRES method and wavelet preconditioning. The error is of 10^{-2}.