SUMMARY OF THE THESIS ENTITLED

A NEW WAVELET TECHNIQUE USING PRECONDITIONING AND SPARSENESS STUDIES FOR THE NUMERICAL SOLUTION OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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1.0 Introduction to wavelets

Wavelet analysis is relatively new mathematical discipline, which has generated much interest in both Pure and Applied Mathematics over the past decade. 'Wavelet' name itself was coined approximately a decade ago (see [31], [61], and [62]). During the last couple of years, the interest in wavelets has grown at an explosive rate. There are several reasons for the success in this area which are mentioned below:

- Quoting from [19], the concepts of wavelets can be viewed as a synthesis during the last thirty years from Engineering (subband coding), Physics (coherent states, renormalization group), and Pure Mathematics (study of Calderon-Zygmund operators). As a consequence of these interdisciplinary origins, wavelets appeal to scientists and engineers.
- Wavelets are simple mathematical tools with a variety of possible applications. Many researchers have done applications in areas like Signal Analysis (see [41], [50], and [51]); Numerical Analysis (see [8]); Wavelet methods for PDE's (see [1], [17], [26], and [35]); Preconditioners and wavelets (see [2], and [45]).

Many other application areas of wavelets are in Statistics, Econometrics, Fractals, Image Processing, Ordinary Differential Equations, Communication Theory, Computer Graphics and some others.

The important properties of wavelets are their ability to analyze different parts of a function at different scales and the fact that they can represent polynomials up to a certain order exactly. As a consequence, functions with fast oscillations, or even discontinuities, in localized regions may be approximated well by a linear combination of relatively few wavelets. In comparison, Fourier expansion must use many basic functions to approximate such a function well. These properties of wavelets lead to some important applications in the above-mentioned fields.

2.0 Wavelets for partial differential equations

Wavelet bases are used in the approximation theory due to remarkable property of separating frequency locally, i.e. the coefficients of wavelet expansion gives information about the frequency content at certain space (or time) location. Even more, there exist orthonormal wavelet bases with compact support, and fast transforms between the classical and wavelet bases.

In recent years, wavelet techniques are applied for solving differential equations. There are several research areas such as:

- **Design of accurate, wavelet based discretization of PDEs**

  In principle, these are Finite Element Method discretizations using wavelets as test functions. The approximation properties of wavelets yield high accurate schemes. The main issues are: stability conditions, and the design of fast quadratures.

- **Wavelets analysis of the solutions**

  Wavelets are used in the original signal analysis context, and to detect the presence of shocks, eddies, etc., in turbulent flow. The knowledge about the singularities of the solution is then used in classical schemes, e.g. adaptive mesh generation, or the control of artificial viscosity.

- **Compression of dense, discrete operators**

  Boundary Element Methods often produces discrete equations with a small condition number. Discrete solution operators are dense matrices, since the solution depends on information from the whole computational domain. However, information usually has a simple structure that can be approximated with a few wavelets. Using wavelets as test functions, or performing basis transformation to wavelet coordinates, induce a sparse structure for the discrete operators after truncation, without destroying the small condition number.

- Traditional schemes that relay on different grids can be rewritten in the wavelet formalism. Concepts such as coarse grids operators have a precious meaning in the wavelet framework.

3.0 Theme of the research

The research presented in the thesis is centered around the SIX themes:

1. The construction of solution operators for elliptic equations in a wavelet basis

The standard methods make use of the integral representation of solution operator. The representation in wavelet basis using different methods like, Boundary Element Method, found in the works of Beylkin, Coifman, and Rohklin [8], David [20], and many others. We adopt a different approach first used by Engquist et al. [25] for parabolic operators. We can regard the solution operator of an elliptic problem as the long time evolution operator of the corresponding parabolic equation. The short time evolution operator is built in wavelet basis and by efficient repeated squaring; the long time operator is obtained.

2. Building efficient preconditioners using wavelet decomposition of discrete elliptic operators

In the $1D$ case, we find the well-known diagonal preconditioners (see [9], and [35]). In the $2D$ case, we find a connection between separation of variables and an extended wavelet transform that permits the incorporation of the $1D$ diagonal preconditioner in to a $2D$ structure. Preconditioners can also be used to efficiently invert the discrete evolution operators of implicit schemes.

3. Derivation of homogenized equations

In the context of wavelet bases, the notion of coarse and fine grids (or function space) becomes precise. There is an exact meaning to be assigned to the homogenized operator. The wavelet transform becomes a systematic method for deriving homogenized equations even if the asymptotic behavior of the solution is unknown. Brewster et al. [12] treats integrated equations in this manner.

The basic tool for our analysis is wavelet transform. We use the orthonormal bases for compactly supported wavelets invented by Daubechies, which have fast direct and inverse transforms. The important properties of the wavelet basis (like orthogonality, vanishing moments) are analytic in their nature. By restricting our self as finite dimensional space, these properties are viewed as algebraic properties. The analytic properties of the differential operator and wavelet bases are used to provide inspiration and confirmation for the algebraic properties of some special discrete linear system and orthogonal matrices.

4. Wavelet transforms solution of elliptic partial differential equation and Green's functions

The wavelet transform framework to solve some elliptic boundary value problem is known. The study applies to wavelet series for the evaluation of analytical solution of elliptic problem in any dimension. The wavelet can be understood as an alternative for the multi-dimensional problem to the standard Fourier series. Beylkin [7], and Glowinski et al. [29] introduced a method to solve elliptic partial differential equations with Dirichlet boundary condition in the wavelet system of coordinates by constructing the Green's function.

5. Wavelets and preconditioners

The solution of partial differential equations using preconditioning has several advantages. In the wavelet system of coordinates, the partial differential equation with boundary conditions are characterized by diagonal preconditioners leading to operations with sparse matrices having the condition number of $O(1)$. Less condition number is very good to avoid instability, minimizing the errors, and speed up the convergence.

6. Finite Pointset Method (FPM) and preconditioning aspects for the solution of elliptic PDE

FPM is a mesh free method for solving partial differential equations. It is based on Least Square (LS) approximation or Moving Least Square (MLS) approximation. It is fully Lagrangian method to handle problems in Fluid Dynamics for flow simulation with complicated as well as rapidly changing geometry (see [44]), involving free surfaces (see [70], and [71]) or phase boundaries (see [33]). FPM has great effect under influence of weight functions. The different weight functions have different impact on FPM simulations from error point of view as well as from condition number of matrix $M^T WM$ point of view (which we get from LS approximation around the central particle).

The problem still remains to find efficient algorithms to extract number of particles from the computational domain to enhance the speed of computation, i.e. how fast the solution converges and how much the system is well-conditioned. The study of the effect of preconditioners like Jacobi preconditioner, Block Jacobi preconditioner, Complete Factorization preconditioner, and some Incomplete Factorization preconditioners of different levels has effect on the improvement of efficiency of FPM. The analysis of effect of different preconditioners on different stationary and non-stationary iterative methods in terms of fast convergence is also essential.

4.0 Structure of the Thesis

The research work, which is incorporated in our thesis, is divided into the following parts:

Part I gives an exposition of the theory of orthogonal, compactly supported wavelets in the context of multiresolution analysis. These wavelets are particularly attractive because they lead to a very stable algorithm namely the Fast Wavelet Transform (FWT). The estimates for the approximation characteristics of wavelets and demonstrate how and why the FWT can be used as a front-end for efficient compression schemes.

Part II deals with vector parallel implementation of several variants of the Fast Wavelet Transform. We develop an efficient and scalable parallel algorithm for the FWT and derived a model for performance.

Part III is an investigation of the numerical methods using the special properties of wavelets for solving partial differential equations numerically. Several approaches are identified and some of them are described in detail. The algorithms developed are applied to the linear and nonlinear elliptic problems. Numerical results reveal that good performance can be achieved provided that the problems are large, solutions are highly localized, and the numerical parameters are chosen appropriately, depending on the problem in question.

Part IV deals with analytical solution of elliptic boundary value problem using wavelet transform approach. The Green's function approach for the solution of elliptic BVP in $1D$ and $2D$ is presented. The error estimates are established for showing advantage of wavelet approach in compare to Fourier series approach. The wavelet-based Green's function approach for solving Helmholtz and Modified Helmholtz equation is presented.

Part V deals with wavelet-based preconditioners for the solution of elliptic boundary value problems. The primary goal of this part is the understanding of the connection between discrete elliptic operators, their inverses, and different wavelet techniques in the framework of efficient computations. Discrete elliptic operators are used in the approximation of the solution of elliptic equations and also in the intermediate steps of iterative solvers for more general nonlinear problems, e.g. the Navier-Stokes equations. It is well known fact that the discrete elliptic solvers need a large computational effort. In computations, the sparsity and the small condition number for the discrete operators are the key to efficiency. Sparseness of the matrix enhances the speed of performing Conjugate Gradient or Multi-grid type iterations, while the small condition number guarantees rapid convergence of such iterations. The classical discretizations of elliptic operators are sparse but have large condition number. There are some standard techniques for building efficient solvers relaying on the scale decomposition of the elliptic operators, its inverses, or the solution itself, e.g. the multi-grid, multi-pole, or domain decomposition methods. The different types of wavelet preconditioning are used to get the fast convergence and small condition number.

Part VI deals with introduction to Finite Pointset Method and its improvement by considering the following aspects:

- Weight function approach
- Filtering algorithms approach
- Preconditioners

Part VII deals with two models: the lubrication model and cooling of coke in a Can model. In this part, we have applied wavelet based methods and Finite Pointset Method to get the accurate solutions.

Part VIII deals with comparison of our wavelet methods and Finite Pointset method by taking the test examples as models of Part VII using our wavelet solver and FPM solver.

5.0 Chapter-wise description

We have divided the whole thesis into ten chapters.

5.1 Chapter 1

The first chapter is introductory.

We have discussed the basic idea of wavelets, its connection with Fourier expansion and advantages of wavelet expansion over Fourier expansion. The idea of removing Gibb's phenomena over Fourier phenomena is greatly explained with our software.

A natural framework for wavelet theory is multiresolution analysis (MRA), which is a mathematical construction that characterizes the wavelets in general way. MRA yields fundamental insights in to wavelet theory and leads to important algorithms as well. The goal of MRA is to express an arbitrary function $f \in L^2(R)$ at various levels of details.

Hence, the concepts of MRA, approximation spaces V_j and W_j, basic scaling function and basic wavelets, expansion of function in V_j, dilation equation and wavelet equation, filtering coefficients, property of vanishing moments, decay of wavelet coefficients, wavelet and Fourier transform, periodizied wavelets, and periodic MRA in $L^2([0,1])$ are discussed in details.

The P vanishing moments have an important consequence for the wavelet coefficients: They decrease rapidly for a smooth function. Furthermore, if a function has discontinuity in one of its derivatives then the wavelet coefficients will decrease slowly only close to that discontinuity and maintain fast decay where the function is smooth. This property makes wavelets particularly suitable for representing *piecewise smooth functions*. In this chapter, we have made an attempt to find estimates for the decay of wavelet coefficients.

5.2 Chapter 2

There are no explicit formulas for the scaling function ϕ and the wavelet function ψ. Hence, most algorithms concerning scaling functions and wavelets are formulated in terms of the filter coefficients. A good example is the task of computing the values of ϕ and ψ. In this chapter, we have developed algorithms to compute ϕ and ψ. We have also described various algorithms for Discrete Wavelet Transform (DWT), Inverse Discrete Scaling Function Transform (IDST), Fast Wavelet Transform (FWT), and Periodic Fast Wavelet Transform (PFWT). The accuracy of multiresolution space is discussed in details.

5.3 Chapter 3

Problems involving the FWT are typically large and wavelet transforms can be time consuming as algorithms complexity is proportional to problem size. Hence, the use of high performance computer is essential. In this chapter, we will describe our efforts to implement the FWT on a selection of high performance computers.

5.4 Chapter 4

Even though the field of wavelet theory has had a great impact in other fields, such as signal processing, it is not yet clear whether it will have a similar impact on numerical methods for solving partial differential equations.

In the early nineties, people were very optimistic because it seemed that the nice properties of wavelets would automatically lead to efficient solution methods for PDEs. The reason for optimism was the fact that many nonlinear PDEs have solution containing local phenomenon (e.g. formation of shocks) and interaction between several scales (e.g. turbulence). Such solutions can often be well-represented in wavelet bases. It was, therefore, believed that efficient wavelet based numerical schemes for solving PDEs would follow from wavelet compression properties (see [4], [13], and [66]).

However, this early optimism remains to be honored. Wavelets have not had the expected impact on differential equations; partly because the computational work is not necessarily reduced by applying wavelet compression – even though the solution is sparsely represented in wavelet basis.

Wavelet based methods for PDEs can be separated in to the following classes:

Class 1: Methods based on scaling function expansions

The unknown solution is expanded in scaling functions at some chosen level J and is solved using a Galerkin approach. Because of their compact support, the scaling functions can be regarded as alternatives to splines or the piecewise polynomials used in Finite Element schemes. By expanding the solution in scaling functions, high frequency components can be filtered away and continuous dependence on the initial condition is restored. In literature, examples of such methods can be found in [24], [36], [67], and [78].

Class 2: Methods based on wavelet expansions

Under this class, the unknown solution is expressed in terms of wavelets instead of scaling functions; so, wavelet compression can be applied. An important aspect of the wavelet approach is that certain operators represented with respect to a wavelet basis become sparser when raised to higher powers. From this property, one can obtain an efficient time-stepping scheme for certain evolution equations. This method has been employed to solve the heat equation (see [4], [6], [13], [26], and [78]).

Class 3: Wavelets and finite differences

In this third approach, wavelets are used to derive adaptive finite difference methods. Instead of expanding the solution in terms of wavelets, the wavelet transform is used to determine where the finite difference grid must be refined or coarsened to optimally represent the solution. Under this class, we have Wavelet Optimized Finite Difference (WOFD) method developed by Leland Jameson (see [37], and [38]).

Class 4: Other methods

There are few other approaches that use wavelets in ways that do not fit into any of the previous classes. Examples are operator wavelets, anti-derivatives of wavelet, the method of traveling wavelet-preconditioning (see [10], [39], [65], and [79]).

On the basis of first two approaches, we have developed a mathematical software to solve elliptic boundary value problem. Based on this software, we have developed numerical solutions of hyperbolic problems (see [13*]), parabolic problems (see [14*]), and Burger's equation (see [4*]). This chapter also includes the solution of elliptic BVP with periodic and non-periodic boundary conditions in $1D$ and $2D$ (see [15*]).

5.5 Chapter 5

Discrete elliptic operators are used in the approximation of uniformly elliptic and possibly variable coefficient differential equations. In computations, the sparseness and small condition number of the discrete operators is the key to efficiency. The sparseness enhances the speed of iterations while small condition number guarantees the rapid convergence of such iterations. The matrices that we obtain using finite difference method are sparse; however, they have large condition number. Using the Galerkin method with Fourier system, we can obtain the bounded condition number but the matrix is no longer sparse. In the Galerkin method with wavelet basis, we obtain both the advantages. In this chapter, we have made an attempt to establish the error estimates to show the advantage of wavelet-Galerkin method over the finite difference method and Fourier-Galerkin method not only in terms of fast computation and rapid convergence but by obtaining better accuracy (see [1*]).

Consider the following theorem:

Theorem – 1: Let V be a Hilbert Space. Suppose

$$\|v\|_E = \sqrt{a(v, v)}; \quad \forall v \in V$$

then
$$\|u - u_s\|_E = \inf\{\|u - v\|_E : \; v \in S\}$$

where $S \subset V$ is any finite dimensional subspace.

This is the basic error estimates for the Ritz-Galerkin method, and it says that the error is optimal in the energy norm. In the next two theorems, we will use this error estimates for deriving more concrete error estimates for the approximate solution in space S.

Theorem – 2: Let $\varepsilon > 0$. Suppose w be the solution of

$$-w'' = u - u_s \quad on \quad [0, 1]; \; with \; \; w(0) = w'(1) = 0,$$

and if
$$\inf_{v \in S} \|w - v\|_E \le \varepsilon \|w''\|$$

then
$$\|u - u_s\| \le \varepsilon \|u - u_s\|_E \le \varepsilon^2 \|u''\| = \varepsilon^2 \|f\|.$$

Theorem – 3: Let

$$\Psi(x)=\begin{cases}1 & ; \ 0\le x\le 1/2\\ -1; & 1/2\le x\le 1\\ 0 & ; \ otherwise.\end{cases}$$

Consider $\Psi_{j,k}(x)=2^{-j/2}\Psi\left(2^{-j}x-k\right)$ where j and k are integers. If u_{ws} be the Wavelet-Galerkin solution in space S, then we have

$$\|u-u_{ws}\|_{L^2}=\inf_{v\in S}\left\{2\|u-v\|_{L^2}+O(1)\right\}\quad v\in S.$$

The numerical techniques such as Finite Difference, Finite Element, and Finite Volume are already known. The wavelet methods have several advantages over these traditional methods. Wavelets have ability to represent functions at different levels of resolution, thereby providing a logical means of hierarchy of solutions. Furthermore, compactly supported wavelets (such as those due to Daubechies) are localized in space which means that the solution can be refined in regions of higher gradients.

In order to demonstrate the wavelet technique using Green's function approach (see [11*]), we consider the following Helmholtz and modified Helmholtz problem in 2D

$$u_{xx}+u_{yy}\pm u=f$$

where u and f are periodic functions in x and y. Let dx and dy be the period in the x and y direction, respectively, then the conditions will be

$$u(0,y)=u(dx,y);\qquad f(0,y)=f(dx,y);$$

$$u(x,0)=u(x,dy);\qquad f(x,0)=f(x,dy).$$

By doing comparison with a simple finite difference solution of this problem with periodic boundary conditions, we have shown how a wavelet technique can be efficiently developed. Dirichlet/ Neumann/ Robin's boundary conditions are then imposed using capacitance matrix method described by Proskurowski, and Widlund [66]. The convergence of wavelet solution is examined and they are compared favorably with the finite difference solutions.

5.6 Chapter 6

G. Beylkin [10] developed a method to represent differential operators in the wavelet basis. This method leads to fast algorithm for evaluating these operators acting on functions. Therefore, he suggests an alternative to common method for the descretization of elliptic equations.

Usually, the descretization of differential operators in the Galerkin method leads to a sparse matrix with large condition number. Typically, for the second order elliptic problem, the condition number is $O(1/h^2)$, where h is the size of discretization. To avoid such ill-conditioning, Beylkin used the preconditioning aspects to obtain the condition number of size $O(1)$.

In this chapter, we have proved the same estimates for non-linear elliptic PDE. We have also made an attempt to find the effect of different wavelet preconditioners for the solution of elliptic BVP in terms of stability and convergence (see [10*]).

5.7 Chapter 7, 8 and 9

The particular type of meshfree method, Finite Pointset Method (FPM), and its effect on elliptic problems is discussed in details in these three chapters.

What are meshfree methods?

Mesh free methods use a set of nodes scattered within the problem domain as well as set of nodes scattered on the boundaries of the domain to represent (not discretize) the problem domain and its boundary. No mesh implies no information on the relationship between the nodes is required. The increasing complexity of real life problems leads towards the development of new methods. The Finite Element Method has better flexibility, effectiveness, and accuracy in problems involving complex geometry in compare to Finite Difference Method. Now, the limitations of FEM and FDM are becoming increasingly evident. For example, large deformations can deteriorate the accuracy because of element distortion. To answer this question, new and more powerful classes of techniques, known as meshfree, or meshless, or gridfree method are emerging.

Why and where meshfree methods?

The traditional mesh based method such as Finite Element Method (FEM) and Finite Difference Method (FDM) often run into problems when the mesh (elements) deteriorates during simulation due to the geometry change. Meshfree methods are originally developed to simulate Fluid Dynamics problems. They are so called particle methods. The appearance and the development of meshfree method is motivated by the challenges in numerical simulation of process involving significant changes of the geometry such as multiphase flows, filling process and other free surface flows in Fluid Dynamics; large displacements, crack propagation in Solid Mechanics, etc.

Meshfree method: Smoothed Particle Hydrodynamics (SPH) method

The classical meshfree Lagrangian method to handle problems in Fluid Dynamics is the Smoothed Particle Hydrodynamics (SPH) method. SPH was initially developed to study phenomena in Astrophysics (see [27], and [49]). Later, it was developed for the flow cases even on earth (see [16], [54], [59], and [60]). In SPH, incompressible flows are approximated by using the compressible approach together with very stiff equations of state. SPH is referred to as the first meshfree method.

The basic steps of the SPH scheme are as follows: first the conservation laws are expressed in the Lagrangian form for primitive variables and the spatial derivatives are approximated; then the governing PDE reduced to a time dependent system of ODE and finally, it is solved by ODE solver. This is a grid free method, where the spatial derivatives of a function at a point, is approximated by discrete values over a set of neighboring points. These neighboring points are the so called particles and their distribution need not be uniform, or regular. Therefore, this method is suitable for fluid dynamical problems with moving boundaries and free surface flows.

In its original formulation, SPH is easy for implementation, but it provides poor accuracy/ convergence (see [23], and [30]). Unfortunately, SPH has poor approximation properties, especially of the second derivatives, required to model the Navier-Stokes equations. Moreover, it is difficult to incorporate boundary conditions of certain types. Further, variety of meshfree methods was proposed in the last decade for solving Fluid Dynamics and Solid Mechanics problems. Detailed discussion on the development of meshfree methods and on their application can be found in [3], [5], [23], [30], [47], [48], and [64]. It should be noted that the terminology in this area is still not well established. We used the term meshfree methods but one can find in literature different names: particle methods, meshless methods, gridfree methods, gridless methods, clouds methods. These names are used as notation for the group including such methods as SPH, Reproducing Kernel Particle Methods (RKPM), Element-Free Galerkin Methods (EFGM), Diffusive Element Methods (DEM), Finite Cloud Methods (FCM), and Generalized Finite Difference Methods (GFDM). Some of these methods are very close or equivalent to each other and great part of them allow a general consideration as Partition of Unity Methods (PUM's).

Since the classical SPH approach is based on the integral interpolate with sufficiently smoothing kernel function, it does not give good approximation of derivatives of a function near boundaries.

The Least Square (LS) method is an alternative approach to approximate spatial derivatives in gridfree structures (see [5], [22], [25], and [42]). In [42], it is shown that the Moving Least Square (MLS) methods give a better approximation of function and its derivative near boundaries. Both of the approaches are similar to the finite difference discretization and show the well-known problem of instability. In order to stabilize the scheme, some sort of viscosity should be introduced. In [55], an artificial viscosity is introduced in the momentum and energy equations for inviscid flows. Similarly, an artificial viscosity is proposed in [42] in all equations of the system of ODEs resulting approximation of the space derivatives of Euler equations. The artificial viscosity used in [42], and [55] stabilizes the numerical scheme.

Fluid Dynamics equations with viscosity, e.g. the Navier-Stokes equations, can not be solved appropriately with these artificial viscosities. In order to solve the Navier-Stokes equations, one needs to approximate the first and second order spatial derivatives. The classical SPH approach as well as MLS approach does not give good approximation of derivatives. In [74], the authors use SPH scheme based on Weighted Least Squares (WLS) method by which they approximate the first and second derivatives. This approach is similar to MLS approach used in [5], and [42]. In [42], the boundary conditions are well treated by the MLS technique. Like in [42], authors have replaced the boundary by particles and prescribe the boundary on the boundary particles. In [74], the authors have considered the full system of Navier-Stokes equations and the solution of compressible Euler equations are obtained by letting the viscosity and the heat conductivity terms tend to zero. The scheme is tested for $1D$ shock tube problem considered in [68]. The scheme is stable and the numerical solution converges to the exact solution of the Euler's equation when the number of particles tends to infinity and the viscosity and the heat conduction coefficient tend to zero.

Numerical simulations of free surface flows have many industrial applications like casting, tank filling and others. Many methods have been developed to simulate free surface flows (see [30], [40], and [72]). In [72] and in [75], authors have simulated the incompressible free surface flows as a limit of the compressible, viscous Navier-Stokes equations with the equation of state such that flow is weakly incompressible. This type of equation of state was first used by Monaghan (see [53], [54], [55], and [56]) to simulate incompressible free surface flows by SPH. Such an equation of state in the framework of SPH has been further used to simulate incompressible viscous flows in [57].

The solution of Poisson equation is necessary for instationary problems in incompressible fluid flows. Here, one considers some projection methods for the Navier-Stokes equations, where the Poisson equation for the pressure has to be solved (see [14]). Several authors have considered the projection method on grid based structure such that Poisson equation can be solved by standard methods like Finite Element or Finite Difference method. The grid based method can be quite complicated if the computational domain change in time or takes complicated shapes. In this case, re-meshing is required and more computational effort is needed. Therefore, a gridfree method has certainly advantages in such cases.

A gridfree method for solving Poisson equation is given in [69]. The idea is to solve the Poisson equation in a gridfree structure such that it can be used in Lagrangian particle projection methods for incompressible flows. The method can be applied to any elliptic problems in complicated geometry where the mesh is poorly constructed. The method given in [69] is a local iteration process. It is based on the LS approximation. A function and its derivatives can be approximated by the LS method at an arbitrary point from its discrete values belonging to the surrounding clouds of points. However, the values of a function on the particle positions are not given. Only the Poisson equation is given. Therefore, we prescribe an initial guess for the values of a function on each particle position. For every iteration step, we enforce the Poisson equation to be satisfied on each particle in the LS ansatz. Boundaries can be replaced by a discrete set of boundary particles. For the Dirichlet boundary condition, boundary values are assigned in every iteration step, on boundary particles. For the

Neumann boundary condition, we again enforce it to be satisfied in the LS ansatz. Therefore, we add one additional equation in the LS approximation. The method is stable and the numerical solution converges to a unique fixed point as the number of iteration steps tends to infinity. The method can be applied to a coarser as well as finer distribution of points. The convergence rate is slow on finer distribution of points.

Meshfree method: Finite Pointset Method (FPM)

A Finite Pointset Method (FPM) is a meshfree method to solve partial differential equations. The computational domain is represented by a finite number of particles, also referred to as numerical points. These points can be arbitrarily distributed; however, they have to provide a neighboring relationship governed by the smoothing length, i.e. each point needs to find sufficiently many neighboring points within a ball of certain radius. Considering the equation of Fluid Dynamics, the numerical points move with fluid velocity and carry all information which completely describes the flow problem concerned. Of course, this is a fully Lagrangian method being appropriate for flow simulations with complicated as well as rapidly changing geometry (see [49]), involving free surfaces (see [71], and [72]), or phase boundaries (see [33]).

The FPM is based on LS approximations, where the higher order derivatives can be approximated and the boundary conditions can be treated in classical sense (see [42]). Computation of several flow problems using the method of LS or MLS are reported by different authors (see [22], [42], [69], [70], [72], [74], and [75]). In [72] and in [75], the authors have performed simulations of incompressible flows as the limit of the compressible Navier-Stokes equations with the quasi-compressible equation of state. This approach was first used in [53] to simulate free surface flows by SPH. The incompressible simulate is obtained by choosing a very large speed of sound in equation of state such that the Mach number is of order 0.1. However, the large value of the speed of the sound restricts the time step to be very small due to the CFL condition.

The Chorins projection method explained in [14] is a widely used approach to solve the incompressible Navier-Stokes equation in grid based structure. In [70], authors have extended Chorins projection method to meshfree framework with the help of the WLS method. The Poisson pressure equation is solved by mesh free method. In [69], it has been shown that Poisson equation can be solved accurately by this approach for any kind of boundary conditions. The Poisson solver can be adopted in the LS approximation procedure with the condition that the Poisson equation and the boundary condition must be satisfied on each particle. This is a local iteration procedure.

In [70], the authors have tested the scheme for channel flows and driven cavity flows. They have performed simulations of steady as well as unsteady flows. In this case, the numerical solution is compared with the one obtained from the Finite Element method. It is found that proposed scheme gives accurate results. In [71], the author have extended the scheme presented in [70] for free surface flows. Numerical experiments are obtained with and without surface tension forces. The Laplace law [46] has been tested for different shapes of bubbles and the scheme produce Laplace law exactly. Finally, the binary drop collision of liquid drops shows that the scheme is suitable for simulation of free surface flows.

The numerical scheme for incompressible and slightly compressible flow phenomena, presented in [73], is based on the classical projection idea of Chorin (see [14], and [70]). Due to that, the solutions of Poisson as well as Helmholtz differential equations, in particular, form a central task of FPM. These equations can be solved directly in the given meshfree structure with Dirichlet, Neumann, or Cauchy boundary conditions in a very accurate way (see [69]).

The literature shows that FPM is applicable to many elliptic problems which arise from fluid models. The main drawback of this method observed by several researchers is in terms of stability, convergence and speed. Hence, we have made an attempt to improve this drawback in chapter 7, 8, and 9. The performance of Finite Pointset Method can be improved by various means such as: *weight functions, preconditioning aspects, and filtering algorithms*.

In chapter 7, we have discussed FPM discretization of general elliptic problem. This chapter also deals with weight function aspects to discuss convergence and stability of elliptic problems (see [5*]).

Chapter 8 deals with preconditioning aspects of FPM method to show the fast convergence and stability of the method (see [6*]). Several preconditioners are used and several iterative and non-iterative methods are also presented to discuss convergence and stability.

In chapter 9, we have proposed different filtering algorithms to discuss the stability and convergence aspects (see [7*]).

5.8 Chapter 10

Chapter 10 deals with two models:

First model is based on pressure distribution in the slider bearing. The problem was posed by Stephen Chapman and Alister Fitt and it is related with slider bearing industry in United Kingdom. In almost every rubbing surface when oil is somewhere present, a lubrication film manages to get between the surfaces to carry the part of the load. So, the lubrication of the machinery is very important to reduce the friction. It is very important to design sophisticated machines which are free from poor lubrication. The slider bearing consisting of surface with viscous fluid as a Newtonian lubricant is analyzed in this chapter. The analysis is based on perturbation technique. We assume only viscous and Newtonian effects whereas inertial terms are neglected. Our initial two models are based with the assumption that pressure is zero at the ends of slider bearing. After constructing the perturbation solutions, the pressure distribution on the load in the slider bearing is calculated by our models and validated by our wavelet method and Finite Pointset Method (see [2*]).

Second model is based on cooling of coke in a Can. Cooling of coke contained in a Can in refrigerator is very much essential during the summer time. In this part of chapter 10, we have made an attempt to present the mathematical model to see the diffusion of heat out of the Can. The main objective is to see how long it will take for the Can of coke in refrigerator to cold enough. We have used Wavelet technique and FPM technique to solve this model (see [3*]).

At the end, we have done the comparison of wavelet method (mesh-based methods) and FPM (meshless methods) in terms of convergence, stability, and speed by using preconditioning as well as non-preconditioning aspects using above two models.