Chapter VIII

Generalized Product Summability of an Orthogonal Series

8.1 Introduction

Let

$$\sum_{n=0}^{\infty} u_n \qquad (8\text{-}1)$$

be a given infinite series and $\{s_n\}$ be its sequence of partial sums.

Let $\{p_n\}$ and $\{q_n\}$ be two sequences of real numbers and let

$$P_n = p_0 + p_1 + \cdots + p_n = \sum_{v=0}^{n} p_v$$

$$Q_n = q_0 + q_1 + \cdots + q_n = \sum_{v=0}^{n} q_v$$

Let p and q represents two sequences $\{p_n\}$ and $\{q_n\}$ respectively.

The convolution between p and q is denoted by $(p * q)_n$ and is defined by

$$R_n \coloneqq (p * q)_n = \sum_{v=0}^{n} p_{n-v} q_v = \sum_{v=0}^{n} p_v q_{n-v}$$

Define

$$R_n^j \coloneqq \sum_{v=j}^{n} p_{n-v} q_v$$

The generalized Nörlund mean of series (8-1) is defined as follows and is denoted by $t_n^{p,q}(x)$.

$$t_n^{p,q} = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_v\,(x)$$

where, $R_n \neq 0$ for all n.

The series (8-1) is said to be absolutely summable (N, p, q) i.e.$|N, p, q|$ summable, if the series

$$\sum_{n=1}^{\infty} \left| t_n^{p,q} - t_{n-1}^{p,q} \right| < \infty$$

If we take $p_n = 1$ for all n then, the sequence-to-sequence transformation $t_n^{p,q}$ reduces to $(\bar{N}, q_n)$ transformation

$$\bar{t}_n \coloneqq \frac{1}{Q_n} \sum_{v=0}^{n} q_v s_v$$

If we take $q_n = 1$ for all n then, the sequence- to- sequence transformation $t_n^{p,q}$ reduces to $(\bar{N}, p_n)$ transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

(See Krasniqi, Xh. Z. 2013(2))

8.2 Product Summability

Das, G. 1969 defined the following transformation

$$U_n := \frac{1}{P_n}\sum_{v=0}^{n}\frac{p_{n-v}}{Q_v}\sum_{j=0}^{v} q_{v-j}s_j \ ,$$

The infinite series (8-1) is said to be summable$|(N,p)(N,q)|$, if the series

$$\sum_{n=1}^{\infty}|U_n - U_{n-1}| < \infty$$

Later on, Sulaiman, W. 2008 considered the following transformation:

$$V_n = \frac{1}{Q_n}\sum_{v=0}^{n}\frac{q_v}{P_v}\sum_{j=0}^{v} p_j\, s_j$$

The infinite series (8-1) is said to be summable$|(\bar{N},q_n)(\bar{N},p_n)|_{k,}\ k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|V_n - V_{n-1}|^k < \infty$$

Krasniqi, Xh. Z. 2013(2) have defined the transformation which as follows:

$$D_n := \frac{1}{R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{R_v}\sum_{j=0}^{v} p_j q_{v-j}s_j,$$

The infinite series (8-1) is said to be $|(N,p_n,q_n)(N,q_n,p_n)|_{k,}\ k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|D_n - D_{n-1}|^k < \infty$$

We have defined the transformation as follows:

$$E_n := \frac{1}{\bar{R}_n}\sum_{v=0}^{n}\frac{p_v q_v}{\bar{R}_v}\sum_{j=0}^{v} p_j q_j s_j,$$

The infinite series (8-1) is said to be summable $|(\bar{N},p_n,q_n)(\bar{N},q_n,p_n)|_{k,}\ k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|E_n - E_{n-1}|^k < \infty$$

8.3 Product Summability of Orthogonal Series

Let $\{\varphi_n(x)\}; n = 0,1,2, \ldots$ be an orthonormal system defined in an interval (a,b). Consider the orthogonal series

$$\sum_{n=1}^{\infty} c_n\varphi_n(x) \qquad (8\text{-}2)$$

where $\{c_n\}$ be a sequence of real numbers. If c_n is according to the following expression

$$c_n = \int_a^b f(x)\,\varphi_n(x)dx, \;\; n = 0,1,...$$

then (8-2) is an orthogonal expansion of $f(x)$ and it is denoted by

$$f(x) \sim \sum_{j=0}^{\infty} c_j\,\varphi_j(x) \tag{8-3}$$

Okuyama, Y. 2002 has proved the following two theorems:

Theorem 8.1

If the series is

$$\sum_{n=0}^{\infty}\left\{\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2 |c_j|^2\right\}^{1/2}$$

converges, then the orthogonal series

$$\sum_{j=0}^{\infty} c_j\varphi_j(x)$$

is summable $|N,p,q|$ almost everywhere.

Theorem 8.2

Let $\{\Omega(n)\}$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is a non-increasing sequence and the series

$$\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$$

converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative sequences. If the series

$$\sum_{n=1}^{\infty}|c_n|^2\Omega(n)\omega^{(i)}(n)$$

converges, then the orthogonal series

$$\sum_{j=0}^{\infty} c_j\varphi_j(x)$$

is $|N,p,q|$ summable almost everywhere, where $\omega^{(i)}(n)$ is defined by

$$\omega^{(i)}(j) := j^{-1}\sum_{n=j}^{\infty} n^2\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2.$$

Krasniqi, Xh. Z. 2013(2) has proved the following theorems:

Theorem 8.3
If for $1 \leq k \leq 2$, the series

$$\sum_{n=1}^{\infty}\left\{n^{2-\frac{2}{k}}\sum_{i=1}^{n}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n}-\frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2\left|c_j\right|^2\right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}c_n\emptyset_n(x)$$

is summable $|(N,p_n,q_n)(N,q_n,p_n)|_k, k \geq 1$ almost everywhere.

Theorem 8.4
Let $1 \leq k \leq 2$ and $\Omega(n)$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is a non-increasing sequence and the series

$$\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$$

converges,
Let $\{p_n\}$ and $\{q_n\}$ be non-negative sequences.
If the series

$$\sum_{n=0}^{\infty}|c_n|^2\Omega^{\frac{2}{k}-1}(n)\mathfrak{R}^{(k)}(n)$$

converges, then the orthogonal series,

$$\sum_{n=1}^{\infty}c_n\varphi_n(x)$$

is $|(N,p_n,q_n)(N,q_n,p_n)|_k$ summable almost everywhere, where

$$\mathfrak{R}^{(k)}(i) \coloneqq \frac{1}{i^{\frac{2}{k}-1}}\sum_{n=i}^{\infty}n^{\frac{2}{k}}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n}-\frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2$$

In this chapter, we have extended the result of Krasniqi, Xh. Z. 2013(2) to $|(\overline{N},p_n,q_n)(\overline{N},q_n,p_n)|_k$; $k \geq 1$ summability. Our theorem is as follows:

Theorem 8A
If for $1 \leq k \leq 2$, the series

$$\sum_{n=1}^{\infty}\left\{n^{2-\frac{2}{k}}\sum_{i=1}^{n}\left(\frac{\overline{R}_n^i\widetilde{\widetilde{R}}_n^i}{\overline{R}_n}-\frac{\overline{R}_{n-1}^i\widetilde{\widetilde{R}}_{n-1}^i}{\overline{R}_{n-1}}\right)^2\left|c_j\right|^2\right\}^{\frac{k}{2}} < \infty \tag{8-4}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \emptyset_n(x)$$

is summable $|(\bar{N}, p_n, q_n)(\bar{N}, q_n, p_n)|_{k,}$ almost everywhere.

Theorem 8B

Let $1 \le k \le 2$ and $\Omega(n)$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is a non-increasing sequence and the series

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$$

converges.

Let $\{p_n\}$ and $\{q_n\}$ be non-negative sequence. If the series

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega^{\frac{2}{k}-1}(n) \widetilde{\mathfrak{R}}^{(k)}(n) < \infty \qquad (8\text{-}5)$$

then the series,

$$\sum_{n=1}^{\infty} c_n \varphi_n(x)$$

is $|(\bar{N}, p_n, q_n)(\bar{N}, q_n, p_n)|_k$ summable almost everywhere, where $\widetilde{\mathfrak{R}}^{(k)}(n)$ is defined by

$$\widetilde{\mathfrak{R}}^{(k)}(i) := \frac{1}{i^{\frac{2}{k}-1}} \sum_{n=i}^{\infty} n^{\frac{2}{k}} \left(\frac{\bar{R}_n^i \tilde{\tilde{R}}_n^i}{\bar{R}_n} - \frac{\bar{R}_{n-1}^i \tilde{\tilde{R}}_{n-1}^i}{\bar{R}_{n-1}} \right)^2$$

Proof of Theorem 8A

First, we shall consider $1 \le k \le 2$.

We use the notations,

$$\tilde{\tilde{R}}_n^i = \sum_{v=i}^{n} \frac{q_v p_v}{\bar{R}_v}; \qquad \tilde{\tilde{R}}_{n-1}^n = 0 \; ; \; \tilde{\tilde{R}}_n^0 = \tilde{\tilde{R}}_{n-1}^0$$

We have;

$$E_n(x) = \frac{1}{\bar{R}_n} \sum_{v=0}^{n} \frac{p_v q_v}{\bar{R}_v} \sum_{j=0}^{v} p_j q_j s_j(x),$$

$$= \frac{1}{\bar{R}_n} \sum_{v=0}^{n} \frac{p_v q_v}{\bar{R}_v} \sum_{j=0}^{v} p_j q_j \sum_{i=0}^{j} c_i \varphi_i(x),$$

$$= \frac{1}{\bar{R}_n} \sum_{v=0}^{n} \frac{p_v q_v}{\bar{R}_v} \sum_{i=0}^{v} c_i \varphi_i(x) \sum_{j=i}^{v} p_j q_j,$$

Refer equation (3-6), we have

$$= \frac{1}{\bar{R}_n} \sum_{v=0}^{n} \frac{p_v q_v}{\bar{R}_v} \sum_{i=0}^{v} \bar{R}_v^i c_i \varphi_i(x)$$

$$= \frac{1}{\bar{R}_n} \sum_{i=0}^{n} \bar{R}_n^i c_i \varphi_i(x) \sum_{v=i}^{n} \frac{p_v q_v}{\bar{R}_v}$$

$$= \sum_{i=0}^{n} \frac{\bar{R}_n^i \tilde{\tilde{R}}_n^i}{\bar{R}_n} c_i \varphi_i(x)$$

Now;

$$\Delta E_n(x) = E_n(x) - E_{n-1}(x)$$

$$= \sum_{i=0}^{n} \frac{\bar{R}_n^i \tilde{\tilde{R}}_n^i}{\bar{R}_n} c_i \varphi_i(x) - \sum_{i=0}^{n-1} \frac{\bar{R}_{n-1}^i \tilde{\tilde{R}}_{n-1}^i}{\bar{R}_{n-1}} c_i \varphi_i(x)$$

$$= \sum_{i=0}^{n} \left(\frac{\bar{R}_n^i \tilde{\tilde{R}}_n^i}{\bar{R}_n} - \frac{\bar{R}_{n-1}^i \tilde{\tilde{R}}_{n-1}^i}{\bar{R}_{n-1}} \right) c_i \varphi_i(x)$$

Using Hölder's inequality and orthogonality, we have

$$\int_a^b |\Delta E_n(x)|^k dx = \int_a^b \left| \sum_{i=0}^{n} \left(\frac{\bar{R}_n^i \tilde{\tilde{R}}_n^i}{\bar{R}_n} - \frac{\bar{R}_{n-1}^i \tilde{\tilde{R}}_{n-1}^i}{\bar{R}_{n-1}} \right) c_i \varphi_i(x) \right|^k dx$$

$$\leq (b-a)^{1-\frac{k}{2}} \left\{ \sum_{i=0}^{n} \left(\frac{\bar{R}_n^i \tilde{\tilde{R}}_n^i}{\bar{R}_n} - \frac{\bar{R}_{n-1}^i \tilde{\tilde{R}}_{n-1}^i}{\bar{R}_{n-1}} \right)^2 |c_i|^2 \right\}^{\frac{k}{2}}$$

$$\sum_{n=1}^{\infty} n^{k-1} \int_a^b |\Delta E_n(x)|^k dx \leq (b-a)^{1-\frac{k}{2}} \sum_{n=1}^{\infty} \left\{ n^{2-\frac{2}{k}} \sum_{i=0}^{n} \left(\frac{\bar{R}_n^i \tilde{\tilde{R}}_n^i}{\bar{R}_n} - \frac{\bar{R}_{n-1}^i \tilde{\tilde{R}}_{n-1}^i}{\bar{R}_{n-1}} \right)^2 |c_i|^2 \right\}^{\frac{k}{2}}$$

$$= M_1 \sum_{n=1}^{\infty} \left\{ n^{2-\frac{2}{k}} \sum_{i=0}^{n} \left(\frac{\bar{R}_n^i \tilde{\tilde{R}}_n^i}{\bar{R}_n} - \frac{\bar{R}_{n-1}^i \tilde{\tilde{R}}_{n-1}^i}{\bar{R}_{n-1}} \right)^2 |c_i|^2 \right\}^{\frac{k}{2}}$$

Using condition (8-4),

$$\sum_{n=1}^{\infty} n^{k-1} \int_a^b |\Delta E_n(x)|^k dx < \infty$$

Hence, by Beppo Levi's theorem,

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta E_n(x)|^k < \infty$$

Hence, the series (8-1) summable $|(\overline{N}, p_n, q_n)(\overline{N}, q_n, p_n)|_k$ almost everywhere.
For $k = 2$; we may apply orthogonality and for $k = 1$, we may apply Schwarz's inequality and the proof follows on the same line as in the case of $1 < k < 2$.
Hence, the proof follows immediately for $1 \leq k \leq 2$.

Proof of Theorem 8B

Consider,

$$\sum_{n=1}^{\infty} n^{k-1} \int_a^b |\Delta E_n(x)|^k dx$$

$$\leq \sum_{n=1}^{\infty} n^{k-1} \left\{ \sum_{i=0}^{n} \left(\frac{\overline{R}_n^i \widetilde{\widetilde{R}}_n^i}{\overline{R}_n} - \frac{\overline{R}_{n-1}^i \widetilde{\widetilde{R}}_{n-1}^i}{\overline{R}_{n-1}} \right)^2 |c_i|^2 \right\}^{\frac{k}{2}}$$

$$= (b-a)^{1-\frac{k}{2}} \sum_{n=1}^{\infty} \left(\frac{1}{n\Omega(n)} \right)^{1-\frac{k}{2}} \left\{ (n\Omega(n))^{\frac{2}{k}-1} n^{2-\frac{2}{k}} \sum_{i=1}^{n} \left(\frac{\overline{R}_n^i \widetilde{\widetilde{R}}_n^i}{\overline{R}_n} - \frac{\overline{R}_{n-1}^i \widetilde{\widetilde{R}}_{n-1}^i}{\overline{R}_{n-1}} \right)^2 |c_i|^2 \right\}^{\frac{k}{2}}$$

Hence, by Holder's inequality

$$\leq (b-a)^{1-\frac{k}{2}} \left[\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)} \right]^{1-\frac{k}{2}} \left\{ \sum_{n=1}^{\infty} (n\Omega(n))^{\frac{2}{k}-1} n^{2-\frac{2}{k}} \sum_{i=1}^{n} \left(\frac{\overline{R}_n^i \widetilde{\widetilde{R}}_n^i}{\overline{R}_n} - \frac{\overline{R}_{n-1}^i \widetilde{\widetilde{R}}_{n-1}^i}{\overline{R}_{n-1}} \right)^2 |c_i|^2 \right\}^{\frac{k}{2}}$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)} < \infty$$

We have

$$\leq (b-a)^{1-\frac{k}{2}} \left\{ \sum_{i=1}^{\infty} |c_i|^2 \sum_{n=i}^{\infty} (n\Omega(n))^{\frac{2}{k}-1} n^{2-\frac{2}{k}} \left(\frac{\overline{R}_n^i \widetilde{\widetilde{R}}_n^i}{\overline{R}_n} - \frac{\overline{R}_{n-1}^i \widetilde{\widetilde{R}}_{n-1}^i}{\overline{R}_{n-1}} \right)^2 \right\}^{\frac{k}{2}}$$

$$\leq (b-a)^{1-\frac{k}{2}} \left\{ \sum_{i=1}^{\infty} |c_i|^2 \sum_{n=i}^{\infty} \left(\frac{\Omega(n)}{n} \right)^{\frac{2}{k}-1} n^{\frac{2}{k}} \left(\frac{\overline{R}_n^i \widetilde{\widetilde{R}}_n^i}{\overline{R}_n} - \frac{\overline{R}_{n-1}^i \widetilde{\widetilde{R}}_{n-1}^i}{\overline{R}_{n-1}} \right)^2 \right\}^{\frac{k}{2}}$$

Since,

$\left\{ \frac{\Omega(n)}{n} \right\}$ is non-increasing

$$\leq (b-a)^{1-\frac{k}{2}}\left\{\sum_{i=1}^{\infty}|c_i|^2\left(\frac{\Omega(i)}{i}\right)^{\frac{2}{k}-1}\sum_{n=i}^{\infty}n^{\frac{2}{k}}\left(\frac{\overline{R}_n^i\widetilde{\widetilde{R}}_n^i}{\overline{R}_n}-\frac{\overline{R}_{n-1}^i\widetilde{\widetilde{R}}_{n-1}^i}{\overline{R}_{n-1}}\right)^2\right\}^{\frac{k}{2}}$$

$$\leq (b-a)^{1-\frac{k}{2}}\left\{\sum_{i=1}^{\infty}|c_i|^2\big(\Omega(i)\big)^{\frac{2}{k}-1}\Omega^{\frac{2}{k}-1}(i)\widetilde{\mathfrak{R}}^{(k)}(i)\right\}^{\frac{k}{2}}$$

where

$$\widetilde{\mathfrak{R}}^{(k)}(i) \coloneqq \frac{1}{i^{\frac{2}{k}-1}}\ \sum_{n=i}^{\infty}n^{\frac{2}{k}}\left(\frac{\overline{R}_n^i\widetilde{\widetilde{R}}_n^i}{\overline{R}_n}-\frac{\overline{R}_{n-1}^i\widetilde{\widetilde{R}}_{n-1}^i}{\overline{R}_{n-1}}\right)^2$$

Hence, by (8-5),

$$\sum_{n=1}^{\infty}n^{k-1}\int_a^b|\Delta E_n(x)|^k dx<\infty$$

Hence by Beppo Levi's theorem

$$\sum_{n=1}^{\infty}n^{k-1}|\Delta E_n(x)|^k<\infty$$

Hence the proof follows.