# Recent Trends in Convergence and Summability of General Orthogonal Series

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## 1.0 INTRODUCTION

A series of the form

$$\sum_{n=0}^{\infty} c_n \, \emptyset_n(x) \,, \qquad (1)$$

where $\{\emptyset_n(x)\}$ is an orthonormal system of functions with respect to a measure μ(x) and $c_0, c_1, c_2, c_3 \dots \dots ....$ are an arbitrary set of real numbers is called an orthogonal series. A system $\{\emptyset_n(x)\}$ is called an ONS if

$$\int_a^b \emptyset_n(x) \, \emptyset_m(x) \, d\mu(x) = \begin{cases} 0 & when\ m \neq n, \\ 1 & when\ m = n. \end{cases}$$

However, if the coefficients $c_n$ are of the form,

$$c_n = \frac{1}{\int_a^b \emptyset_n^2 d\mu(x)} \int_a^b f(x) \, \emptyset_n(x) d\mu(x), \text{ n=0, 1, 2, 3, } \dots\dots\dots\dots\dots$$

then the series (1) is said to be an orthogonal expansion of a function f(x) and the numbers $c_n$ are expansion coefficients of f(x). The orthogonal expansion of a function f(x) is written as

$$f(x) \sim \sum_{n=0}^{\infty} c_n \emptyset_n(x)$$

The difference between orthogonal expansion and orthogonal series is characterized by the following minimum property given by Gram (Szego, 1939):

Let f(x) ∈ $L^2_{\mu(x)}$ and $\{\emptyset_n(x)\}$ be an arbitrary ONS. Among all the partial sums of the form

$$S_n(x) = \sum_{k=0}^{n} c_k \emptyset_k(x)$$

the integral

$$\int_a^b [f(x) - S_n(x)]^2 \, d\mu(x)$$

attains its minimum value when

$$S_n(x) = \sum_{k=0}^{n} c_k \emptyset_k(x)$$

that is, $S_n(x)$ is the $n^{th}$ partial sum of the orthogonal expansion of function f(x)(Alexits G., 1961).

The well known Bessel's inequality

$$\sum_{n=0}^{\infty} c_n^2 \leq \int_a^b f^2(x)d\mu(x)$$

is the immediate consequences of the Gram(Gram, J.P., 1883) result. The important consequence of Bessel's inequality is that the expansion coefficient of $L_\mu^2$- integrable function converges to zero as n→ ∞.

This fact is an important starting point for the discussion of the convergence of orthogonal series.

The Fundamental Riesz - Fischer theorem in the theory of orthogonal series (Riesz, 1907, Fischer, 1907) reads as follows:

A necessary and sufficient condition that $\{c_n\}$ be the sequence of expansion coefficients of a function f(x) $\in L^2_{\mu(x)}$ is

$$\sum_{n=0}^{\infty} c_n^2 < \infty \qquad (2)$$

The partial sums

$S_n(x) = \sum_{k=0}^{n} c_k \emptyset_k(x)$ of the expansion of f(x) then converge in the mean to the generating function f(x).

Since the 18th century, the research of L. Euler, D. Bernoulli, A. Legendre, P. Laplace, F. Bessel contains special orthonormal systems and expansion of a functions with respect to orthonormal systems in the subjects like mathematics, astronomy, mechanics, and physics. Researches of some of the researchers mentioned below suggest the strong requirement of development of theory of orthogonal series:

- The research of J. Fourier on the Fourier method for solving the boundary value problems of mathematical physics
- The research of J. Sturm and J. Liouville
- The research of P.L. Chebyshev on the problem of moments and interpolation
- The research of D. Hilbert on integral equation with symmetric kernel
- The research by H. Lebesgue for measure theory and the Lebesgue integral

There was a remarkable progress of the theory of general orthogonal series during 20th century. Several researchers have make use of orthonormal systems of functions and orthogonal series in the most areas like mathematical physics, operational calculus, functional analysis, quantum mechanics, mathematical statistics, computational mathematics, etc..

Looking to this the question of convergence and summability of general orthogonal series have made an important impact on several researchers. The researchers like Fejer, Hardy, Hilbert, Hobson, Lebesgue, Riesz F., Riesz M., Weyl H., Alexits, Kacmarz, Steinhaus, Menchoff D., Zygmund, Lonrentz L., Meder M., Tandori K., Leindler L., and many other leading mathematicians have made an important contribution in the areas of convergence, summation and approximation problems of general orthogonal series.

## 2.0 BACKGROUND OF THE PROPOSED RESEARCH

The convergence of orthogonal series was studied by Jerosch, F., Weyl H(1909), Weyl, H. (1909), Hobson, E.W. (1913), Plancherel, M. (1910), Rademacher, H(1922), Menchoff, D. (1923), Gaposkin, V.F.(1964), Salem, R(1940), Talalyan, A.A.(1956), Walfisz, A. (1940), Tandori, K(1975) , Kolmogoroff, A., Seliverstoff, G. (1925) and Plessner, A.(1926). In recent years significant contribution in this area is observed.

Besides the question of convergence, Menchoff, D. (1925), Menchoff, D.(1926), Kaczmarz, S.(1929) and several others have discussed the Cesáro summability of orthogonal series.

The order of approximation was studied by Snouchi, G(1944), Alexits, G., Kralik, D. (1965), Leindler, L.(1964), Bolgov, Efimov(1971), Kantawala (1986), Wlodzimierz, L.(2001) and several others.

Absolute summability of orthogonal series was studied by Prasad(1960), Tsuchikura(1953), Tandori K.(1957), Leindler L.(1961), Leindler L.(1963), Grepacevskaja(1964), Patel, R.K.(1975) and Bhatgnar(1973), Alexits, G., Kralik, D. (1965) , Moricz(1969), Srivastava, P.(1967), Meder(1958) and others.

It is important to note that Nörlund summability, Riesz summability, $(\overline{N},p_n)$ summability, Euler summability, and de-la Vallée summability are discussed by several researchers. However, it is observe that certain summability requires more study for research in orthogonal series.

## 2.0.1 BANACH SUMMABILITY OF ORTHOGONAL SERIES

Let $l_\infty$ denotes the linear spaces of all bounded sequence on R. If all the Banach limits of sequence x are same, a sequence x ∈ $\boldsymbol{l}_\infty$ is said to be Banach summable.

A series

$$\sum_{n=1}^{\infty} u_n \qquad (3)$$

with the sequence of partial sums $\{s_n = \sum_{m=0}^{n} u_m\}$ is said to be Banach summable if and only if $\{s_n\}$ is Banach summable. Let the sequence $\{t_m(n)\}$ is defined by

$$t_m(\mathrm{n}) = \frac{1}{\mathrm{m}} \sum_{v=0}^{m-1} s_{v+n}\ ; m \in N$$

Then, $t_m(n)$ is called the $m^{th}$ element of the Banach transformed sequence.

If $\lim_{m\to\infty} t_m\,(n) = s$, a finite number, uniformly for all n∈ N, then $\sum_{n=1}^{\infty} u_n$ is said to be Banach summable (Lorentz, 1948). Thus, if

$$\sup_n |t_m(n) - S\,| \to 0\ as\ m \to \infty$$

then $\sum_{n=1}^{\infty} u_n$ is said to be is Banach summable to s.

Further, if the series

$$\sum_{m=1}^{\infty} |t_m(n) - t_{m+1}(n)| < \infty$$

uniformly for all n∈ $N$, then $\sum_{n=1}^{\infty} u_n$ is said to be absolutely Banach summable or |B| summable.

Bosanquet and Hyslop (1937) proved the following theorem for absolute Cesàro summability of the conjugate series of a Fourier series.

**Theorem:** If $0 < \propto < 1$, $\Psi_\propto(+0) = 0$ and $\int_0^{\pi} \frac{\mathrm{d}\Psi_\propto(\mathrm{t})}{\mathrm{t}^\propto} < \infty$, then

$$\sum_{\mathrm{n}=1}^{\infty} \mathrm{B_n(t)}$$

is summable |C, β | at t = x, $\beta > \propto$. Here, $\Psi_0(t) = \Psi(t)$, $\Psi_\propto(t) = \frac{1}{\Gamma(\propto)} \int_0^t (t-u)^{\propto-1}\,\Psi(u)du$ , $\propto > 0$; $\Psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$ and

$$\sum_{\mathrm{n}=1}^{\infty} \mathrm{B_n(t)}$$

is conjugate Fourier series of a function f.

Swamy(1980) extended the above theorem to generalized absolute Cesàro summability. Similar theorems have been established by Misra and Sahoo (2002) as well as Paikray, Misra and Sahoo (2011) for Banach summability. Paikray, Misra and Sahoo (2012) proved the theorem for a factored conjugate Fourier series for absolute Banach Summability.

### 2.0.2 NÖRLUND MEAN OF THE ORTHOGONAL EXPANSION

We say the series (1) is (N, $p_n$)-summable to s, if

$$\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\, s_k \to s \; as \; n \to \infty$$

where $\{p_n\}$ is a sequence of numbers with $p_0 > 0$ and $p_n \geq 0$ for all n and $P_n = \sum_{k=0}^{n} p_k$.

Strong approximation of Cesáro means of order $\propto > 0$ is obtained by Sunouchi G (1965, 1967), Leindier L. (1967,1971, 1996) and Kantawala P.S.(1991,1995) have obtained the strong approximation of Nörlund and Euler means of orthogonal series.

Sunouchi G. (1965) proved the strong (C, $\propto$ )-summability of orthogonal series and theorem is as follows:

**Theorem:** If

$$\sum_{m=1}^{\infty} c_m^2 (loglogm)^2 < \infty ,$$

then, there exists a square integrable function f (x) such that

$$\lim_{n\to\infty} \frac{1}{A_n^{\alpha}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} \, |S_{n_v}(x) - f(x)|^{r} = 0$$

for any $\propto > 0$ and r > 0 a.e. in [a,b] and for increasing sequence $\{ n_v \}$ .

Tiwari S.K. and Kachhara D. K.(2011) have generalized the same theorem for the Strong Nörlund summability of the orthogonal expansion.

### 2.0.3 APPROXIMATION BY NÖRLUND MEAN OF WALSH-FOURIER SERIES:

Given a function f $\in L^1$, its Walsh-Fourier series is defined by $\sum_{n=1}^{\infty} c_n w_n(x)$, where

$$c_n = \int_0^1 f(t) w_n(t) dt$$

The $n^{th}$ partial sum of the above series is

$$S_n(f,x) = \sum_{k=0}^{n-1} c_k w_k(x) , \; n \geq 1$$

Moricz F. and Siddiqui A.H. (1992) worked the rate of approximation by Nörlund means for Walsh-Fourier series of a function in $L^p$ as well as in Lip(∝, p) over the unit interval [O, 1), where ∝≥ 0 and 1≤p < ∞.

They have obtained the results of Yano Sh.(1951), Jaxfsbova M. A(1966), and Skvorcov V. A. (1981) as a particular case for the rate of approximation by Cesáro means. The result of Moricz F. and Siddiqui A.H. (1992) is as follows:

**Theorem :** Let f∈ $L^p$, l≤ $p \le \infty$, let n = $2^m$+k, 1≤ $k \le 2^m$, m≥ 1, and let {$q_k$: k ≥ 0} be a sequence of nonnegative numbers such that

$$\frac{n^{\gamma-1}}{Q_n^{\gamma}} \sum_{k=0}^{n-1} q_k^{\gamma} = O(1) \ \ for\ some\ 1 < \gamma \le 2$$

If {$q_k$} is non decreasing, then

$$\| t_n(f)-f\|_p \ll \frac{5}{2Q_n} \sum_{j=0}^{m-1} 2^j \ q_{n-2^j} w_\rho \ \left(f, 2^{-j}\right) + O\{w_\rho\,(f, 2^m)\}$$

while if {$q_k$ } is non increasing

$$\| t_n(f)-f\|_p \ll \frac{5}{2Q_n} \sum_{j=0}^{m-1} (q_{n-2^j+1} - q_{n-2^{j+1}+1}) w_\rho \ \left(f, 2^{-j}\right) + O\{w_\rho\,(f, 2^{-m})\}$$

**2.0.4 APPROXIMATION OF DOUBLE ORTHOGONAL SERIES**

Let $\varphi = \{\varphi_i(x)$: i∈ $z_+^2$ } be an orthonormal system on the unit interval (0, 1) and Φ the set of all such systems. Consider the double orthogonal series

$$\sum_{i \ge 0} a_i\, \emptyset_i\,(x) = \sum_{i_1=0}^{\infty} \ \ \sum_{i_2=0}^{\infty} a_{i_1,i_2}\, \varphi_{i_1,i_2}(x),$$

Where $a = \{a_i : i \in z_+^2\}$ is a double sequence of real numbers (coefficients). Also, $z_+$ be set of all nonnegative integers and $z_+^2$ be the set of all 2-tuples i = $(i_1, i_2)$ with nonnegative integral coordinates for which

$$\sum_{i \ge 0} a_i^2 = \sum_{i_1=0}^{\infty} \ \ \sum_{i_2=0}^{\infty} a_{i_1 i_2}^2 < \infty$$

Let a positive double sequence $\lambda(n) = \lambda(n_1, n_2)$ be such that $\lambda(n) \uparrow \infty$ as $n \to \infty$

Put $\Delta_n^{\propto}$(x) = |f(x) - $\sigma_n^{\propto}(x)$| , n $\epsilon\ z_+^2$ , x$\epsilon$ (0, 1), ∝= ($\propto_1, \propto_2$), where $\sigma_n^{\propto}(x)$ is the Cesáro means of the rectangular partial sums with respect to n.( Andrienko V. A. and Kovalenko I. G. (2004))

Andrienko, V. A.(1989) have proved the following theorem for single series and the sequence $\lambda(n)$.

**Theorem:** Let a sequence $\{\lambda(n) > 0 : n \in z_+\}$ be such that $\frac{\lambda(n)}{loglogn} \uparrow$ and $\frac{\lambda(n)}{loglogn} \downarrow 0$. Then if the condition $\sum_{i\geq 0} a_i^2 \lambda_i^2 < \infty$ ; $i \in z_+$ is satisfied, then for any $\propto > 0$ and $\varphi \epsilon \Phi$, the following estimate holds a. e. in (0, 1):

$$\Delta_n^{\propto}(x) = o_x\{ loglogn/\lambda(n) \}, \ n \to \infty \qquad (4)$$

If one of the following condition is satisfied,

A. $\lambda(n)\exp(-log^{\gamma} n)$ decreases for some $\gamma \in (0,1)$

B. $\mu(n) = n/\lambda(n) \uparrow \infty$, but $\mu(n)\exp(-log^{\gamma}(n))$ decreases for some $\gamma \in (0,1)$

C. $\frac{n}{\lambda(n)} \downarrow$

then, the statement (4) holds.

Estimates of the quantity $\Delta_n^{(1,1)}(x)$ for double orthogonal series were obtained in Móricz, F.(1983), Móricz, F.(1984), and Móricz, F.(1987).

Andrienko, V. A., Kovalenko, I. G. (2004) have obtained the estimates for $\Delta_n^{(\propto_1,\propto_2)}(x)$.

## 2.0.5 TAUBERIAN THEOREMS FOR ORTHOGONAL SERIES

Assume that $\{p_k\}$ is a fixed sequence of positive numbers such that

$$P_n = \sum_{k-0}^{n} p_k \to \infty \ as \ n \to \infty \qquad (5)$$

and

$$p(t) = \sum_{k=0}^{\infty} p_k t^k < \infty \ \ for \ 0 \leq t < 1 \qquad (6)$$

Given a sequence $\{s_k\}$ of complex numbers, method (Mp) is defined by

$$s_n \to s(M_p) \ \text{ if}$$

$$\frac{1}{P_n}\sum_{k=0}^{n} p_k\, s_k(x) \to s \ as \ n \to \infty$$

while the method (Jp) is defined by

$$s_n \to s(J_p) \ \text{ if}$$

$$\sum_{k=0}^{n} p_k\, s_k(x) r^k \ \ converges\ for\ \ 0 \leq t < 1$$

and if

$$\frac{1}{p(t)}\sum_{k=0}^{n} p_k\, s_k(x) t^k \to s \ \text{ as } \ t \uparrow 1$$

Borwein D. and Kratz W.(1989) have introduced the following notion:

$$\Delta_n := \inf_{0<t<1} p(t) t^n \ ; \ n=0, 1, 2, 3, 4, \ \ldots\ldots$$

The sequence $(\Delta_n)$ is closely related to $(P_n)$. We always have $P_n \leq \Delta_n$ and for many sequences $(p_n)$ we have $\Delta_n \approx Pn$.

Móricz, F. and Stadtmüller, U. (1999) proved the following theorems. Theorem 1 and Theorem 2 deals with numerical sequence and Theorem 3 deals with general orthogonal series.

**Theorem 1:** Assume that conditions (5) and (6) are satisfied and that $\frac{p_n}{P_n}$ is a moment sequence. Then $s_n \to s(Jp)$ implies $sn \to s(Jp \circ Mp)$, that is $Jp \subseteq Jp \circ Mp$.

**Theorem 2:** Assume that the conditions of above theorem are satisfied and that

$$\frac{P_{2n}}{P_n} = O(1)$$

Furthermore, suppose that (1) is an orthogonal series with coefficients $\{c_k\}$ satisfying (2) and that E is a measurable set of positive measure. Then we have

$$s_n(x) \to s\ (Mp) \text{ a.e. on } E$$

if and only if

$$s_n(x) \to s\ (Jp) \text{ a.e. on } E$$

**Theorem 3:** Assume that conditions (5) and (6) hold.

Then

$s_n \to s\ (Jp)$ and $\sum_{k=0}^{\infty} \frac{\Delta_n}{p_k} |u_k|^2 < \infty$ imply $s_n \to s$ as $n \to \infty$, where $s_n$ is the sequence of partial sums of the series $\sum_{n=1}^{\infty} u_n$.

### 2.0.6 $|N, p, q|_k$ SUMMABILITY OF ORTHOGONAL SERIES

Let (3) be a given infinite series with its partial sums $\{s_n\}$. Let p denotes the sequence $\{p_n\}$ and q denotes the sequence $\{q_n\}$. For two given sequences p and q, the convolution $(p * q)_n$ is defined by

$$(p * q)_n = \sum_{m=0}^{n} q_{n-m}\, p_m = \sum_{m=0}^{n} q_m\, p_{n-m}.$$

When $(p * q)_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_n^{p,q} = \frac{1}{(p * q)_n} \sum_{m=0}^{n} p_{n-m}\, q_m s_m.$$

The infinite series (3) is absolutely summable $(N, p, q)_k$ of order k, if for $k \geq 1$ the series

$$\sum_{n=0}^{\infty} n^{k-1} |t_n^{p,q} - t_{n-1}^{p,q}|^k$$

converges, and we write

$$\sum_{n=1}^{\infty} u_n \in |N, p, q|_k.$$

We note that for k = 1, $|N, p, q|_k$ summability is the same as |N, p, q| summability introduced by Tanaka M.(1978).

Okuyama Y. (2002) adopted the following notions:

$$R_n := (p * q)_n, \ R_n^j := \sum_{m=j}^{n} p_{n-m}\, q_m$$

and

$$R_n^{n+1} = 0$$

$$R_n^0 = R_n$$

Also we put

$$P_n := (p * 1)_n = \sum_{m=0}^{n} p_m \quad \text{and } q_n := (1 * q)_n = \sum_{m=0}^{n} q_m$$

Krasniqi, X.Z. (2010) studied the $|N, p, q|_k$ summability of the orthogonal series (1), for $1 \le k \le 2$, and he deduced as corollaries of all results of Okuyama Y.(2002).

He proved the following theorem:

**Theorem:** If for $1 \le k \le 2$, the series

$$\sum_{n=0}^{\infty} \{ n^{2-\frac{2}{k}} \sum_{j=1}^{n} \left( \frac{R_j^n}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \}^{\frac{k}{2}}$$

converges then the orthogonal series (1) is summable $|N, p, q|_k$ almost everywhere.

### 2.0.7. $|\bar{N}, p_n, \delta|_k$, $(1 \le k \le 2)$ SUMMABILITY OF ORTHOGONAL SERIES

Let (3) be a given infinite series with its partial sums $\{s_n\}$. Let $p_n$ be a sequence of numbers such that, $\sum_{k=0}^{n} p_k \to \infty$ $as$ $n \to \infty (P_{-i} = p_{-i} = 0, i \ge 1)$.

The sequence to sequence transformation

$$T_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k\, s_k$$

defines the sequence ($T_n$) of the Riesz mean or simply the ( $\bar{N}$ , $p_n$) mean of the sequence ($s_n$), generated by the sequence of coefficients ($p_n$) (Tandori K,1960). The series (3) is said to be summable $|\bar{N}, p_n|_k$ or summable $|R, P_n, 1|_k$, $k \ge 1$, if (Bor H.(1993))

$$\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} | \Delta T_{n-1}|^k < \infty$$

The case k = 1 is reduced to the absolute Riesz summability | R, Pn, 1 | and further, in the special case $p_n$ = 1/n + 1, the summability | R, Pn, 1 | is the same as the absolute logarithmic summability.

Dealing with the absolute Riesz summability of orthogonal series, Moricz F.(1962) have generalized the result of Tandori K(1960). Okuyama, Y., Tsuchikura T.(1981) as well as Leindler L. (1983) proved some results in the same direction. Okuyama Y.(1988) proved the following theorem:

**Theorem**: Let $1 \leq k \leq 2$ and $\{p_n\}$ be a positive sequence. If the series

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n\, P_{n-1}{}^{k}} \left(\sum_{j=1}^{n} P_{j-1}^{2} |a_j|^2\right)^{k/2} < \infty$$

then the orthogonal series (1) is summable $| R, p_n, 1 |_k$ almost everywhere.

## 2.0.8 LAMBERT SUMMABILITY OF ORTHOGONAL SERIES

A series $\sum a_n$ is Lambert summable to A, written $\sum a_n = A(L)$ , if

$$\lim_{r \to 1-} (1 - r) \sum_{n=1}^{\infty} \frac{n a_n r^n}{1 - r^n} = A$$

Bellman, R. (1943) have proved the result on Lambert summability of orthogonal series.

## 2.0.9 MATRIX SUMMABILITY OF ORTHOGONAL SERIES

Let (3) be a given infinite series with its partial sums $\{s_n\}$ and let A := $(a_{nv})$ be a normal matrix, i.e. a lower triangular matrix of non-zero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence s:= $\{s_n\}$ to As := {An(s)}, where

$$A_n(s) := \sum_{v=0}^{n} a_{nv}\, s_v \quad n = 0,1,2,3 \ldots \ldots$$

The series $\sum_{n=1}^{\infty} u_n$ is said to be summable $|A|_k$, k ≥1, if (Tanović-Miller N. (1979))

$$\sum_{n=1}^{\infty} n^{k-1} |\bar{\Delta} A_n(s)|^k$$

converges, where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s)$$

Krasniqi, X. Z et.al. (2012) proved the following theorem for matrix summability of orthogonal series.

**Theorem.** If the series

$$\sum_{n=0}^{\infty}\{n^{2-\frac{2}{k}}\ \sum_{j=1}^{n}|\ \hat{a}_{n,j}|^2\ \ |c_j|^2\}^{\frac{k}{2}}$$

converges for $1\leq\ k \leq 2$, then the orthogonal series (1) is summable $|A|_k$ almost everywhere.

## 2.0.10 SUMMABILITY OF ORTHOGONAL SERIES IN COMMUTATIVE L2 SPACE

Given a sequence of real numbers $0=\lambda_0 < \lambda_1 < \lambda_2 < \dots\dots\dots < \lambda_n \to \infty\ as\ n \to \infty$ . A Series (3) is said to be Riesz summable or (R, $\lambda_n$, 1) to a sum s if

$$\lim_{n\to\infty}\sum_{k=0}^{n}(1-\frac{\lambda_k}{\lambda_{n+1}})u_k = s$$

Clearly, s is uniquely determined if exists.

Let (X, F, μ) be an arbitrary positive measure space, $\{\zeta_n : n = 0,1,2 \dots\dots\dots\dots\}$ be a sequence of pairwise orthogonal functions in $L_2 = L_2(X, F,\mu)$ and set

$s_n = \sum_{k=0}^{n}\zeta_k$ , $\sigma_n = \sum_{k=0}^{n}(1-\frac{\lambda_k}{\lambda_{n+1}})\zeta_k$   n=0,1,2,3……..

The following theorem is due to Zygmund A. (1927) in commutative $L^2$ space:

**Theorem 1:**

If a sequence $\{\zeta_n : n = 0,1,2 \dots\dots\dots\dots\}$ of pair wise orthogonal functions in $L_2 = L_2(X, F,\mu)$ over a positive measure space is such that

$$\sum_{n:\lambda_n\geq 4}(\mathrm{loglog}\lambda_n)^2||\zeta_n||^2 < \infty$$

Then

$$\lim_{n\to\infty}\sigma_n\,(x) = s(x)\quad a.e.,$$

where s is the sum of the series $\sum_{n=0}^{\infty}\zeta_n$ in the norm of $L_2$.

Gac B.L. and Móricz F. (2011) have extended the theorem for von Neumann algebra in non-commutative space.

**Theorem 2**: Let M be a von Neumann algebra, Φ a faithful and normal state acting on **M** and $\{\lambda_n : n = 0, 1, 2 \dots\dots\dots\dots\}$ a sequence of real numbers satisfying conditions $\lim_{n\to\infty} sup\frac{\lambda_{n+1}}{\lambda_n} = C < \infty, where\ 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\dots\dots < \lambda_n \to \infty\ as\ n \to \infty$ and if $\{A_n : n = 0,1,2 \dots\dots\dots\dots\}$ is a sequence of pair wise orthogonal operators in M such that

$$\sum_{n:\lambda_n\geq 4}(\mathrm{loglog}\lambda_n)^2\emptyset(|A_n|^2) < \infty$$

Then

$$\sum_{k=0}^{n}\left(1-\frac{\lambda_k}{\lambda_{n+1}}\right)\pi(A_k)w \to \sigma \ as\ n \to \infty$$

Where $\sigma$ is the sum of the series $\sum_{n=0}^{\infty}\pi(A_n)w$ in the norm of $L_2 = L_2(M, \Phi)$

Similar result was proved by Gac B.L. and Móricz F. (2011) for $L_2$ space.

**Theorem 3**: Let M be a von Neumann algebra, Φ a faithful and normal state acting on **M** and $\{\lambda_n: n = 0, 1, 2 \ldots\ldots\ldots\ldots\}$ be a sequence of real numbers satisfying conditions $\lim_{n\to\infty} sup\frac{\lambda_{n+1}}{\lambda_n}$, $\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2} < \infty$ and $where\ 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \ldots\ldots < \lambda_n \to \infty\ as\ n \to \infty$ and if $\{\zeta_n: n = 0,1,2 \ldots\ldots\ldots\ldots\}$ is a sequence of pair wise orthogonal vectors in $L_2 := L_2(M, \Phi)$ such that

$$\sum_{n:\lambda_n\geq 4}(\mathrm{loglog}\lambda_n)^2||\zeta_n||^2 < \infty$$

Then

$$\sum_{k=0}^{n}\left(1-\frac{\lambda_k}{\lambda_{n+1}}\right)\xi_k \to \sigma \ as\ n \to \infty$$

here $\sigma$ is the sum of the series $\sum_{n=0}^{\infty}\xi_n$ in the norm of $L_2$

**3.0 OUTLINE OF CHAPTERS**

Present thesis will consist of eight chapters.

**Chapter 1** will be an introduction.

**Chapter 2** will cover the results on absolute Banach and Nörlund Summability of orthogonal series.

Paikray S. K. et al.(2012) have obtained the result for factored conjugate Fourier series for absolute Banach Summability. We have generalized his result for orthogonal series.

Sunouchi G.(1965) obtained the strong approximation of orthogonal Series. Sunouchi G.(1967) proved the approximation of Fourier series and orthogonal Series. In this chapter, we have obtained the result with general Strong Nörlund summability of orthogonal expansion on the line of Tiwari S.K. and Kachhara D. K.(2011) .

**Chapter 3** will deal with rate of summability of double orthogonal series and rate of approximation by Nörlund means for Walsh-Fourier series.

Moricz F. and Siddiqi A. H. (1992), studied the rate of approximation by Nörlund means for Walsh-Fourier series of a function in $L^p$. In this chapter we have made an attempt to generalize the same results for different Nörlund means.

Andrienko V. A. and Kovalenko I. G. (2004) obtained the estimates on the rate of almost everywhere summability of orthogonal expansions of square integrable functions by Cesáro methods of positive order for double orthogonal series.

In this chapter, we have made an attempt to investigate the rate of Nörlund Summability of double orthogonal series in parallel to the work of Andrienko V. A. and Kovalenko I. G. (2004).

**Chapter 4** will deal with certain Tauberian theorems related with general orthogonal series.

Móricz, F. and Stadtmüller, U.(1999) proved Tauberian theorems from *Jp*-summability methods of power series type and *Mp*-summability methods of weighted means to ordinary convergence for general orthogonal series.

In this chapter we have made an attempt to generalize the results of Móricz, F. and Stadtmüller, U.(1999).

**Chapter 5** deals with summability of general orthogonal series.

Krasniqi X. Z(2010) gave some general theorems on the $|N, p, q|_k$, $(1 \leq k \leq 2)$ summability of orthogonal series. In this chapter we have generalized the theorems to $|\overline{N}, p, q|_k$ summability.

Sönmez A. (2008) present some results on $|\overline{N}, p_n, \delta|_k$, $(1 \leq k \leq 2)$ summability of orthogonal series. In this chapter we have made an attempt to generalize the result of Sönmez A (2008) for $|\overline{N}, p_n, \delta|_k$, $(1 \leq k \leq 2)$ summability.

**Chapter 6** will be devoted for Lambert summability of general orthogonal series.

In this chapter we have made an attempt to generalize the theorem of Bellman R.(1943).

**Chapter 7** deals with matrix summability of general Orthogonal Series.

Krasniqi X.Z.(2012) et al. proved theorems on absolute matrix summability of general orthogonal series. In this chapter we have generalized the result of Krasniqi X. Z. (2012) for different Nörlund means.

**Chapter 8** will discuss the results on Nörlund Summability of orthogonal series in commutative and non-commutative $L^2$ spaces. These results are an extension of Zygmund A.(1927), Le Gac B., Móricz F.(2011).

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