

Chapter II

Absolute Banach and

$$|N, p_n^\alpha, q_n^\alpha|, \alpha > -1$$

Summability of General

Orthogonal Series

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2.1 Introduction

Let $\{\varphi_n(x)\}; n = 0,1,2 \dots$ be an orthonormal system of functions defined in the interval $[a, b]$. We shall consider the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x), \quad (2-1)$$

where $\{c_n\}$ is a sequence of real numbers.

Let $\{s_n\}$ be the sequence of partial sums of the series (2-1). We may write it as,

$$s_n(x) = \sum_{k=0}^n c_k \varphi_k(x).$$

The Banach mean, (N, p_n) mean, (\bar{N}, p_n) mean, and $(N, p_n^\alpha, q_n^\alpha)$ mean of the series (2-1) are denoted by

$$t_k^*(n) = \frac{1}{k} \sum_{v=0}^{k-1} s_{n+v}; k \in N$$

$$t_n(x) = \frac{1}{P_n} \sum_{k=0}^n p_{n-k} s_k(x),$$

$$\bar{t}_n(x) = \frac{1}{P_n} \sum_{k=0}^n p_k s_k(x),$$

and

$$t_n^{p^\alpha, q^\alpha}(x) = \frac{1}{R_n^\alpha} \sum_{r=0}^n p_{n-r}^\alpha q_r^\alpha s_r(x)$$

respectively.

We may refer to equations (1-13), (1-16), (1-18), and (1-26) for more information.

2.2 Absolute Banach Summability of Orthogonal Series

Let $f(t)$ be a periodic function with period 2π and Riemann integrable over $(-\pi, \pi)$. Suppose,

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t)$$

be the Fourier series of $f(t)$. Then, the series

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t)$$

be the conjugate Fourier series of $f(t)$.

Let

$$\psi(t) := \frac{1}{2} \{f(x+t) - f(x-t)\},$$

$$\Psi_\alpha(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \psi(u) du, \quad \alpha > 0,$$

$$\psi_\alpha(t) = \Gamma(\alpha + 1)t^{-\alpha}\Psi_\alpha(t), \quad \alpha \geq 0,$$

$$\psi_0(t) = \psi(t)$$

$[x]$ = The greatest integer which does not exceed x .

Bosanquet, L. et al. 1937 proved the following theorem for absolute Cesàro summability of order β ($0 < \beta < 1$) for conjugate Fourier series:

Theorem 2.1

If, $0 < \alpha < 1$,

$$\Psi_\alpha(+0) = 0 \text{ and } \int_0^\pi \frac{d\Psi_\alpha(t)}{t^\alpha} < \infty$$

then

$$\sum_{n=1}^{\infty} B_n(t)$$

is summable $|C, \beta|$ at $t = x, \beta > \alpha$.

In the same direction Swamy, N. et al. 1980 proved the following theorem for generalized absolute Cesàro summability of conjugate Fourier series.

Theorem 2.2

If, $0 < \alpha < 1, \Psi_\alpha(+0) = 0$,

$$\int_0^\pi \frac{d\Psi_\alpha(t)}{t^\alpha} < \infty$$

then the conjugate series of the Fourier series of $f(t)$ is summable $|C, \delta, \beta|$ at $t = x, \delta > \alpha$.

Misra, S. et al. 2002 have generalized the Theorem 2.2 on absolute Banach Summability. The theorem is as follows:

Theorem 2.3

If, $0 < \alpha < 1$, $\Psi_\alpha(+0) = 0$

and

$$\int_0^\pi \frac{d\Psi_\alpha(t)}{t^\alpha} < \infty$$

then the conjugate series of Fourier series of $f(t)$ is $|B|$ summable at $t = x$.

Moreover, Paikray, S. et al. 2012 have proved the following theorem on absolute Banach summability of factored conjugate Fourier series.

Theorem 2.4

Let $\Psi_\alpha(+0) = 0$, $0 < \alpha < 1$, such that

$$\int_0^\pi \frac{d\Psi_\alpha(u)}{u^\alpha \log(n+U)} < \infty$$

then the series

$$\sum_{n=1}^{\infty} \frac{B_n(t)}{\log(n+1)}$$

is $|B|$ summable at $t = x$ if

$$\sum_{k \leq \frac{1}{u}} \log(n+U) k^{\alpha-1} = O(U^\alpha \log(n+2)); U = \left[\frac{1}{u} \right]$$

The Theorem 2.1, Theorem 2.2, Theorem 2.3, and Theorem 2.4 are based on summability of conjugate Fourier series.

Tsuchikura, T. 1953, have proved the following theorem on Cesàro summability of order α for orthogonal series.

Theorem 2.5

Let $\{\varphi_n(x)\}$ be orthonormal system defined in the interval (a, b) and let $\alpha > 0$. If the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}} \left[\sum_{k=1}^n k^2 (n-k+1)^{2(\alpha-1)} a_k^2 \right]^{\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{|a_n|}{n^\alpha}$$

converges, then the orthogonal series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x)$$

is summable $|C, \alpha|$ for almost every x .

In this chapter, we have extended the Theorem 2.5 of Tsuchikura, T. 1953 for the absolute Banach summability.

Our theorem is as follows:

Theorem 2A

Let $\{\varphi_n(x)\}$ be an orthonormal system defined in (a, b) .

If,

$$\sum_{k=1}^{\infty} \frac{1}{k+1} \left\{ \sum_{v=1}^k c_{n+v}^2 \right\}^{\frac{1}{2}} < \infty$$

for all n , then orthogonal series (2-1) is absolutely Banach summable i.e. $|B|$ summable for every x .

2.3 Absolute $(N, p_n^\alpha, q_n^\alpha), \alpha > -1$ Summability of General Orthogonal Series

Strong approximation and strong summability problems were studied by Kantawala, P. et al. 1991, Kantawala, P. et al. 1995, Leindler, L. 1966, Leindler, L. 1967, Sunouchi, G. 1966, Sunouchi, G.1967.

Sunouchi, G. 1966 has proved the following theorem for $(C, \alpha), \alpha > 0$ summability of orthogonal series.

Theorem 2.6

If

$$\sum_{m=1}^{\infty} c_m^2 (\log \log m)^2 < \infty$$

then, there exists a square integrable function $f(x)$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{A_n^\alpha} \sum_{v=0}^n A_{n-v}^{\alpha-1} |s_{n_v}(x) - f(x)|^r = 0$$

for any $\alpha > 0$ and $r > 0$ a.e. in $[a, b]$ and for increasing sequence $\{n_v\}$.

Tiwari. S. et al. 2011, obtained the following result on strong Nörlund summability of orthogonal series.

Theorem 2.7

If the series

$$\sum_{n=1}^{\infty} \left\{ \sum_{j=1}^n \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \right\}^{\frac{1}{2}}$$

converges, then the orthogonal series (1-8) is summable $|N, p_n, q_n|$ almost everywhere. Refer to equations (3-3) and (3-5) for R_n and R_n^j respectively.

We have generalized the Theorem 2.7 for $|N, p_n^\alpha, q_n^\alpha|$, $\alpha > -1$ summability of an orthogonal series. Our theorem is as follows:

Theorem 2B

If the series

$$\sum_{n=1}^{\infty} \left\{ \sum_{v=1}^n \left(\frac{R_n^{\alpha v}}{R_n^\alpha} - \frac{R_{n-1}^{\alpha v}}{R_{n-1}^\alpha} \right)^2 |c_v|^2 \right\}^{\frac{1}{2}}$$

converges, then the orthogonal expansion

$$\sum_{v=0}^{\infty} c_v \varphi_v(x)$$

is summable $|N, p_n^\alpha, q_n^\alpha|$, $\alpha > -1$ almost everywhere.

The corollary related to Theorem 2B is as follows:

Corollary 2B

If the series,

$$\sum_{n=1}^{\infty} \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \left\{ \sum_{v=1}^n \left(\frac{P_n^\alpha}{p_n^\alpha} - \frac{P_{n-v}^\alpha}{R_{n-v}^\alpha} \right)^2 P_{n-v}^{\alpha-2} |c_v|^2 \right\}^{\frac{1}{2}}$$

converges, then orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable (N, p_n^α) almost everywhere.

2.4 Proof of Theorems

We shall use the following lemma to prove Theorem 2A;

Lemma 2A (Paikray, S. et al. 2012)

Let

$$\sum_{n=0}^{\infty} u_n$$

be an infinite series and $\{s_n\}$ be a sequence of its partial sums. Let $\{t_k^*(n)\}$ be a sequence defined by

$$t_k^*(n) = \frac{1}{k} \sum_{v=0}^{k-1} s_{n+v}; \quad k \in \mathbb{N}$$

then,

$$t_k^*(n) - t_{k+1}^*(n) = \frac{-1}{k(k+1)} \sum_{v=1}^k v u_{n+v} \quad (2-2)$$

Proof of Theorem 2A

Now,

$$t_k^*(n) - t_{k+1}^*(n) = \frac{-1}{k(k+1)} \sum_{v=1}^k v u_{n+v}$$

where, $u_{n+v} = c_{n+v} \varphi_{n+v}$

$$= \frac{-1}{k(k+1)} \sum_{v=1}^k v c_{n+v} \varphi_{n+v}$$

Now

$$\sum_{k=1}^{\infty} \int_a^b |t_k^*(n) - t_{k+1}^*(n)| dx = \sum_{k=1}^{\infty} \int_a^b 1 \cdot |t_k^*(n) - t_{k+1}^*(n)| dx$$

Hence, by Schwarz's inequality,

$$\begin{aligned} &\leq \sum_{k=1}^{\infty} \left\{ \int_a^b 1^2 dx \right\}^{\frac{1}{2}} \left\{ \int_a^b |t_k^*(n) - t_{k+1}^*(n)|^2 dx \right\}^{\frac{1}{2}} \\ &= \sqrt{b-a} \sum_{k=1}^{\infty} \left\{ \int_a^b |t_k^*(n) - t_{k+1}^*(n)|^2 dx \right\}^{\frac{1}{2}} \\ &= \sqrt{b-a} \sum_{k=1}^{\infty} \left\{ \int_a^b \frac{1}{k^2(k+1)^2} \left(\sum_{v=1}^k v c_{n+v} \varphi_{n+v}(x) \right)^2 dx \right\}^{\frac{1}{2}} \end{aligned}$$

Hence, by orthonormality we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \int_a^b |t_k^*(n) - t_{k+1}^*(n)| dx &\leq \sqrt{b-a} \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \left\{ \int_a^b \left(\sum_{v=1}^k v c_{n+v} \varphi_{n+v}(x) \right)^2 dx \right\}^{\frac{1}{2}} \\
&= \sqrt{b-a} \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \left\{ \sum_{v=1}^k v^2 c_{n+v}^2 \right\}^{\frac{1}{2}} \\
&\leq \sqrt{b-a} \sum_{k=1}^{\infty} \frac{1}{(k+1)} \left\{ \sum_{v=1}^k c_{n+v}^2 \right\}^{\frac{1}{2}} \\
&= M \sum_{k=1}^{\infty} \frac{1}{(k+1)} \left\{ \sum_{v=1}^k c_{n+v}^2 \right\}^{\frac{1}{2}}
\end{aligned}$$

where, $M := \sqrt{b-a}$ is some constant.

Hence, by hypothesis of Theorem 2A

$$\sum_{k=1}^{\infty} \int_a^b |t_k^*(n) - t_{k+1}^*(n)| dx < \infty$$

Therefore, by Beppo Levi's Theorem

$$\sum_{k=1}^{\infty} |t_k^*(n) - t_{k+1}^*(n)| < \infty$$

This completes the proof of theorem.

Proof of Theorem 2B

Now, $t_n^{p^\alpha, q^\alpha}(x)$ be the $(N, p_n^\alpha, q_n^\alpha)$ mean of the series (2-1).

Therefore,

$$\begin{aligned}
t_n^{p^\alpha, q^\alpha}(x) &= \frac{1}{R_n^\alpha} \sum_{k=0}^n p_{n-k}^\alpha q_k^\alpha s_k(x) \\
&= \frac{1}{R_n^\alpha} \sum_{k=0}^n p_{n-k}^\alpha q_k^\alpha \sum_{v=0}^k c_v \varphi_v(x)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{R_n^\alpha} \sum_{v=0}^n c_v \varphi_v(x) \sum_{k=v}^n p_{n-k}^\alpha q_k^\alpha \\
&= \frac{1}{R_n^\alpha} \sum_{v=0}^n R_n^{\alpha v} c_v \varphi_v(x),
\end{aligned}$$

where,

$$R_n^{\alpha v} = \sum_{k=v}^n p_{n-k}^\alpha q_k^\alpha$$

Now,

$$\begin{aligned}
t_n^{p^\alpha, q^\alpha}(x) - t_{n-1}^{p^\alpha, q^\alpha}(x) &= \frac{1}{R_n^\alpha} \sum_{v=0}^n R_n^{\alpha v} c_v \varphi_v(x) - \frac{1}{R_{n-1}^\alpha} \sum_{v=0}^{n-1} R_{n-1}^{\alpha v} c_v \varphi_v(x) \\
&= \sum_{v=1}^n \left(\frac{R_n^{\alpha v}}{R_n^\alpha} - \frac{R_{n-1}^{\alpha v}}{R_{n-1}^\alpha} \right) c_v \varphi_v(x) \\
\sum_{n=1}^{\infty} \int_a^b |t_n^{p^\alpha, q^\alpha} - t_{n-1}^{p^\alpha, q^\alpha}| dx &= \sum_{n=1}^{\infty} \int_a^b \left| \sum_{v=1}^n \left(\frac{R_n^{\alpha v}}{R_n^\alpha} - \frac{R_{n-1}^{\alpha v}}{R_{n-1}^\alpha} \right) c_v \varphi_v(x) \right| dx \\
&= \sum_{n=1}^{\infty} \left\{ \int_a^b \sum_{v=1}^n \left(\left(\frac{R_n^{\alpha v}}{R_n^\alpha} - \frac{R_{n-1}^{\alpha v}}{R_{n-1}^\alpha} \right) c_v \varphi_v(x) \right)^2 \right\}^{1/2}
\end{aligned}$$

By Schwarz's inequality and orthonormality gives

$$\begin{aligned}
&= \sqrt{b-a} \sum_{n=1}^{\infty} \left\{ \sum_{v=1}^n \left(\frac{R_n^{\alpha v}}{R_n^\alpha} - \frac{R_{n-1}^{\alpha v}}{R_{n-1}^\alpha} \right)^2 |c_v|^2 \right\}^{1/2} \\
&= M \sum_{n=1}^{\infty} \left\{ \sum_{v=1}^n \left(\frac{R_n^{\alpha v}}{R_n^\alpha} - \frac{R_{n-1}^{\alpha v}}{R_{n-1}^\alpha} \right)^2 |c_v|^2 \right\}^{1/2}
\end{aligned}$$

Hence, by hypothesis of Theorem 2B

$$\sum_{n=1}^{\infty} \int_a^b |t_n^{p^\alpha, q^\alpha} - t_{n-1}^{p^\alpha, q^\alpha}| dx < \infty$$

Hence, by Beppo Levi's Theorem

$$\sum_{n=1}^{\infty} |t_n^{p^\alpha, q^\alpha} - t_{n-1}^{p^\alpha, q^\alpha}| < \infty$$

Hence, series (2-1) is summable $|N, p_n^\alpha, q_n^\alpha|$ almost everywhere.

2.5 Proof of Corollaries

Proof of Corollary 2B

Let $q_n^\alpha = 1$ and we may use our Theorem 2B to prove the Corollary.

Now,

$$\begin{aligned} \frac{R_n^{\alpha v}}{R_n^\alpha} - \frac{R_{n-1}^{\alpha v}}{R_{n-1}^\alpha} &= \frac{P_{n-v}^\alpha}{P_n^\alpha} - \frac{P_{n-1-v}^\alpha}{P_{n-1}^\alpha} \\ &= \frac{1}{P_n^\alpha P_{n-1}^\alpha} (P_{n-v}^\alpha P_{n-1}^\alpha - P_{n-1-v}^\alpha P_n^\alpha) \\ &= \frac{1}{P_n^\alpha P_{n-1}^\alpha} (P_{n-v}^\alpha P_n^\alpha - P_{n-v}^\alpha p_n^\alpha - P_{n-v}^\alpha P_n^\alpha + p_{n-v}^\alpha P_n^\alpha) \\ &= \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \left(\frac{P_n^\alpha}{p_n^\alpha} - \frac{P_{n-v}^\alpha}{p_{n-v}^\alpha} \right) p_{n-v}^\alpha \end{aligned}$$

Now,

$$\begin{aligned} \int_a^b |t_n^{p^\alpha, q^\alpha}(x) - t_{n-1}^{p^\alpha, q^\alpha}(x)| dx &= \int_b^a \left| \sum_{v=1}^n \left(\frac{R_n^{\alpha v}}{R_n^\alpha} - \frac{R_{n-1}^{\alpha v}}{R_{n-1}^\alpha} \right) c_v \varphi_v(x) \right| dx \\ &= \int_b^a \left| \sum_{v=1}^n \left(\frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \left(\frac{P_n^\alpha}{p_n^\alpha} - \frac{P_{n-v}^\alpha}{p_{n-v}^\alpha} \right) p_{n-v}^\alpha \right) c_v \varphi_v(x) \right| dx \end{aligned}$$

Hence, Schwarz's inequality, and orthonormality gives

$$\begin{aligned} &\leq \sqrt{b-a} \left\{ \sum_{v=1}^n \frac{p_n^{\alpha^2} p_{n-v}^{\alpha^2}}{P_n^{\alpha^2} P_{n-1}^{\alpha^2}} \left(\frac{P_n^\alpha}{p_n^\alpha} - \frac{P_{n-v}^\alpha}{p_{n-v}^\alpha} \right)^2 |c_v|^2 \right\}^{1/2} \\ \sum_{n=1}^{\infty} \int_a^b |t_n^{p^\alpha, q^\alpha}(x) - t_{n-1}^{p^\alpha, q^\alpha}(x)| dx &\leq \sqrt{b-a} \sum_{n=1}^{\infty} \left\{ \sum_{v=1}^n \frac{p_n^{\alpha^2} p_{n-v}^{\alpha^2}}{P_n^{\alpha^2} P_{n-1}^{\alpha^2}} \left(\frac{P_n^\alpha}{p_n^\alpha} - \frac{P_{n-v}^\alpha}{p_{n-v}^\alpha} \right)^2 |c_v|^2 \right\}^{1/2} \\ &= M \sum_{n=1}^{\infty} \frac{p_n^\alpha}{P_n^\alpha P_{n-1}^\alpha} \left\{ \sum_{v=1}^n \left(\frac{P_n^\alpha}{p_n^\alpha} - \frac{P_{n-v}^\alpha}{p_{n-v}^\alpha} \right)^2 P_{n-v}^{\alpha^2} |c_v|^2 \right\}^{1/2} \end{aligned}$$

Hence by hypothesis of Corollary 2B

$$\sum_{n=1}^{\infty} \int_a^b |t_n^{p^\alpha, q^\alpha}(x) - t_{n-1}^{p^\alpha, q^\alpha}(x)| dx < \infty$$

Hence by Beppo Levi's Theorem

$$\sum_{n=1}^{\infty} |t_n^{p^\alpha, q^\alpha}(x) - t_{n-1}^{p^\alpha, q^\alpha}(x)| dx < \infty$$

Therefore, series (2-1) is absolutely (N, p_n^α) summable almost everywhere.