Chapter 2

Wavelet Theory

Contents

In this chapter, the prerequisites to understand wavelets are described in brief.

2.1 Introduction

Wavelets have been extensively used in data analysis, data compression and image processing. Its multiresolution property lead to the application of wavelet to estimate the differential operator. As described in section 1.1 of chapter 1, numerical solution of a PDE is handled with various algorithms. The recent development in this direction is the use of wavelets in solving PDEs.

As compared to these approaches, wavelets represents surface as a solution of PDE. This approach however offers considerable advantages over alternative basis sets and allows us to attack problems not accessible with conventional numerical methods.

So first we shall briefly mention what is wavelet? And why we are required to study wavelets?

As shown in the figure 2.1, wavelets are small waves which satisfies the following two properties in time domain:

- Wavelet has small concentrated burst of finite energy in the time domain (Wavy).
- It exhibits some oscillations in time. (Little).

Wavelets exhibits rapid oscillatory behaviour in some interval and then decays rapidly to zero outside the interval.

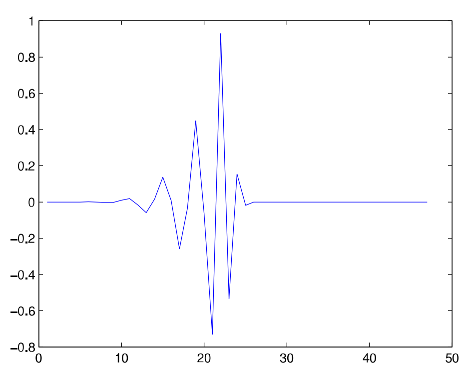

Figure 2.1: Plot of a wavelet (db12)

Basic Wavelet [20]: If $\psi \in L^2(R)$ satisfies the admissibility condition:

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \tag{2.1.1}$$

then ψ is called a basic wavelet or mother wavelet. Here $\hat{\psi}(\omega)$ represents Fourier transform of $\psi(t)$. From the continuity of function $\hat{\psi}(t)$, we say that C_ψ is finite and which implies $\hat{\psi}(0) = 0$ i.e.,

$$\int_{-\infty}^{\infty} \psi(t) dt = 0 \tag{2.1.2}$$

hence ψ is called a wavelet.

This definition of basic wavelet indicates that the following statements are equivalent:

1. $\int_{-\infty}^{\infty} |\psi(t)|^2 dt < \infty$
2. $\int_{-\infty}^{\infty} \psi(t) dt = 0$
3. $\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$

Mother wavelet function ψ is used to generate an orthonormal basis for $L^2(R)$. These basic functions form a double infinite family $\{\psi_{j,k} | j, k \in Z\}$ where,

$$\psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k), \tag{2.1.3}$$

which are translates and dilates of ψ.

A function $\psi \in L^2(R)$ is called an orthonormal wavelet, if the family $\{\psi_{j,k}\}$ forms orthonormal basis of $L^2(R)$ i.e. for $j, k, l, m \in Z$,

$$\langle \psi_{j,k}, \psi_{l,m} \rangle = \delta_{j,l}\delta_{k,m} = \begin{cases} 1 & \text{if } j = l \text{ and } k = m \\ 0 & \text{if } j \neq l \text{ or } k \neq m \end{cases} \tag{2.1.4}$$

2.1.1 Wavelet transform classification

Wavelets are classified [11] as,

- Continuous wavelet transforms (CWT)
- Discrete wavelet transforms (DWT)

Continuous wavelet transforms CWT [21][68]: Morlet gave the concept of wavelet family being constructed by translation and dilation of a single mother wavelet $\psi(t) \in L^2(R)$,

$$\psi_{s,u}(t) = \frac{1}{\sqrt{s}}\psi(\frac{t-u}{s})|_{u \in R, s \in R^+} \tag{2.1.5}$$

where u is the translating parameter, indicating which region we concern and s is the scaling parameter greater than zero because negative scaling is undefined. Conceptually, the continuous wavelet transform is the coefficient of the basis $\psi_{s,u}(t)$. It is defined as,

$$\begin{aligned} Wf(s,u) &= \langle f(t), \psi_{s,u} \rangle & (2.1.6) \\ &= \int_{-\infty}^{\infty} f(t)\bar{\psi}_{s,u}(t)dt & (2.1.7) \end{aligned}$$

Using this transformation, one can map a one-dimensional signal $f(t)$ to a two dimensional coefficients $Wf(s,u)$. This two variables can perform the time frequency analysis. We could locate a particular frequency (s) at a certain time instant (u). If $f(t)$ is a $L^2(R)$ function. The inverse wavelet transform is,

$$f(t) = \frac{1}{C_\psi}\int_0^\infty \int_{-\infty}^\infty Wf(s,u)\frac{1}{\sqrt{s}}\psi(\frac{t-u}{s})du\frac{ds}{s^2} \tag{2.1.8}$$

where C_ψ is defined as in equation 2.1.1.

Discrete time wavelet transform (DTWT) [110]: The DTWT of $f(k) \in l^2(Z)$ is given by

$$DTWTf(m,n) = a_0^{-\frac{m}{2}}\sum_k f(k)\psi(a_0^{-m}k - n\tau_0) \tag{2.1.9}$$

which is time discretization, with $t = kT$ and sampling interval $T = 1$. Here a_0, τ_0 are sampling intervals and m, n are integers. ψ is the basic wavelet.
If $a_0 = 2$ there is an output only at every 2^m sample whenever $2^{-m}k$ is an integer. This leads to the following definition,

Discrete wavelet transform (DWT) [87]: The DWT of $f(k) \in l^2(Z)$ is given by

$$DWTf(m,n) = 2^{-\frac{m}{2}}\sum_k f(k)\psi(2^{-m}k - n) \tag{2.1.10}$$

where the discrete wavelet $\psi(k)$ can be (but not necessarily) a sampled version of a continuous counterpart. When $\psi(k)$ is a discretization of a $\psi(t)$, the DWT is identical to DTWT, with $\psi(t)$ as given in equation 2.1.9.

In real life, f is not given as a function, but in a sample version. One can compute the integrals in equations 2.1.6 and 2.1.7, using quadrature formulas. For a smaller values of s (with specific interest of study), this will not involve many samples of f, and one can do the computations quickly. For larger values of s however one faces huge integrals, which might slow down the computation of the wavelet transform of any given function.

Multiresolution overcomes this computational slowdown. In the next section multiresolution analysis is discussed in brief.

2.1.2 Multiresolution analysis (MRA)

We look for $\psi \in L^2(R)$ where the collection of $\psi_{j,k}(t) = 2^{\frac{j}{2}}\psi(2^j t - k)_{j,k \in Z}$ constitutes an orthogonal basis for $L^2(R)$.

MRA decomposes space $L^2(R)$ into a set of approximate closed subspaces $\{V^j\}_{j \in Z}$ and mutually orthogonal subspaces $\{W^j\}_{j \in Z}$ where,

$$L^2(R) = \bigoplus_{j \in Z} W^j$$

and

$$V^j = \bigoplus_{m=-\infty}^{j-1} W^m$$

and therefore

$$V^{j+1} = \bigoplus_{m=-\infty}^{j} W^m = \bigoplus_{m=-\infty}^{j-1} W^m \bigoplus W^j = V^j \bigoplus W^j$$

Wavelet subspace W^j is orthogonal complement of V^j that lies in V^{j+1}.

Finally the process of MRA decomposes the space $L^2(R)$ into approximate subspaces and mutually orthogonal subspaces which satisfies the following axioms:

- Monotonicity: $\{0\} \subset \cdots \subset V^{-1} \subset V^0 \subset V^1 \subset \cdots \subset L^2(R)$ for all $j \in Z$.
- Dilation property $f(t) \in V^j \Leftrightarrow f(2t) \in V^{j+1}$ for $j \in Z$.
- Intersection property
$$\bigcap_{j \in Z} V^j = \{0\}.$$
- Dense property
$$\overline{\cup_{j \in Z} V^j} = L^2(R).$$
- Existence of a scaling function. There exists a function $\phi \in V^0$ such that $\phi(t-n)$ where $n \in Z$ is an orthogonal basis for V^0

$$V^0 = \{\sum_{n \in Z} \alpha_n \phi(t-n) | \{\alpha_n\}_{n \in Z} \in l^2(Z)\}. \tag{2.1.11}$$

Now using the dilation property of MRA,

$$V^0 \subset V^1$$

so each vector in V^0 belongs to V^1. So

$$\phi(t) \in V^0 \Rightarrow \phi(2t) \in V^1.$$

Hence $\phi(t)$ could be expressed as a linear combination of the basis from V^1 i.e. $\{\phi(2t-n) \quad \text{where} \quad n \in Z\}$.

$\phi(t)$ satisfies the dilation equation [80],

$$\phi(t) = \sum_{n \in Z} c_n \phi(2t-n) \tag{2.1.12}$$

here, $c_n \in l^2(Z), n \in Z$.

The equation 2.1.12 is also referred as two scale difference equation as here $\phi(t)$ is expressed in terms of its own dyadic dilation and translation.

Compactly supported An interval $I \subset R$ is compact if I contains both of its end points: $I = [a, b]$, for some $a, b \in R$. Also, a function $f : R \to C$ has compact support if there is a compact interval $I = [a, b]$ such that $f(x) = 0$ for every $x \notin I$.

Compactly supported wavelets: If the mother wavelet ψ is compactly supported then it is compactly supported wavelet.

$\phi, \quad \psi$ have compact support $\Leftrightarrow$ Finitely many $c_n \neq 0$ where c_n is from equation 2.1.12.

2.1.3 Wavelet families

There are two functions that play a primary role in wavelet analysis, the scaling function $\phi_i(t)$ (father wavelet) and the wavelet function $\psi_i(t)$ (mother wavelet). Here we mention some well known scaling functions and wavelet functions (for some wavelets) [42],

(1) Haar scaling function in $[0, 1)$ is defined as,

$$\phi_i(t) = \begin{cases} 1, & t \in [\alpha, \gamma) \\ 0 & \text{otherwise.} \end{cases}$$

and Haar wavelet function is defined as

$$\psi_i(t) = h_i(t) = \begin{cases} 1 & \text{if } t \in [\alpha, \beta) \\ -1 & \text{if } t \in [\beta, \gamma) \\ 0 & \text{elsewhere.} \end{cases} \tag{2.1.13}$$

here,

$$\alpha = \frac{k}{m} \quad \beta = \frac{k + 0.5}{m} \quad \gamma = \frac{k+1}{m} \tag{2.1.14}$$

here $m = 2^j, j = 0, 1, ..., J$ indicates the levels of wavelet where integer $k = 0, 1, .., (m - 1)$, is the shift parameter. Maximum resolution is J, and $i = m + k + 1$. Incase of minimum value $m = 1$ we get $k = 0, i = 2$. The maximal value of i is $i = 2M = 2J + 1$.

The graphical representation of Haar scaling and wavelet function is given in figure 2.2

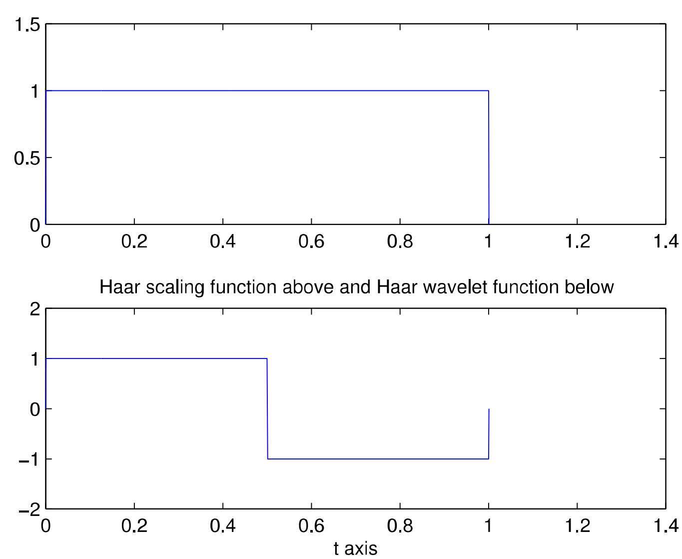

Figure 2.2: Haar scaling and wavelet function

(2) Morlet family has no scaling function. Morlet wavelet (also known as Gabor wavelet) has a basic wavelet representation as,

$$\psi(t) = e^{iw_0 t} e^{-0.5t^2}$$

with $i = \sqrt{-1}, \quad w_0 \geq 5$.

The graphical plot is given in figure 2.3.

Orthogonal and compactly supported wavelets include Daubechies, Symlet

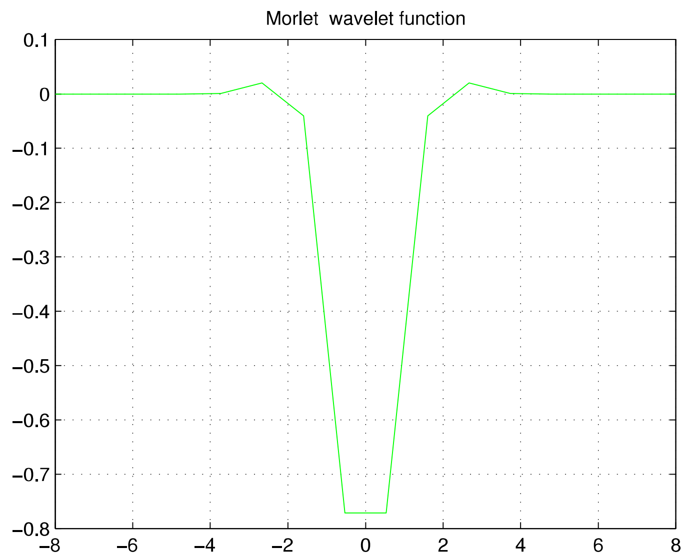

Figure 2.3: Morlet wavelet

and Coiflet. Their scaling and wavelet functions are compactly supported.

(3) Symlet wavelet families have wavelets ranging from symlet 2 to 20. In this study we have utilized symlet 4 with properties of near symmetric, orthogonal and biorthogonal. Symlet wavelet scaling and wavelet plots are given in figure 2.4.

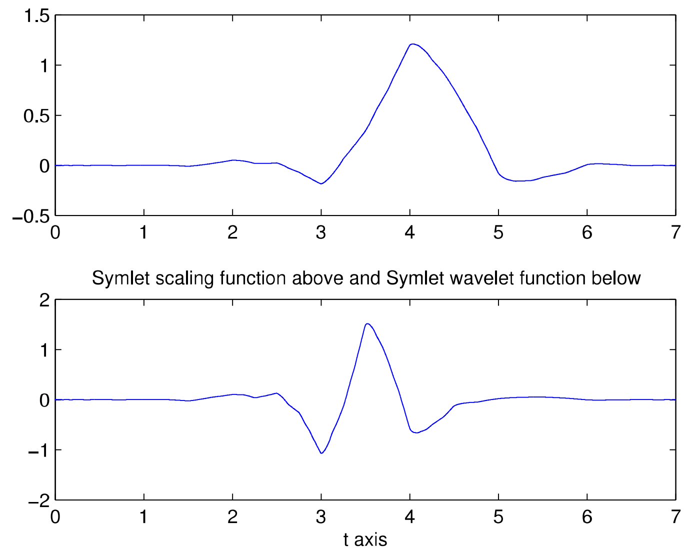

Figure 2.4: Plot of scaling and wavelet function for Symlet wavelet

(4) Coiflet family consist of range Coiflets 1 to Coiflets 5. It has properties like near symmetric, orthogonal and biorthogonal. The figure 2.5 gives the graph of scaling and wavelet functions utilized in this study.

(5) Harmonic scaling function and wavelet function proposed by Newland [73] in 1993 is defined as,

$$\phi(t) = \frac{e^{i2\pi t} - 1}{i2\pi t}, \quad \psi(t) = \frac{1}{i2\pi t}(e^{i4\pi t} - e^{i2\pi t}).$$

The graphical plot for wavelet function [88] within $(-1, 1)$ with 100 points is given in figure 2.6.

(6) Shannon wavelet families have scaling and wavelet functions which is given by,

$$\phi(t) = sinc(\pi t) = \frac{\sin(\pi t)}{\pi t}, \quad \psi(t) = sinc(\frac{t}{2}) \cos(\frac{3\pi t}{2})$$

.

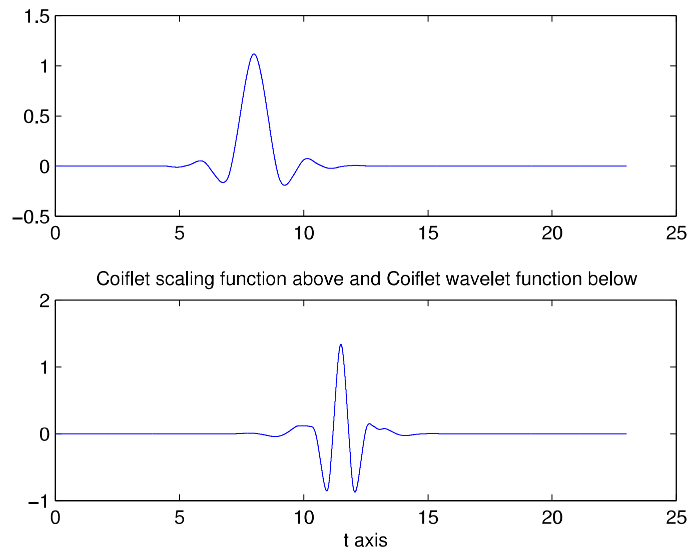

Figure 2.5: Plots of Coiflet scaling and wavelet function

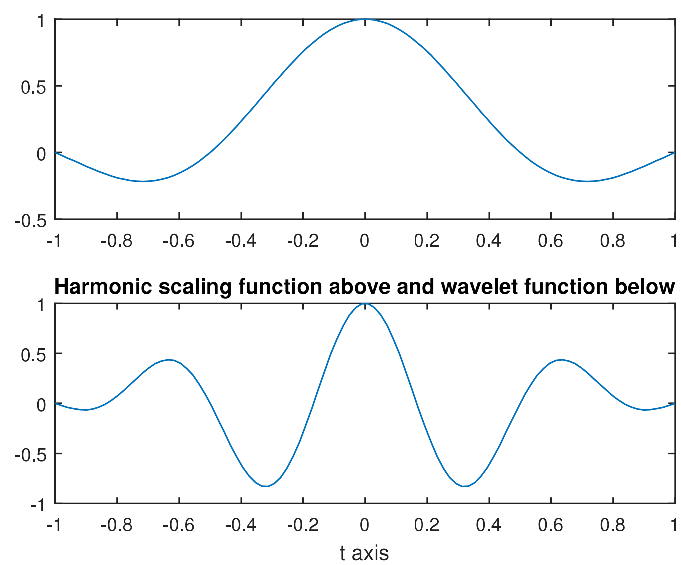

Figure 2.6: Plots of Harmonic scaling and wavelet function

The graphical plot is given in figure 2.7

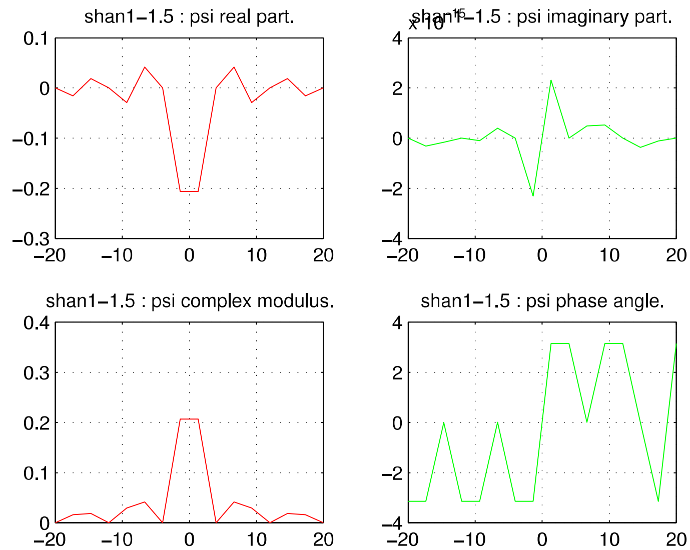

Figure 2.7: Shannon wavelet

(7) Legendre multiwavelets utilize the two scale functions in the construction of their family of wavelets which is given by scaling functions,

$$\phi^0(t) = 1 \quad \phi^1(t) = \sqrt{3}(2t-1) \quad 0 \le t < 1$$

The corresponding wavelet is given as,

$$\psi^0(t) = \begin{cases} -\sqrt{3}(4t-1), & 0 \le t \le \frac{1}{2} \\ \sqrt{3}(4t-3), & \frac{1}{2} \le t < 1. \end{cases}$$

$$\psi^1(t) = \begin{cases} 6t-1, & 0 \le t \le \frac{1}{2} \\ 6t-5, & \frac{1}{2} \le t < 1. \end{cases}$$

Gaussian wavelets, Morlet and Mexican hat wavelet families do not have scaling function. Analysis is not orthogonal and their wavelet function is not compactly supported.

(8) The second derivative of a Gaussian is denoted as,

$$\psi(t) = (1 - t^2)e^{-\frac{t^2}{2}}$$

The graphical representation is given in figure 2.8

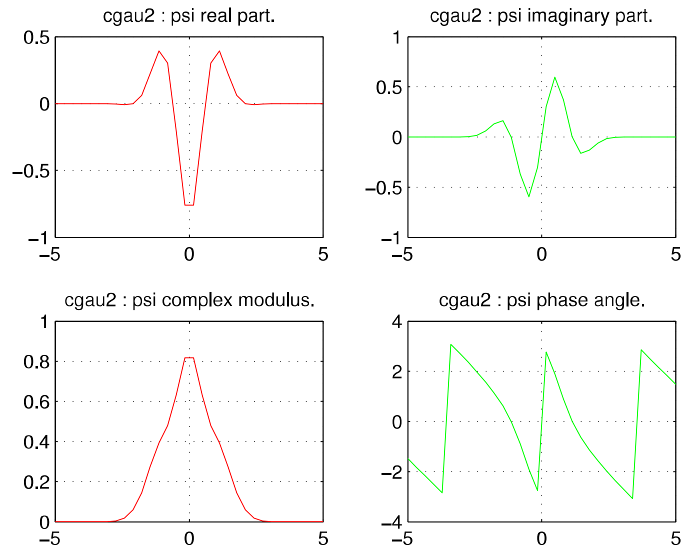

Figure 2.8: Complex gaussian wavelet

(9) Mexicanhat wavelet has no scaling function and its wavelet function is derived from a function that is proportional to the second derivative function of the Gaussian probability density function. It is also known as the Ricker wavelet. Mexican hat wavelet is given by,

$$\psi(t) = (1 - 2t^2)e^{-t^2}$$

Graph is represented by figure 2.9

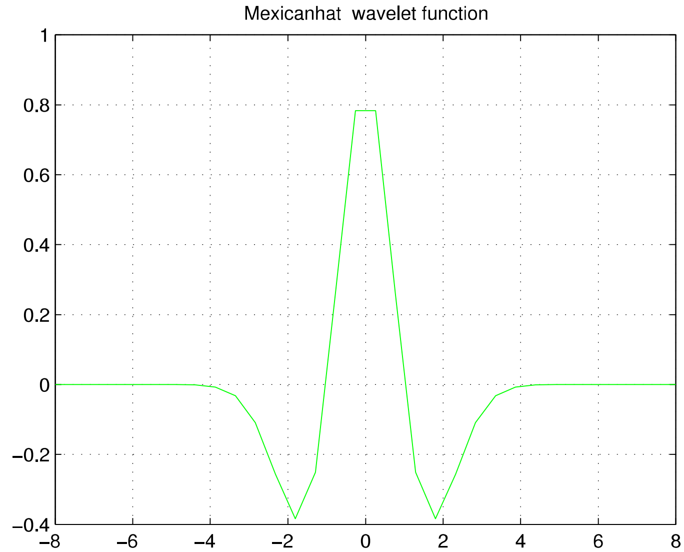

Figure 2.9: Mexican hat wavelet

Few of the wavelet families do not have an explicit expression. Wavelet families include a wide variety of choices, graphical representations are specified in the following figures:

Daubechies family of wavelets do not have an explicit expression except for db1 which is Haar wavelet.

Meyer wavelet is infinitely regular wavelet. Their scaling and wavelet functions are indefinitely derivable. They do not have compact support.

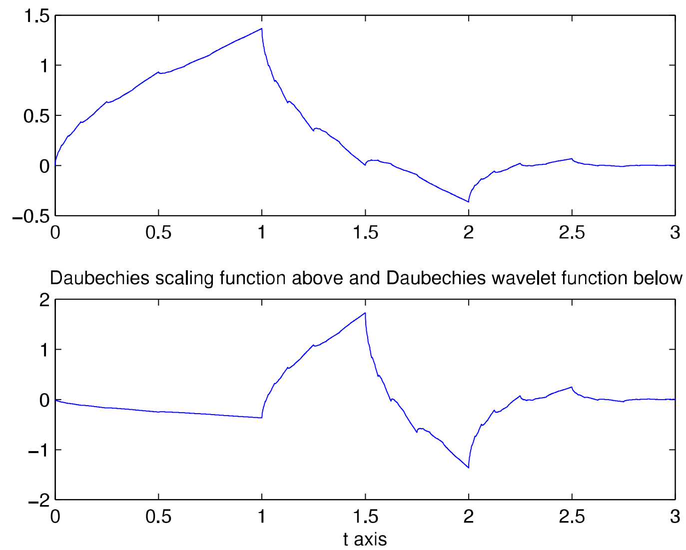

Figure 2.10: Plot of Daubechies scaling function and wavelet function

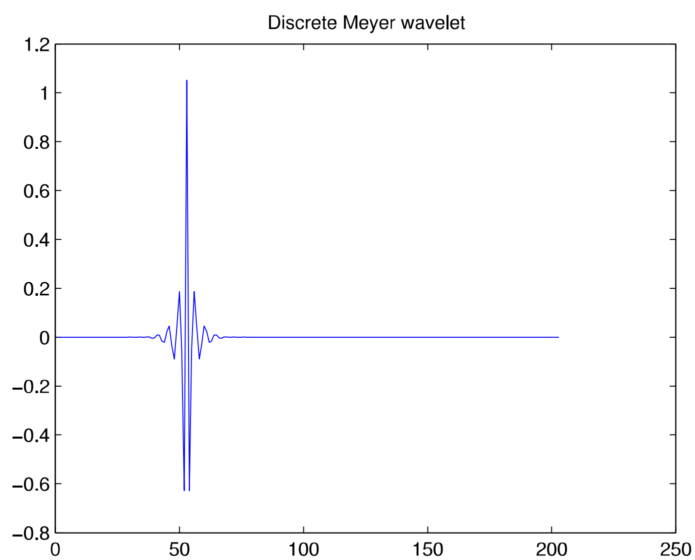

Figure 2.11: Discrete Meyer wavelet

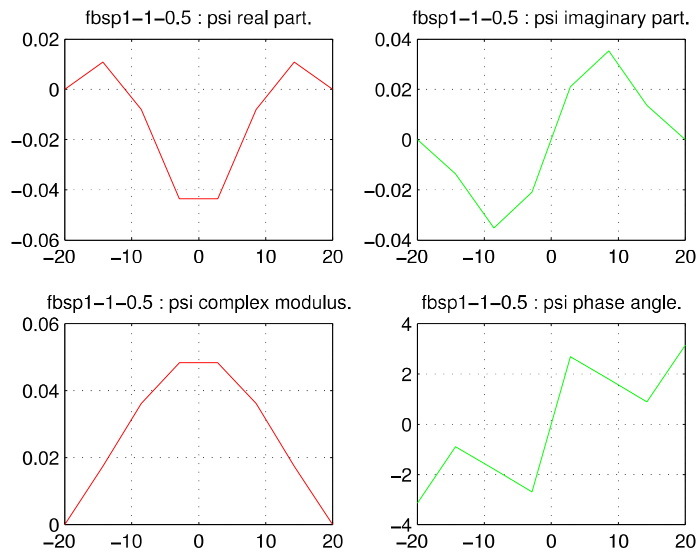

Figure 2.12: B-Spline (fbsp) wavelet

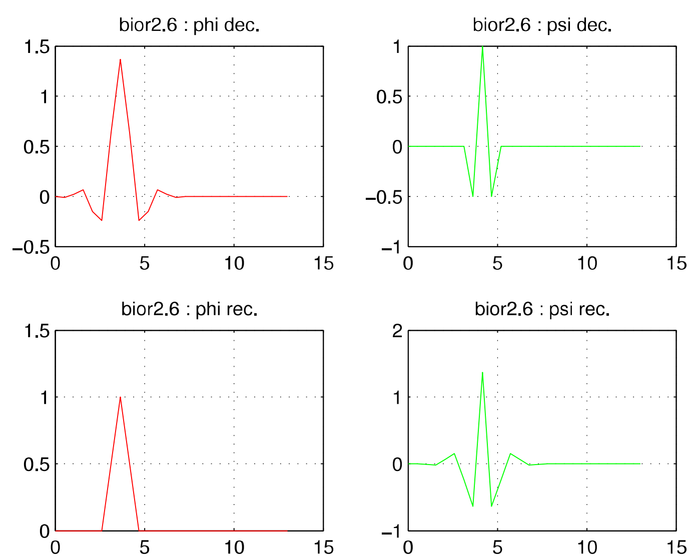

Figure 2.13: Biorthogonal spline wavelet

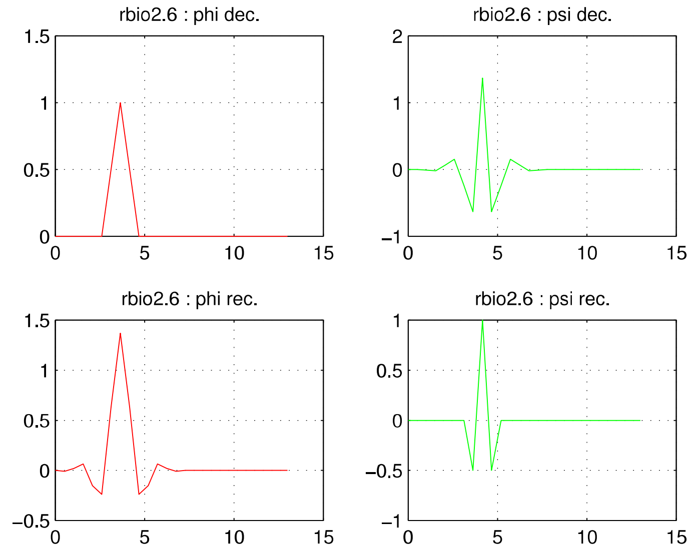

Figure 2.14: Reverse biorthogonal wavelet

Wavelet analysis represents a time scale view of the input function.

The salient features of wavelet includes

- Localization in both time and scale.
- The position of the wavelet allows to locate the location of event in time.
- The shape of the wavelet allows to incorporate the details or resolutions.
- It can easily adapt to discrete discontinuous function representation.

Now we discuss the orthogonal wavelets on real line for Daubechies wavelet family. Here we have also mentioned decomposition of function using single scale and also with a multiscale using Haar wavelet which is $db1$. A similar projection is utilized later in chapter 5 and chapter 6 in the discussion of combined proposed algorithm of wavelet based finite volume method.

2.2 Orthogonal wavelets on real line

A wavelet decomposition involves two families of functions the scaling functions and the wavelets. The two sets of functions are linked together to perform the multiresolution analysis. The advantages for using wavelet as basis are

1. Different resolutions can be used in different regions of space.
2. The coupling between different resolution levels is easy.
3. The numerical effort scales linearly with respect to computational efforts.

The wavelet function is defined in terms of the scaling function for $x \in R$ as,

$$\psi(x) = \sqrt{2} \sum_{m=-\infty}^{\infty} g_m \varphi(2x - m) \tag{2.2.1}$$

where g_m is to be determined. Example let,

$$\varphi(k) = \begin{cases} 1 & k \in [0, 1) \\ 0 & otherwise. \end{cases}$$

where φ will satisfy the two scale relation equation 2.1.12 on real axis.
We can build an orthonormal basis for Hilbert space $L^2(R)$ of square integrable functions φ and ψ by dilating and translating them to obtain basis functions:

$$\varphi_{j,k}(x) = 2^{j/2} \varphi(2^j x - k) \tag{2.2.2}$$

For a function $f \in L^2(R)$, there exit a sequence $\{d_{jk}\}$ such that

$$\begin{aligned} f(x) &= \sum_{j\in Z}\sum_{k\in Z} d_{jk}\psi_{j,k}(x) \\ where \quad d_{jk} &= \int_{-\infty}^{\infty} f(x)\overline{\psi_{j,k}}(x)dx \end{aligned} \tag{2.2.3}$$

In this study we have utilized wavelet families namely Haar wavelet, Daubechies wavelet, Coiflet wavelet and Symlet wavelets. The Daubechies wavelet family consist of db 1 to 20. db1 represents Haar wavelet as in figure 2.2. For this study we have utilized db 2 and db 4.

In particular, the MRA of $L^2[-1,1]$ is taken which is an increasing sequence of mutually orthogonal closed linear subspaces [33].

Such an infinite nested sequence of subspaces is given by,

$$\{V_k^0 \subset V_k^1 \subset .. \subset V_k^n \subset ... \subset L^2[-1,1]\} \tag{2.2.4}$$

where,

$$V_k^n = \{f : f is \text{ a polynomials of degree} \leq k\} \tag{2.2.5}$$

where f has support in the interval of $(-1 + 2^{(-n+1)}j, -1 + 2^{(-n+1)}(j+1))$, for $j = 0, 1, ..., 2^n - 1$. In particular, this property is valid for a function space,

$V_k^0 = \{$ f : f is a polynomial of degree $\leq k$ with support on $[-1,1]\}$ with dimension $k+1$ that can be spanned by a scaling basis ϕ as in figure 2.10.

From the father basis ϕ , it is possible to span any sub-space V_k^n via dilation and translation [60], as

$$\phi_j^n(x) = 2^{(n/2)}\phi(2^n(x+1) - 2j - 1) \tag{2.2.6}$$

with $n = 0, 1, 2, ..N, j = 0, 1, ...2^n - 1$, where n is dilation index and j is translation index.

The wavelet sub-spaces $W_k^n \ \ (n \geq 0)$ is the orthogonal complement of V_k^n in V_k^{n+1} and they satisfy the conditions:

$$V_k^{n+1} = V_k^n \oplus W_k^n, \qquad V_k^n \perp W_k^n \tag{2.2.7}$$

Taking Daubechies wavelet ψ as the mother wavelet given in figure 2.10, it spans the space W_k^0 and any subspace W_k^n can be spanned by it translation and dilation as,

$$\psi_j^n(x) = 2^{(n/2)}\psi(2^n(x+1) - 2j - 1). \tag{2.2.8}$$

Using equation 2.2.3 any function can be linearly expressed in single scale decomposition as an orthogonal projection in V_p^n with respect to the bases $\phi_{l,j}^n$ [107] as,

$$P_p^n f(x) = \sum_{j=0}^{2^n-1} \sum_{l=0}^{p} s_{l,j}^n \phi_{l,j}^n \tag{2.2.9}$$

p is the order of legendre polynomial used in the generation of scaling space, with $j = 0, 1, ..., 2^n - 1$ as the resolution.

The legendre multiscaling bases $\phi_{l,j}^n$ are obtained by dilation and translation in the interval $[-1, 1]$, followed by $L^2[-1, 1]$ normalization [107]. $\phi_{l,j}$ is the scaling function and the scaling coefficient is given by,

$$s_{l,j}^n(i) = \langle f, \phi_{l,j} \rangle \tag{2.2.10}$$

which is single scale decomposition.

Now using the fact,

$$V_p^N = V_p^0 + W_p^0 + W_p^1 + ... + W_p^{N-1} \tag{2.2.11}$$

The multi scale decomposition could be given as,

$$P_p^N f(x) = \sum_{l=0}^{p} s_{l,0}^0 \phi_{l,0}^0 + \sum_{n=0}^{N-1} \sum_{j=0}^{2^n-1} \sum_{l=0}^{p} d_{l,j}^n \psi_{l,j}^n(x) \tag{2.2.12}$$

with $\psi_{l,j}$ as the wavelet function and its corresponding detail coefficient as $d_{l,j}^n = \langle f, \psi_{l,j} \rangle$. The space spanned by polynomial of degree zero for example is the Haar wavelet family which is utilized as basis [28]. The basis functions are selected from the above wavelet families to obtain different scaling ϕ and wavelet ψ functions. Here we illustrate for Haar wavelet family the construction of scaling functions. We denote the scaling space V_k^n as a space of piecewise polynomial functions as, in equation 2.2.5.

It satisfies all the conditions of MRA as the scaling function choosen ϕ are selected to be orthogonal.

Example for wavelet decomposition and reconstruction of a function

To observe the effect of change in resolution, projection of a function $f(x) = \sin(2\pi x)$ given in figure 2.15 is studied. Projection formula of equation 2.2.9 is implemented to obtain $V(0, 2)$, which is projection of function f onto the space V_0^2. The plot of

$V(0,2)$ is given in figure 2.16.

Now using equation 2.2.12, we obtain equation 2.2.13,

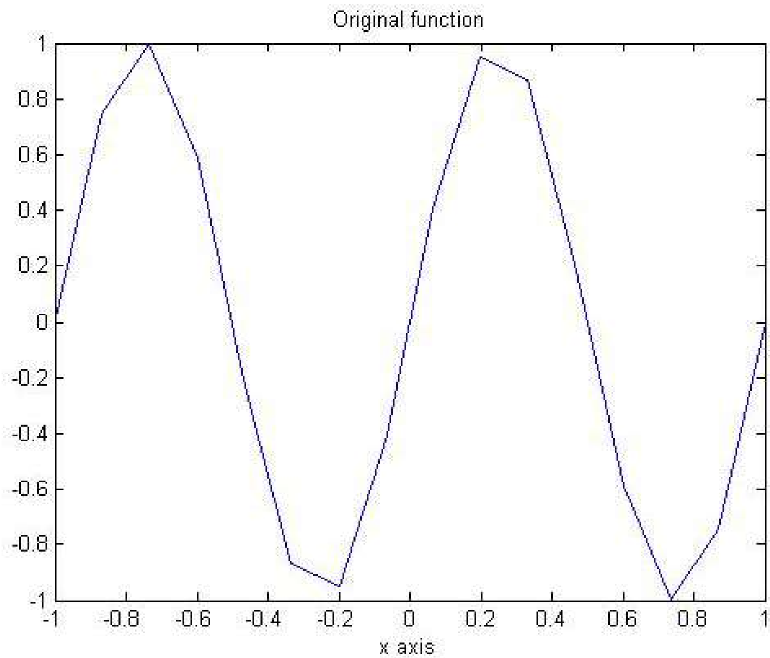

Figure 2.15: Original function $\sin(2\pi x)$

$$P_p^N f(x) = \sum_{l=0}^{2} s_{l,0}^0 \phi_{l,0}^0 + \sum_{n=0}^{N-1} \sum_{j=0}^{2^n-1} \sum_{l=0}^{2} d_{l,j}^n \psi_{l,j}^n(x). \tag{2.2.13}$$

$V(2,2)$ which is projection of f onto the space V_2^2 is obtained by implementing equation 2.2.13. The plot of $V(2,2)$ is given in figure 2.17, which denotes the representation of projection of original function onto space V_2^2.

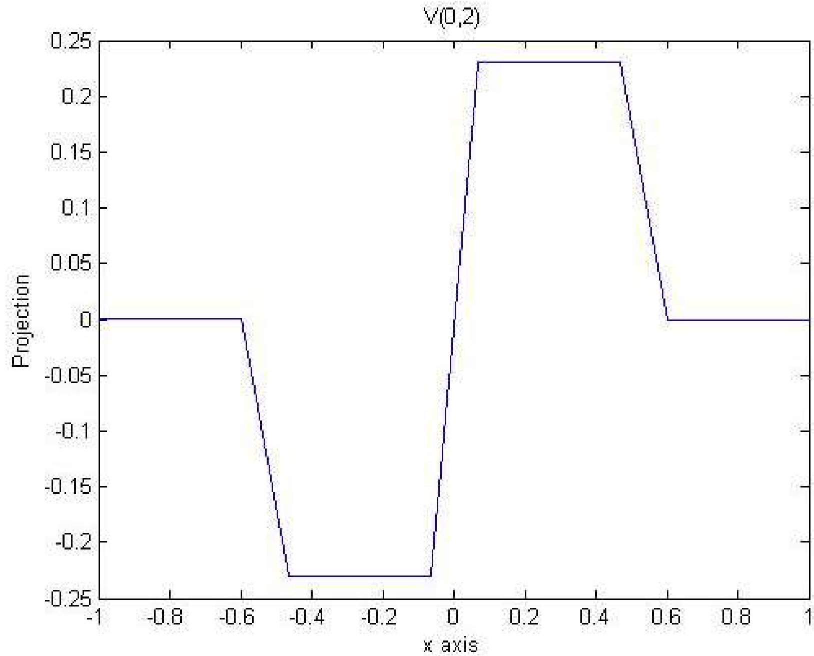

Figure 2.16: Projection of $\sin(2\pi x)$ onto V_0^2

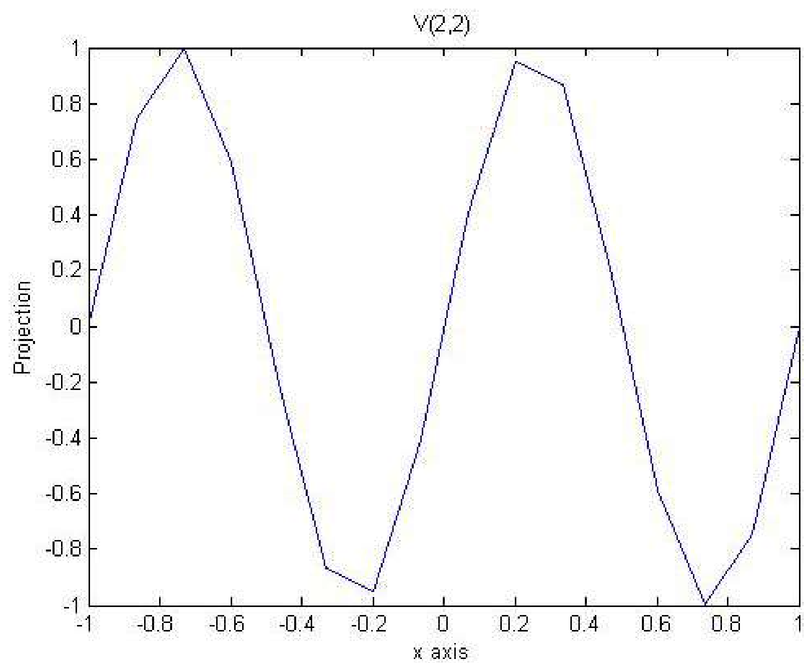

Figure 2.17: Projection of $\sin(2\pi x)$ onto V_2^2

2.3 Literature survey Current status

Over the last two decades wavelets are being effectively used for signal processing, fingerprint verification [68], Storing fingerprint electronically using wavelet, denoising data, musical tones, etc [70] and solution of differential equations [12], [40], [76], [67].

Wavelets have several properties which are encouraging their use for numerical solutions of PDEs [85]. The Wavelet-Galerkin method [12], [43] is a powerful tool for solving partial differential equations. The orthogonal, compactly supported wavelet basis of Daubechies exactly approximates polynomial of increasingly higher order. These wavelet bases can provide an accurate and stable representation of differential operations even in region of strong gradients or oscillations. In addition, the orthogonal wavelet bases have the inherent advantage of multi resolution analysis over the traditional methods [67]. The adaptive wavelet collocation method is able to dynamically track the evolution of the solution's irregular features and to allocate higher grid density to the necessary regions. Therefore, the number of collocation points needed is optimized, without damaging the accuracy of the solution [76]. Haar wavelet is also used in solving PDEs. The benefit of Haar wavelet approach are their sparse matrices representation, fast transformation and possibility of implementation of fast algorithms [70]. PDE that encounters either singularities or steep changes require non-uniform time spatial grids or moving element .Wavelet analysis is an efficient method for solving such PDE. Work has been done using the wavelet transform which can track the position of a moving steep front and increase the local resolution of the grid by adding higher resolution wavelets. In the smoother region, a lower resolution can be used. The wavelet transform is used in signal analysis, e.g. for compression, denoising and feature extraction. For control applications wavelets are used in motion tracking, robot positioning, identification and both linear and nonlinear control purposes. Wavelets are also a powerful tool for the analysis and adjustment of audio signals [70].

Wavelet analysis implemented to problems in physics as discussed in book by Fang and Thews [22]. The reason for use of wavelets in solving PDE lies in the fact that, when the solution have intermittency both in space and time, a very fine resolution and small time steps are necessary to capture small scale structures of the solution. Wavelet methods are capable to capture small scale structure of the solution while the large scale structures are computed automatically with a coarser resolution.

Following are the important advantages of wavelet considerations [40] :

- The basis set can be improved in a systematic way.
- Different resolutions can be used in different regions of space.
- The coupling between different resolution levels is easy.
- There are few topological constraints for increased resolution regions.
- The regions of increased resolution can be chosen, the only requirement being that a region of higher resolution be contained in a region of the next lower resolution. If one uses for instance generalized plane waves in connection with curvilinear coordinates to obtain varying resolution one has the requirement that the varying resolution grid can be obtained by a mapping from a equally spaced grid.
- The numerical effort scales linearly with respect to system size.

Our approach is to study the complexities involved in numerical algorithms of wavelet based methods, their implementation and improvements. We start with the application of Haar wavelet to solve ordinary differential equation in the next chapter.