Chapter 3

Wavelet approaches with finite difference framework

Contents

3.1 Introduction

The chapter discusses approximation of solution for ordinary differential equation with wavelets. It presents a unified approach for wavelet collocation method, applied to solve both the initial value problems as well as boundary value problems. The method was found to give good agreement with the analytical solutions. We have directly solved the boundary value problem as against the traditional shooting methods as given in section 1.2.3, where the boundary value problem itself is approximated by the initial value problem. This motivates us to implement wavelet collocation methods [103], [105] as discussed in section 3.2.

3.2 Wavelet approximation for ODE

Haar wavelets were employed in solution of differential equations by various researchers. Chen and Hsiao [18] were first to derive the operational matrix for integrals of Haar wavelet. Siraj ul Islam, Imran Aziz, Fazal Hak [94] used Haar wavelet and hybrid functions in numerical integration. Linear stiff systems were solved using the Haar wavelets by Hsiao [17]. Nonlinear stiff system was solved using wavelets by Wang [16]. Lepik [103] applied Haar wavelet in solving differential equations. Lepik and Tamme [104] used Haar wavelets for solving differential equations and integral equations. Glabisz solved boundary valued problems using Walsh wavelet [109]. We propose a unified approach to the solution of both initial value and boundary value problems by detailing the algorithms proposed by Lepik [103], Siraj [94] and Mishra

[105]. Here the extension of region of solution is formulated. We give examples with plots to illustrate the implementation of the unified approach and its generalization.

Chen and Hsiao [18] had given the approach of integral of the basic vector $\phi(t)$ as

$$\int_0^t \phi(\tau)d\tau \cong P\phi(t)$$

with $\phi(t) = [\phi_0(t), \phi_1(t), ..., \phi_{m-1}(t)]^T$ with elements $\phi_0(t), \phi_1(t), ..., \phi_{m-1}(t)$ being the basic functions orthogonal on certain interval $[a, b]$ where a and b could be 0 and 1. The simplest Haar wavelet functions are utilised in the system analysis. The integrals are expanded in terms of Haar series.
Considering the integral of the first four Haar wavelets,

$$\int_0^t H_4(\tau)d\tau = P_4H_4(t).$$

In particular for $m = 2^j$ with $j \in Z^+$, P_m is given for an m^{th} order system as

$$P_m = \frac{1}{2m}\begin{pmatrix} 2mP_{m/2} & -H_{m/2} \\ H^{-1}_{m/2} & 0 \end{pmatrix} \quad with \quad P_m = [\int_0^t H_m(\tau)d\tau]H_m^{-1} \tag{3.2.1}$$

and $H_m(t) = \begin{pmatrix} h_0(t) \\ h_1(t) \\ . \\ . \\ h_{m-1}(t) \end{pmatrix}$. Hsiao and Chen solved a lumped parameter linear system with n states $x(t)$, p inputs $u(t)$ and q outputs $y(t)$ which is described by the state equation given as,

$$x(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \qquad and$$

$$y(t) = Cx(t) + Du(t), \quad 0 \leq t < 1$$

where $u(t)$ is expressed as a Haar series. We utilize a similar integral form in terms of Haar series in the algorithm proposed.

Lepik [103] has also utilized the integral form and discussed the solution of evolution equation on collocation points,

$$x_l = \frac{l - 0.5}{2M}$$

where $l = 1, 2, ..., 2M$.

Mishra [105] solved the initial value problem

$$y''(x) + y(x) = u(x), \quad y(0) = y'(0) = 0,$$

where $x \in [0, 1]$, where $y^n(x)$ is approximated using Haar series as,

$$y^n(x) = \sum_{i=1}^{2M} a_i h_i(x) \quad with \quad A \leq x \leq B.$$

For $\alpha < n$,

$$y^n(x) = \sum_{i=1}^{2M} a_i P_{n-\alpha,i}(x) + \sum_{\sigma=1}^{n-\alpha-1} \frac{1}{\sigma!}(x - A)^\sigma y_0^{(\alpha+\sigma)}.$$

where $P_n(x)$ is given by equation 3.2.1. He had implemented the procedure of solving initial value ODE which consist of substituting various obtained derivatives to the given ODE in Haar series form. Then calculation of $a_i's$ are performed which leads to the numerical solution. Mishra restricted the algorithms to the domain with interval $[0, 1]$.

The procedure suggested by Mishra [105] and Lepik [103] are combined to formulate a unified approach [56], as discussed in this chapter article 3.2.1.

3.2.1 Unified collocation approaches for solving IVP and BVP problems

We brief the collocation method used for solving initial value problem and boundary value problem with simple examples to establish a common unified approach of approximating derivative using wavelet function as basis. As discussed in section 3.2, in this method higher order derivative is approximated using wavelet function and the lower order derivatives and functions itself are expressed by repeated integration. The orthogonal set of Haar wavelet function is used. This group of square waves has magnitude in some interval and zero elsewhere. These zeros make Haar transform faster than other square functions such as Walsh functions. Haar wavelet basis lacks differentiability and hence here integration approach is used instead of the differentiability for calculation of coefficients. Due to the local property of the powerful Haar wavelet the new method is simpler.
The Haar wavelet family for, $x \in [0, 1)$ which is defined by equation 2.1.13 is utilized. Using equation 2.1.13, the functions h_1 and h_2 is given by,

$$h_1(x) = \begin{cases} 1 & x \in [0,\ 1) \\ 0 & elsewhere. \end{cases} \tag{3.2.2}$$

$$h_2(x) = \begin{cases} 1 & \text{if } x \in (0, 0.5) \\ -1 & \text{if } x \in [0.5, 1) \\ 0 & \text{elsewhere.} \end{cases} \tag{3.2.3}$$

h_2 can be graphically vizualized as figure 3.1. In order to perform wavelet transform,

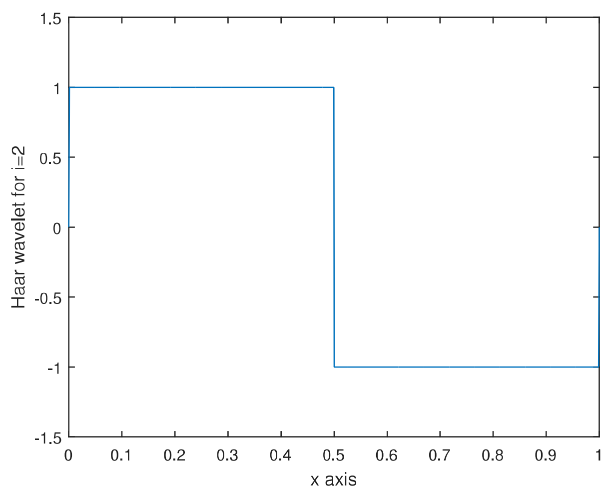

Figure 3.1: Haar wavelet for specific interval

Haar wavelet uses translations and dilations of the function, i.e. the transformation uses the following relation:

$$h_{j,k}(x) = 2^{j/2} h(2^j x - k), \quad k = 0, 1, .., (m-1), \quad j = 0, 1, ..., J. \tag{3.2.4}$$

We can obtain coefficient matrix H of order $2m \times 2m$ as,

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

with $m = 2$.

Haar wavelets being orthogonal leads to,

$$\int_0^1 h_i(x)h_l(x)dx = \begin{cases} \frac{1}{m} & \text{for} \quad i = l \\ 0 & \text{for} \quad i \neq l \end{cases}$$

The operational matrix p which is a $2m$ square matrix is define as in [105] by

$$p_{i,1}(x) = \int_0^x h_i(x')dx' \tag{3.2.5}$$

and the recurrence relation is given by

$$p_{i,v+1}(x) = \int_0^x p_{i,v}(x')dx' \quad where\ v = 1,\ 2,\ \ ... \tag{3.2.6}$$

We will need the integral

$$p(x) = \underline{\int_A^x \int_A^x ... \int_A^x h_i(t)dt^u} = \frac{1}{(u-1)!} \int_A^x (x-t)^{u-1} h_i(t)dt \tag{3.2.7}$$

with $u = 2, 3, ..., n$ and $i = 1, 2, ..., 2m$.

The above integrals can be evaluated using equation 3.2.6 and 3.2.5, the first two values are given by

$$p_{i,1}(x) = \begin{cases} x - \alpha & \text{for } x \in [\alpha, \beta) \\ \gamma - x & \text{for } x \in [\beta, \gamma) \\ 0 & \text{elsewhere.} \end{cases} \tag{3.2.8}$$

$$p_{i,2}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2, & \text{for } x \in [\alpha, \beta) \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2, & \text{for } x \in [\beta, \gamma) \\ \frac{1}{4m^2}, & \text{for } x \in [\gamma, 1) \\ 0 & \text{elsewhere.} \end{cases} \quad (3.2.9)$$

and so on will be utilized in representing the derivative and function values in the further discussion. Graphically the successive integrals of Haar wavelet function is given in figure 3.2.

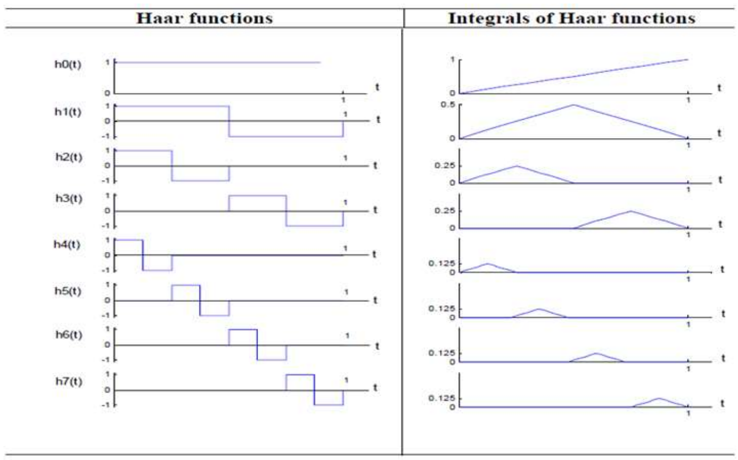

Figure 3.2: Integral representation of Haar wavelets

3.3 Haar approximation

Considering the fact that Haar wavelets are orthogonal, we may take any function $f(x)$ which is square integrable in the interval $[0, 1)$ as an infinite sum of Haar wavelets,

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x) \quad (3.3.1)$$

where a_i are Haar coefficients and $h_i(x)$ are Haar wavelet functions.
$f(x)$ has finite terms and so $f(x)$ is piecewise constant or can be approximated as piecewise constant during each subinterval [105] as,

$$f(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{3.3.2}$$

In this scheme the highest order derivative of the function is approximated by Haar wavelets and the consecutive lower order derivatives and function itself is obtained by repeated integration as explained in next section.

3.4 Initial Value Problem IVP

3.4.1 Algorithm-IVP

Consider the general n^{th} order linear differential equation

$$N_1 y^n(x) + N_2 y^{(n-1)}(x) + ... + N_n y(x) = f \tag{3.4.1}$$

for $x \in [A, B]$ with initial condition,

$$y^{(n-1)}(A) = Y_{n-1}, y^{(n-2)}(A) = Y_{n-2}, ..., y(A) = Y_0 \tag{3.4.2}$$

Now taking the r^{th} order derivative of y as,

$$y^r(x) = \sum_{i=1}^{2M} a_i p_{i,n-r}(x) + \sum_{\sigma=0}^{n-r-1} \frac{1}{\sigma!}(x-A)^{\sigma} y_0^{(r+\sigma)} \tag{3.4.3}$$

we obtain $y^{(n-1)}(x), y^{(n-2)}(x), ...\quad and \quad y(x)$ at the collocation points,

$$x_p = \frac{(p - \frac{1}{2})}{2M}, \quad p = 1,\ 2,\ \dots, 2M \tag{3.4.4}$$

The expressions of $y^n(x)$, $y^{(n-1)}(x)$, and $y(x)$ are substituted in differential equation. Discretization is applied along the points given by equation 3.4.4 resulting in a linear or non linear system of $2M \times 2M$. Solving the system for Haar coefficients the approximate solution is achieved.

3.5 Boundary value problems BVP

Consider a second order boundary valued problem

$$y''(x) = f(x, y, y') \tag{3.5.1}$$

for $x \in [0, 1]$. For such second order ordinary differential equations, there are four different types of boundary conditions possible.

They are treated differently as follows:

Case 1 $y(0) = R$ and $y(1) = Q$,

by integrating equation (3.5.1) between 0 $\quad to \quad x$ yields,

$$y'(x) = \sum_{i=1}^{2M} a_i p_{i,1}(x) + y'(0)$$

as $p_{i,1}(0) = 0$. Integrating again and using condition $y(0) = R$ we get,

$$y(x) = R + y'(0)x + \sum_{i=1}^{2M} a_i p_{i,2}(x) \tag{3.5.2}$$

Now utilizing second condition $y(1) = Q$ we obtain

$$y'(0) = (Q - R) - \sum_{i=1}^{2M} a_i c_{i1}$$

with $c_{i1} = \int_0^1 p_{i,1}(x)dx$ further simplifying we get,

$$y(x) = R + (Q - R)x + \sum_{i=1}^{2M} a_i(p_{i,2}(x) - xc_{i1}) \tag{3.5.3}$$

and

$$y'(x) = Q - R + \sum_{i=1}^{2M} a_i(p_{i,1}(x) - c_{i1}) \tag{3.5.4}$$

Case 2 $y'(0) = R_1$ and $y(1) = Q_1$

Integrating equation 3.5.1 and using boundary condition $y'(0) = R_1$ we get

$$y'(x) = R_1 + \sum_{i=1}^{2M} a_i p_{i,1}(x) \tag{3.5.5}$$

$$y(x) = Q_1 - R_1(1 - x) - \sum_{i=1}^{2M} a_i(c_{i1} - p_{i,2}(x)) \tag{3.5.6}$$

Case 3 $y(0) = R_2$ and $y'(1) = Q_2$

where successive integration leads to,

$$y'(x) = Q_2 - a_1 + \sum_{i=1}^{2M} a_i p_{i,1}(x) \tag{3.5.7}$$

and

$$y(x) = R_2 + (Q_2 - a_1)x + \sum_{i=1}^{2M} a_i(p_{i,2}(x)) \tag{3.5.8}$$

Case 4 $y'(0) = R_3$ and $y'(1) = Q_3$

where by integrating and applying the first condition we obtain,

$$\begin{aligned} &y'(x) = R_3 + \sum_{i=1}^{2M} a_i p_{i,1}(x) \quad using\ y'(1), \\ &(Q_3 - R_3) = a_1 \quad as \quad p_{i,1}(1) = 1 \end{aligned} \tag{3.5.9}$$

so we get

$$y''(x) = (Q_3 - R_3)h_1(x) + \sum_{i=2}^{2M} a_i h_i(x) \tag{3.5.10}$$

$$y'(x) = R_3 + (Q_3 - R_3)p_{11}(x) + \sum_{i=2}^{2M} a_i p_{i1}(x) \tag{3.5.11}$$

$$y(x) = y(0) + R_3 x + (Q_3 - R_3)p_{12}(x) + \sum_{i=2}^{2M} a_i p_{i2}(x) \tag{3.5.12}$$

which is obtained by equation 3.5.9, by integrating from 0 to x. We have extended the approach to second order differential equations with boundary conditions.

3.5.1 Algorithm-BVP

i The highest order derivative is approximated by Haar wavelet function.

ii The successive lower order derivatives and the function itself is replaced by the expressions obtained by repeated integration obtained in (i)

iii The algebraic expression in terms of Haar coefficients is represented in matrix form.

iv The matrix is solved to obtain the Haar coefficients a_i which are then substituted in the expression of solution function.

Separate MATLAB routines are generated for computation of the matrix P and C which appear in the algebraic representation. P represents the matrix formed by $p_{i,k}$ and C represents the matrix formed by required $c'_{i,k}s$, where k depends on the order of the equation handled. Now to get a clear idea of the methods we give examples, one each for second order initial value problem and second order boundary value problem in the next sections.

3.6 Numerical examples

Example 1 Consider an initial value ordinary differential equation

$$y'' + y = \sin(x) + x\cos(x) \tag{3.6.1}$$

with $x \in [0, 1]$, $y(0) = 1$ and $y'(0) = 1$.
The analytic solution is given by

$$y(x) = \cos x + \frac{5}{4}\sin x + \frac{1}{4}(x^2 \sin x - x\cos x)$$

Wavelet formulation is obtained by substituting

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{3.6.2}$$

and integrating twice equation 3.6.2 with respect to x we get,

$$y(x) = \sum_{i=1}^{2M} a_i p_{i2}(x) + 1 + x \tag{3.6.3}$$

The differential equation 3.6.1 gets converted into,

$$\sum_{i=1}^{2M} a_i(h_i(x) + p_{i,2}(x)) = \sin x + x\cos x - 1 - x.$$

Solving the system $A[H+P] = B$, we obtain the wavelet coefficient matrix A, where H is the Haar matrix, P is the matrix consisting with rows $p_{i,k}$'s and B is right hand side vector obtained by considering values of x at collocation points. From the expression of $y(x)$ given in equation 3.6.3, substituting the wavelet coefficients we obtain the solution.

For $2M = 16, j = 3$ we obtain $m = 16$. Results are compared with its analytical solution using matlab program and the plot is given in figure 3.3. Now figure 3.4 represents the graph of the Haar, analytical and inbuilt function implementation of matlab results for the initial valued problem (example 1).

Example 2 Consider a second order boundary value problem as

$$y'' = y' + y + e^x(1 - 2x), \tag{3.6.4}$$

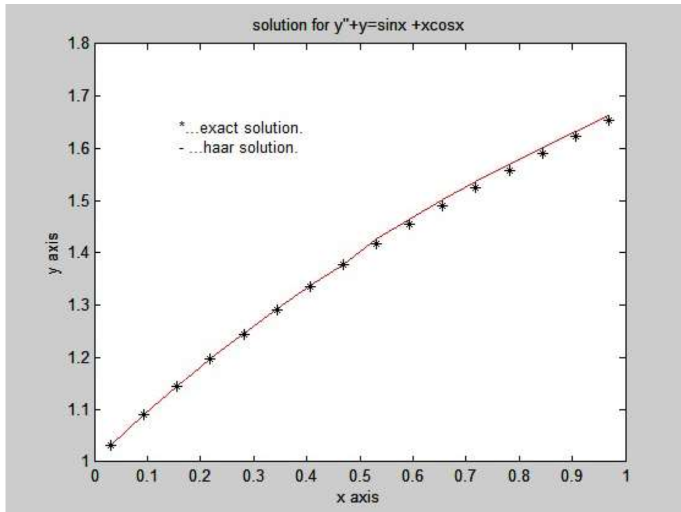

Figure 3.3: Comparison of Haar solution with exact solution for example 1 using j=3

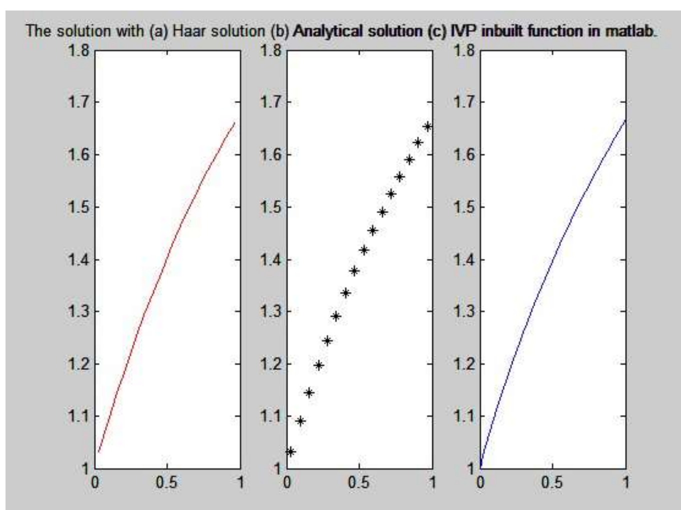

Figure 3.4: Plot for Haar implementation, analytic and inbuilt solution

$x \in [0,1]$ with boundary conditions $y(0) = 1$ and $y(1) = 3e$ as mentioned in Case 1. The analytical solution of this boundary value problem is

$$y = e^x(1 + 2x).$$

Taking the Haar wavelet approximation for second derivative then successive integration leads to,

$$y'(x) = 3e - 1 + \sum_{i=1}^{2M} a_i(p_{i,1}(x) - c_{i1}) \quad and \tag{3.6.5}$$

$$y(x) = 1 + (3e - 1)x + \sum_{i=1}^{2M} a_i(p_{i,2}(x) - xc_{i1}). \tag{3.6.6}$$

Therefore the boundary valued problem equation 3.6.4 gets transformed as,

$$\sum_{i=1}^{2M} a_i(h_i(x) - p_{i,1}(x) + c_{i1}(1 + x) - p_{i,2}(x)) = x(3e - 1) + e^x(1 - 2x) + 3e. \tag{3.6.7}$$

The system of equations are solved to obtain the Haar coefficients. By substituting the coefficients a_i so obtained in the equation for $y(x)$ we get the required solution. This result is compared with the analytical solution which is exactly similar at $2M = 16, j = 3$ that is at $m = 16$, as given in figure 3.5. The graph of the Haar solution, analytical solution and inbuilt function implementation of matlab results for the boundary value problem in (example 2) given by equation 3.6.4, for $j = 3$ is given in figure 3.6.

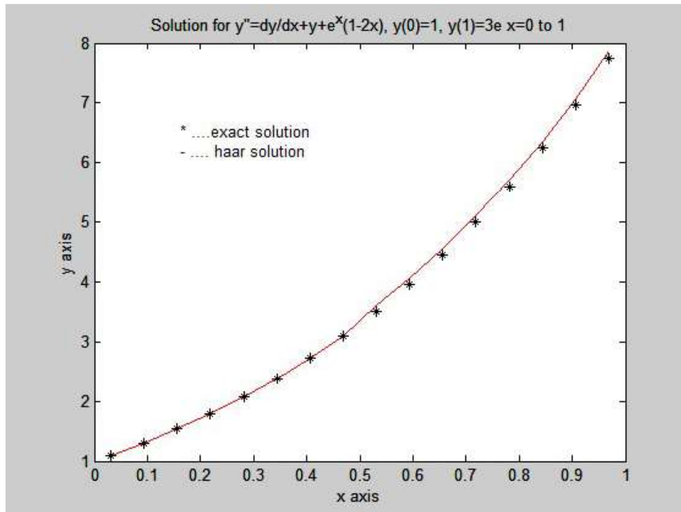

Figure 3.5: Comparison of Haar solution with exact solution for example 2

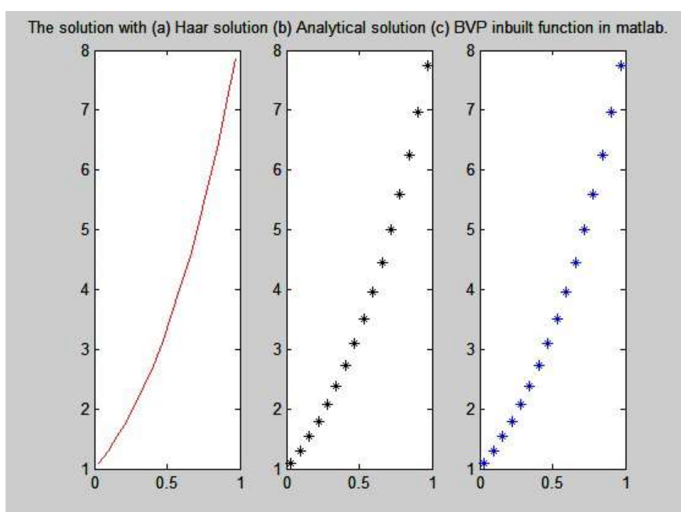

Figure 3.6: Comparison of Haar solution with exact solution for BVP

3.6.1 Observations

We have presented a unified way for solving the initial valued problem and the boundary valued problem using the wavelet collocation method. As we stated earlier this method gives accurate solution for boundary value problem and easy to implement compared to the traditional shooting methods. To improve the accuracy and optimize the computation an appropriate dynamic resolution adaptive scheme could be formulated.
In case of nonlinear differential equation with relatively less non linearity, it generates manageable non linear algebraic equations, otherwise it becomes a complicated system. The method is more amicable for linear differential equations. We propose to generalize the approach to an interval $[A, B]$, which is discussed in the next section and illustrated by an example.

3.7 Generalization proposed for the approach discussed

Since Haar wavelet function is defined in $[0, 1]$ the collocation method with Haar basis function can be used for obtaining solution in interval $[0, 1]$, but when we seek solution either for initial value or boundary value ordinary differential equation in domain $[A, B]$, we need to carryout transformation.
Consider a boundary valued ordinary differential equation 3.5.1, $x \in [A, B]$ with conditions specified at any random points A and B as $y(A)$ and $y(B)$. We transform the variable x to x_1 such that x_1 lies in the interval $[0, 1]$ with the transformation, $x_1 = \frac{x-A}{B-A}$ which leads to a change in the differential equation and boundary

conditions. Derivatives will change with this transformation as,

$$\frac{dy}{dx} = \frac{dy}{dx_1}(\frac{dx_1}{dx}), \tag{3.7.1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx_1}(\frac{dy}{dx})\frac{dx_1}{dx} \tag{3.7.2}$$

Accordingly the boundary conditions are modified and obtained between $[0, 1]$. Once the above formulation is done, the equation is solved as specified in the section 3.7. Finally the solution is again transformed back to the original variable with specified parametric changes to obtain the results in the domain $[A, B]$. We have also extended the solution for an arbitrary interval $[A, B]$ with initial conditions specified at 0. Here we have converted the interval first to an interval $[B - A, 0]$ which is further transformed back to obtain the original region of solution. This conversion helps in converting the problem to the required domain $[0, 1]$ where Haar collocation is implemented. Reverse conversion to $[A, B]$ gives the required solution. Here the variable $x_1 = B - x$ gives the first derivative and second derivative in the form as,

$$\frac{dy}{dx} = \frac{d}{dx_1}(\frac{dx_1}{dx}) \tag{3.7.3}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx_1}(\frac{-dy}{dx_1})\frac{dx_1}{dx} = \frac{d^2y}{dx_1{}^2} \tag{3.7.4}$$

The above concept is implemented in an example as discussed in the section 3.7 .

Example 1 Consider a simple second order boundary valued problem as

$$y''(x) = 1 \tag{3.7.5}$$

$$x \in [-2, 2], \qquad y(-2) = y(2) = 4. \tag{3.7.6}$$

Here a change in variable as mentioned in the section 3.7 is done with $x_1 = \frac{x+2}{4}$ and the modified equation obtained using the change of variable both for the differential operator and the boundary conditions as,

$$\frac{y''(x)}{16} = 1 \quad with \quad y(0) = y(1) = 4.$$

is solved. The analytical solution of this boundary valued problem is

$$y(x) = 8x^2 - 8x + 4.$$

The wavelet formulation is obtain as,

$$\begin{aligned} y''(x) &= \sum_{1=1}^{2M} a_i h_i(x) \\ y'(x) &= \sum_{i=1}^{2M} a_i (p_{i,1}(x) - c_{i1}) \\ y(x) &= 4 + \sum_{i=1}^{2M} a_i (p_{i,2}(x) - x c_{i1}) \end{aligned} \tag{3.7.7}$$

Therefore the boundary valued problem gets transformed as

$$\sum_{1=1}^{2M} a_i h_i(x) = 16.$$

For $2M$ collocation points we get $2M$ linear algebraic equations with unknowns $2M$ haar coefficients, which are obtained by solving the system of equations. By substituting the coefficients a_i so obtained in the equation for $y(x)$ we get the required solution. This result is compared with the analytical solution which is exactly similar at $j = 4$. The comparison of Haar solution with exact solution for $x \in [0, 1]$ for

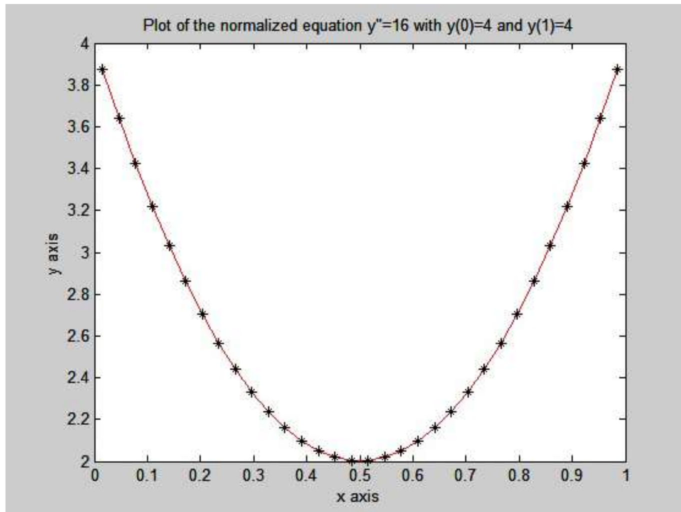

Figure 3.7: Comparitive study of normalized approach

the boundary valued ordinary differential equation 3.7.5, with $j = 4$ is shown in figure 3.7. Now figure 3.8 represents the graph of the Haar solution after transforming back to the original domain, with its analytic solution.

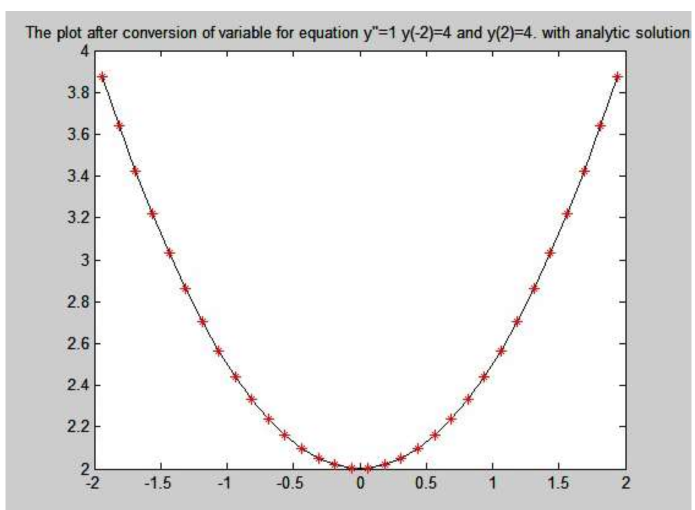

Figure 3.8: Haar solution after transforming for example 1

Example 2 Considering the boundary value problem,

$$\varepsilon y'' - y = -(\varepsilon\pi^2 + 1)\cos(\pi x), \quad y(-1) = y(1) = -1 \tag{3.7.8}$$

Analytic solution is given by $y(x) = \cos(\pi x)$ for $\varepsilon = 1$. The variable is changed as $x_1 = \frac{x+1}{2}$. The equation takes the form,

$$\frac{1}{4}\frac{d^2y}{dx_1^2} - y = (\pi^2 + 1)\cos(\pi(2x_1 - 1))$$

Haar approximation as considered in section 3.5 simplifies the equation 3.7.8 as,

$$\frac{1}{4}\sum_{i=1}^{2M} a_i h_i(x) - 1 + 2x - \sum_{i=1}^{2M} a_i(p_{i2}(x) - xc_{i1}) = -(\pi^2 + 1)\cos(\pi(2x - 1)). \tag{3.7.9}$$

The Haar approximation leads to the solution,

$$y(x) = 1 - 2x + \sum_{i=1}^{2M} a_i(p_{i,2}(x) - xc_{i1})$$

The Haar solution along with analytic solution in normalized form is given in figure 3.9. The solution after conversion into actual domain is given in figure 3.10 for example 2.

Extension in the region of solution is demonstrated with an example in section 3.8 having initial conditions specified at origin. Two cases are considered, the extension is done in case 1 for positive domain and then the solution for negative domain is shown in case 2.

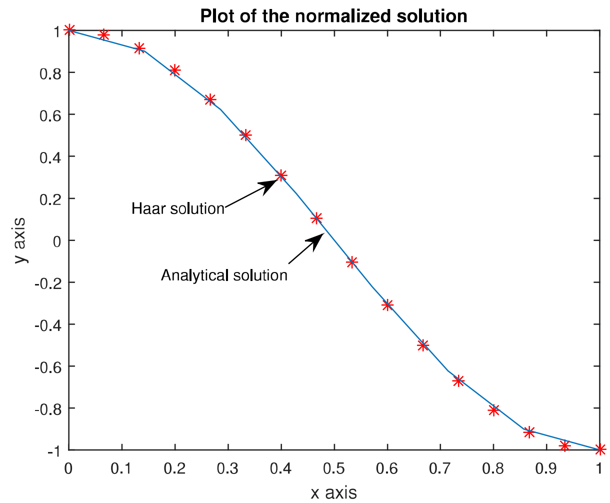

Figure 3.9: Haar solution along with analytic solution

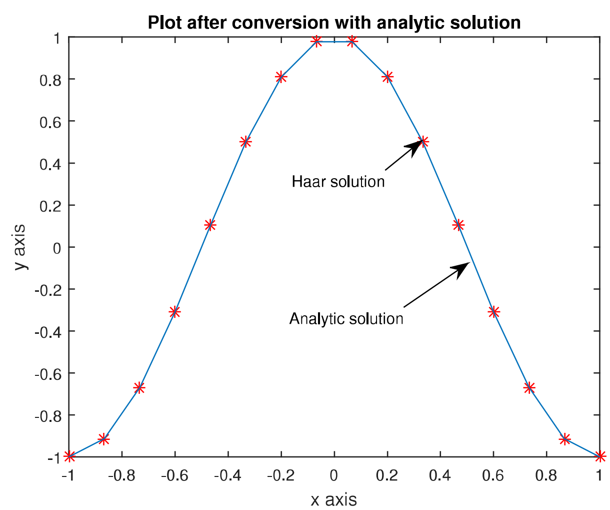

Figure 3.10: Haar solution after transforming for example 2

3.8 Extension of region of solution using Haar wavelets

Consider an equation

$$y'' + y = \cos wt \tag{3.8.1}$$

with $y'(0) = 0$ and $y(0) = 0$ with $t \in [-10, 10]$. The analytic solution for the equation is

$$y(t) = \frac{2}{1-w^2} \sin(\frac{(1+w)t}{2}) \sin(\frac{(1-w)t}{2}) \tag{3.8.2}$$

- Case 1 The Haar solution is done for $w = 0.9$ and the solution is then extended from $(0, 1)$ to $(0, 10)$, as shown in figure 3.11.

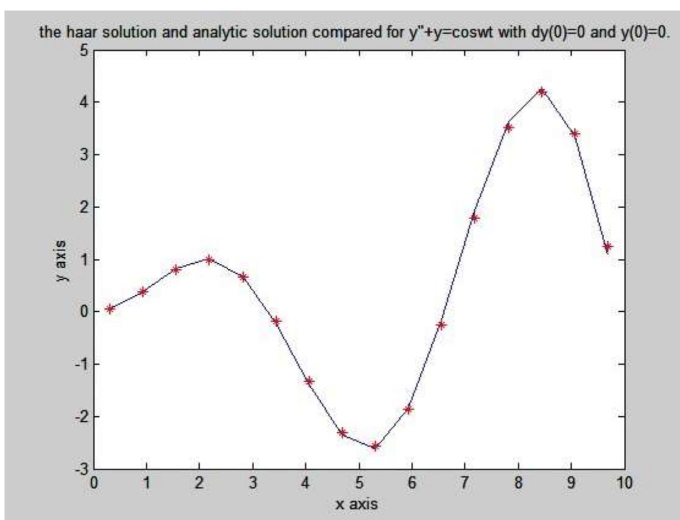

Figure 3.11: Generalized solution for case 1 in the domain $(0, 10)$

- Case 2 The Haar solution for solving the equation in the interval $[-10, 0]$ with conditions specified at 0.

The conversions to the differential equation is done as specified in previous section 3.7. After reconversion of the domain Haar solution and its comparison with analytic solution is shown in figure 3.12. Solutions are mapped and

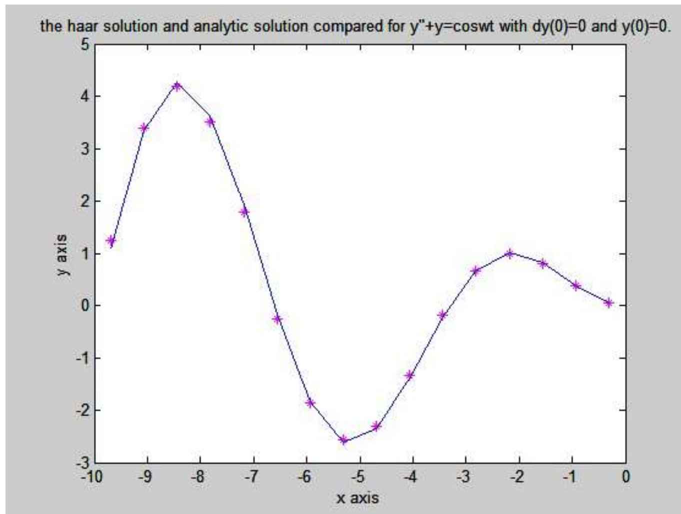

Figure 3.12: Extended solution case 1 in the domain $(-10, 0)$

compared. The dotted values represent the Haar solution and lines represent the analytic solution. The pattern agrees with the required solution even after change of domain in Haar approach.