Chapter 1

Introduction

Beginning of the 20th century marked as a major milestone in the world of physics. The inadequacy of classical physics in explaining the systems at microscopic level brought a major revolution and a new branch of physics "Quantum Physics" was born. Subsequently emerged the field of quantum statistical mechanics which deals with the statistical mechanics of these quantum systems, and is now an integral part of theoretical physics [1]. Since then the question of out of equilibrium dynamics and thermalization of isolated finite interacting many-body quantum systems has been of some interest and has been addressed from theoretical perspective. Recently magnificent experiments have been developed on cooling and trapping of ultra-cold quantum gases [2, 3] and electrons in solids [4]. These experimental developments suddenly resulted in a lot of interest on theoretical investigations of thermalization of isolated finite interacting many-body quantum systems mainly because one can simulate these systems using these experiments [5–11]. These ultra-cold gases are highly isolated in character and hence are perfect to be used for this kind of simulation compared to the other condensed matter experiments where this isolation is almost impossible [12]. The problem of thermalization of isolated finite interacting quantum systems is of great fundamental importance, however a complete understanding of the same is not known so far. Various groups have addressed the problem of thermalization using different models and perspectives. It is now known that integrable quantum systems do not thermalize due to a phenomena called many-body localization. In contrast to these, the non-integrable quantum systems thermalize and the underlying mechanism of thermalization in these systems is given by eigenstate thermalization hypothesis (ETH) [5, 13]. Many groups have studied role of localization and chaos, statistical relaxation, eigenstate thermalization, ergodicity principle and so on using lattice models of interacting spins both for fermionic and bosonic systems [8, 14–21]. Thermalization has also been studied using embedded random matrix models for femionic as well as bosonic systems [10, 11, 22–27]. The role played by the structure of chaotic wavefunctions in the process of thermalization is shown in [28–30].

The investigations on isolated finite interacting many particle quantum systems like atomic nuclei, atoms, mesoscopic systems (quantum dots, small metallic grains), interacting spin systems modeling quantum computing core, ultra-cold atoms and quantum black holes with SYK model and so on are useful in addressing open problems of quantum statistical mechanics. In order to address open problems of quantum statistical mechanics such as Bose-Einstein (BE) condensation, quantum many-body chaos and thermalization, it is important to analyze the spectral and wavefunction properties of these systems [8–11, 31–36]. It is now well established that Random Matrix Theory (RMT), due to its universality [37], successfully describes the spectral as well as wavefunction properties of isolated finite interacting many-particle quantum systems [31]. These investigations not only help us in understanding and making formulations of many-body quantum mechanics but also aid in building new quantum technologies like quantum information science. Both many-body quantum mechanics and building new quantum technologies depend on each other. For instance developing quantum computers requires the knowledge of quantum many-body systems and these quantum computers can be used to simulate the quantum many-body systems [38].

1.1 Random Matrix Theory

In 1928 Wishart's historical paper [39] on multivariate statistics, gave birth to the field of RMT. Later in 1950, Wigner introduced RMT in the field of physics, while addressing compound nucleus resonances [40]. Wigner worked on the spectra of neutron excitation of heavy nuclei obtained using neutron resonances and on the basis of his work he pointed out that for many-body systems whose interaction is complex enough, the Hamiltonian representing the system should behave like a large random matrix. The matrix elements of this matrix are random numbers drawn from some probability distribution (usually Gaussian). However it doesnot depend on the probability distribution but only on the symmetries present in the quantum system. These matrix elements should be statistically independent and identically distributed (which means they should be drawn from the same distribution).

Dyson gave the tripartite classification of classical random matrices (Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE) and Gaussian Symplectic Ensemble (GSE)) [41] on the basis of the type of symmetry preserved by the system viz. time reversal, rotational, etc. Over the coming years, rigorous research in the field by Dyson, Mehta, Pandey, Bohigas, Gaudin, Porter, Rosenzweirg, French, Berry, Tabor and many others resulted in tremendous development of RMT. In 1984, Bohigas, Giannoni and Schmit came up with their famous Bohigas-Giannoni-Schmit (BGS) conjecture which stated that

"Spectra of all quantum systems whose classical analog is chaotic follow RMT" [37]. This conjecture established the universality of RMT.

This opened doors to application of RMT to various other quantum chaotic systems like quantum dots, chaotic quantum billiards, metallic grains, etc. Over the years, it is now established as a good model to describe such complex quantum systems and stochastic systems. Due to its universality, accompanied with a great deal of mathematical work done over the years, RMT has now found applications in almost all possible fields of science like wireless communications, brain functional networks, number theory, quantum information science, quantum chromodynamics [8, 31–36, 42]. However its application is not only limited to fields of science but also has emerged as a multidisciplinary research area with numerous applications in fields like economics, finance, stock markets, etc. [43, 44]. One can learn in more detail about RMT and its applications to quantum chaotic systems from these very famous books by Pioneers of RMT, Mehta [45], Haake [46], Stoeckmann [47] and Forrester [48]. Other good books on the subject include those by Wright and Weiver [49], Bai and Silverstein [50] and by Couillet and Debbah [51].

1.2 Classification of Classical Random Matrix Ensembles Based on Space-Time Symmetries

Knowledge of symmetries present in the quantum systems is very important as it corresponds to quantum numbers which help in understanding the system qualitatively. However for quantitative understanding of these systems one requires knowledge of the interactions. The symmetries present in the quantum systems are reflected in their respective Hamiltonian. Any space-time symmetry present in the Hamiltonian matrix can be represented by an equivalent matrix (real, Hermitian or quaternion) using group theory and it is invariant under a specific transformation. A collection of such Hamiltonian matrices whose independent matrix elements are Gaussian random numbers is called a random matrix ensemble.

Dyson gave the three fold classification of these random matrix ensembles based on the symmetries present in the quantum system. Dyson's three fold classification on the basis of symmetries and angular momentum values of particles is as follows:

1. Gaussian Orthogonal Ensemble (GOE):
   If a quantum system preserves time-reversal symmetry as well as a good rotational symmetry with any spin interaction, then its Hamiltonian matrix can be represented by a real matrix invariant under orthogonal transformation i.e $O\widetilde{O} = I$.

(a)

- Symmetries: Time-reversal symmetry, No rotational symmetry
- Angular Momentum: Integer
- Hamiltonian: Real symmetric

(b)

- Symmetries: Time-reversal symmetry, Rotational symmetry
- Angular Momentum: Any (integer/half-integer)
- Hamiltonian: Real symmetric

2. Gaussian Unitary Ensemble (GUE):
If a quantum system does not preserve time-reversal symmetry, then its Hamiltonian matrix can be represented by a Hermitian matrix invariant under unitary transformation i.e. $UU^{\dagger} = I$.

    - Symmetries: No time-reversal symmetry
    - Angular Momentum: Any (integer/half-integer)
    - Hamiltonian: Hermitian

3. Gaussian Symplectic Ensemble (GSE):
If a quantum system preserves time-reversal symmetry and has a half-integer spin interaction, then its Hamiltonian matrix can be represented by a quaternion real matrix invariant under symplectic transformation
i.e. $SZ\widetilde{S} = Z, SS^{\dagger} = I$.

    - Symmetries: Time-reversal symmetry, No rotational symmetry
    - Angular Momentum: Half-integer
    - Hamiltonian: Quaternion real

1.3 Embedded Random Matrix Ensembles

Constituents of isolated quantum systems interact via few-body interactions whereas the classical random matrix ensembles (and in particular the GOE) take into account many-body interactions. This motivated French and co-workers to introduce random matrix model accounting for few-body interactions, called *Embedded ensemble* (EE) [52, 53]. The orthogonal variant of these EE with two body interactions is denoted by EGOE(2). These systems also possess an additional one body part corresponding to mean-field generated by all other constituents in these systems. With two-body interaction and in the presence of mean-field one-body part they are called EGOE(1+2). It is now well established that these EGOE(1+2) models are paradigmatic models to study the dynamical transition from integrability to chaos in isolated finite interacting many-body quantum systems [8, 24, 31, 54]. For spin-less fermion systems these models are denoted by EGOE(1+2) and for spin-less boson systems they are denoted by BEGOE(1+2) (with 'B' for bosons). These models were initially analyzed for isolated finite interacting spin-less fermion systems by many groups [24, 55–58]. Also, isolated finite interacting spin-less boson systems were analyzed in [27, 59–63].

Consider m particles (fermions/bosons) distributed in N single particle (sp) states. Then two limiting situations exist. One is the dilute limit which is defined by $m \to \infty$, $N \to \infty$ and $m/N \to 0$, which allows only one particle per sp state. The other one is the dense limit which is defined by $m \to \infty$, $N \to \infty$ and $m/N \to \infty$, which allows more than one particle to occupy a particular sp state. The dilute limit exists for both fermion and boson systems while the dense limit is feasible only for boson systems. The dense limit is not feasible for fermion systems because of the fact that fermions obey Pauli's exclusion principle which allows only one particle per sp state. Since in the dilute limit both fermions and bosons show same behavior, the investigations on BEGOE carried out in past focused on the dense limit [22, 27, 59–63].

Finite interacting particle systems preserve various symmetries like particle number, spin, angular momentum, parity, spin-isospin SU(4) symmetry and so on [31]. These symmetries give rise to various quantum numbers. In the later years, EGOE(1+2) models preserving various symmetries were introduced. EGOE(1+2) models with spin $1/2$ degree of freedom are the simplest ones. They are denoted by EGOE(1+2)-s for fermion systems [64] and BEGOE(1+2)-F for two species boson systems with a fictitious (F) spin $1/2$ degree of freedom [65]. Moving further for boson systems with spin one degree of freedom, EGOE(1+2) model is denoted by BEGOE(1+2)-$S1$ [66]. EGOE(1+2) model preserving parity is denoted by EGOE(1+2)-π [67]. More details on all these EE can be found in a book on the subject by Kota [31], in a recent review article by Kota and Chavda [34] and also in [68–70]. In this thesis we have used the spinless EGOE(1+2) and BEGOE(1+2) models as well as EGOE(1+2)-s, BEGOE(1+2)-F and BEGOE(1+2)-$S1$. The definition and construction of the basic Hamiltonian of all these EE used to model isolated finite interacting fermion and boson systems in this thesis is given in chapter 2.

Spectral fluctuations arising from various complex quantum systems help in understanding the chaotic dynamics of these systems. Over the years various measures have been developed in the field of RMT to study spectral fluctuations. Various spacing distributions like nearest neighbor spacing distribution (NNSD), distribution of ratio of consecutive level spacings, Dyson-Mehta Δ_3 statistic and so on are these measures. In chapter 3, we study spectral fluctuations using the spectra (i.e. eigenvalues) obtained from the Hamiltonian of various EE described in chapter 2. For this purpose we use two spacing distributions. In the first part of this chapter, we have studied the closest neighbor spacing distribution $P_{CN}(s)$ and the farther neighbor spacing distribution $P_{FN}(s)$ for interacting fermion and boson systems with and without spin degree of freedom. A very good correspondence between the numerical EE results and the recently derived analytical expressions using a 3×3 random matrix model and other related quantities is obtained. The construction of this spacing distribution involves a cumbersome and non-trivial procedure called unfolding, which can be avoided if we use the method of ratio of spacings. Going beyond the method of ratio of spacings, in the second part of this chapter, we study the probability distribution of higher orders of the ratio of spacings for both interacting fermion and boson systems with and without spin degree of freedom. Our numerical results demonstrate a very good consistency with the recently proposed generalized Wigner surmise like scaling relation. These results confirm that the recently derived analytical expressions using classical random matrix model for both these distributions are universal in understanding spectral fluctuations in complex quantum systems. This conclusively establishes that results of local spectral fluctuations generated by EE follow that of given by the classical Gaussian ensembles. Further the higher order spacing ratio distributions can also reveal quantitative information about the underlying symmetry structure (examples are isospin in lighter nuclei and scissors states in heavy nuclei). We use spin ensembles to demonstrate this.

In the chapters so far, the main focus was on one- plus two-body part of the interaction as inter-particle interaction is known to be only one-body and two-body in nature. However, it is seen from recent studies that the higher body interactions $k > 2$ play an important role in strongly interacting quantum systems [71, 72], nuclear physics [73], quantum black holes [35, 74] and wormholes [75] with SYK model and also in quantum transport in disordered networks connected by many-body interactions [76–78]. Therefore, it is necessary to extend the analysis of EE to higher k-body interactions in order to understand these problems. They are represented by EGOE(k) (or BEGOE(k)) for fermion (or boson) systems. In the presence of mean-field one body part they are represented by EGOE(1+k) (or BEGOE(1+k)) for fermion (or boson) systems.

From the previous studies, it is known that with EGOE(k)(or BEGOE(k)), the eigenvalue density for a system of m fermions/bosons in N spx states changes from Gaussian form to semi-circle as k changes from 2 to m [31, 34, 55, 79]. The generating function of q-Hermite polynomials also demonstrates this Gaussian to semi-circle transition. Due to this property of q-Hermite polynomials they have been employed in recent studies on spectral densities of the so-called SYK model [80,81] and quantum spin glasses [82]. Apart from this they have been used recently to study the strength functions (also known as local density of states (LDOS)) and fidelity decay (also known as survival or return probability) in EE, both for fermion as well as boson systems [79]. Formulae for parameter q in terms of m, N and k are derived for fermionic and bosonic EE in [79] which explain the Gaussian to semi-circle transition in spectral densities and strength functions and also fidelity decay in many-body quantum systems as a function of rank of interaction. Recently, the lower-order bivariate reduced moments of the transition strengths are examined for the action of a transition operator on the eigenstates generated by EGOE(k) and it is shown that the ensemble averaged distribution of transition strengths follows a bivariate q-normal distribution [83]. Also using the bivariate q-normal form, a formula for the chaos measure, number of principal components (NPC), in the transition strengths from a state is presented in [83].

In the beginning of chapter 4 the method of construction of Hamiltonian of EGOE(1+k) and BEGOE(1+k) for isolated finite interacting fermion and boson systems is described. Further the q-Hermite polynomials, their generating functions and properties are introduced. We have also defined the so-called q-normal distribution f_{qN}, conditional q-normal distribution f_{CqN} and bivariate q-normal distribution f_{biv-qN}. In the past the formulae of parameter q in terms of m, N and k for both EGOE(k) and BEGOE(k) were derived in [79]. Going beyond this, we have derived an analytical formula of parameter q considering only the mean-field one-body part for both fermions and bosons. Furthermore, for a fixed body rank k, the variation of parameter q is studied as the interaction strength λ varies in EGOE(1+k) and BEGOE(1+k). In the end of this chapter we use q-Hermite polynomials to study the spectral density for EGOE(1+k) and BEGOE(1+k). It is shown that the spectral density of both EGOE(1+k) and BEGOE(1+k) demonstrate Gaussian to semi-circle transition and this transition is well described by f_{qN}.

The chapters so far involved the spectral properties of isolated finite interacting many-particle quantum systems. The spectral properties were analyzed using two measures viz. ordered level spacing distribution and higher order spacing ratio distribution. These spectral properties are analyzed using the eigenvalues of these systems. However more information about the chaotic dynamics of these systems is embedded in the eigenfunctions. In the upcoming two chapters 5 and 6, we study the structure of wavefunction of these systems using their eigenfunctions. Now in order to study wavefunction properties, it is necessary to have knowledge of the strength functions as they play a crucial role in the analysis of wavefunction properties. They give information about how a particular basis state spreads over the eigenstates.

In chapter 5, we analyze the strength function and its width for both fermions and bosons using EGOE(1+k) and BEGOE(1+k) respectively. Recently, it is shown that in the strong coupling limit (i.e. $\lambda > \lambda_t$), the Gaussian to semi-circle transition in strength functions can be represented by f_{CqN} [84]. We show that for both EGOE(1+k) and BEGOE(1+k), this transition in strength functions as a function of body rank of interaction k is well described by f_{CqN}. Further we have also derived a complete analytical description of the variance of the strength function in terms of the correlation coefficient ζ, as a function of λ and k. The analytical expression of ζ is utilized to obtain the analytical expression of marker λ_t which defines the thermalization region in terms of the system parameters m, N and k. Very recently, analytical formulae for the lowest four moments of the strength functions for fermion systems modeled by EGOE(1+k) are derived in [84]. Lastly in chapter 5, we have analyzed the lower order moments of the strength functions.

The strength functions are a basis in the study of various chaos quantifiers. In chapter 6, we study two such chaos quantifiers viz. number of principal components (NPC) and information entropy (S^{info}) using the strength function analysis presented in chapter 5. We use the interpolating conditional q-normal form f_{CqN} of the strength functions and f_{qN} form to study NPC and the localization length l_H related to S^{info}. An analytical formula for NPC as a function of energy is derived in terms of two parameters ζ and q for k-body interaction. For strong enough interaction strength, this formula is tested and found to be in good agreement with the numerical EE results for both EGOE(1+k) and BEGOE(1+k). The other chaos quantifier l_H is studied numerically using f_{CqN} for both EGOE(1+k) and BEGOE(1+k) as a function of energy for k-body interaction and these results are in good agreement with the EE results. We have also studied time evolution of these systems using an important quantity called fidelity decay in boson systems after the application of k-body interaction quench. This is studied for BEGOE(1+k) using the interpolating form of strength functions.

We have seen earlier that k-body interactions play an important role in quantum transport across disordered networks connected by many-body interactions. In chapter 7, we use all the knowledge of EE with k-body interactions acquired from the work presented in this thesis as well as from past studies to study the role of centrosymmetry in quantum transport across disordered fermionic and bosonic networks connected by many-body interactions. In this regard, we study the influence of centrosymmetry on transport efficiencies of an initial localized excitation in disordered finite fermionic and bosonic network modeled by EGOE(k) and BEGOE(k) respectively. The disordered fermionic network of d sites is modeled by three different ensembles that includes many-body interactions: (i) EGOE(k) without centrosymmetry, (ii) EGOE(k) with centrosymmetry present in both k as well as m particle space (denoted by csEGOE(k)) and (iii) EGOE(k) with centrosymmetry present in k particle space only (denoted by EGOE(k-cs)). Similarly for bosonic network, we used BEGOE(k), csBEGOE(k) and BEGOE(k-cs) ensembles. We found that the quantum efficiency is enhanced when centrosymmetry is imposed and the results are in good agreement with those obtained in past [76].

Finally the last chapter 8, presents the conclusions of the entire work carried out in this thesis. Also the future directions of the work presented in this thesis are discussed.