Chapter 2

Embedded Ensembles for Fermion and Boson Systems

In random matrix theory, the statistical properties of isolated finite many-particle complex quantum systems can be investigated by representing its Hamiltonian by a random matrix that contains all the information about the system. Depending upon the symmetries imposed on the system, we have tripartite classification of classical random matrices viz. Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE) and Gaussian Symplectic Ensemble (GSE). This classification was given by Dyson on the basis of symmetries and angular momentum values of particles.

However, constituent particles of various finite interacting many-particle quantum systems like nuclei, atoms, quantum dots, etc. interact via few body interactions in the presence of an average field generated by other particles. This motivated French and co-workers to introduce random matrix ensembles with few body interactions called Embedded Ensembles (EE) which are now well established models to represent these systems [52, 53]. For two body interactions in the presence of an average field, their orthogonal variant called Embedded Gaussian Orthogonal Ensemble (EGOE) is denoted by EGOE(1+2). In these ensembles, the two particle Hamiltonian matrix is defined using the classical GOE and the $m > 2$ particle Hamiltonian matrix is generated using the concepts of direct product space and Lie algebra. They are generically called EE because of the fact that the two particle Hamiltonian matrix is embedded in the m particle Hamiltonian matrix. Recently these models have also been used successfully in understanding high energy physics related problems. EGOE(k) for complex fermions are known as complex Sachdev-Ye-Kitaev models in this area [85–87]. In this chapter we define and describe the construction of Hamiltonian matrix of various EE for fermion and boson systems with and without spin degree of freedom used in this thesis. For spinless fermion and boson systems they are denoted by

EGOE(1+2) and BEGOE(1+2) respectively.

2.1 EGOE for Spinless Systems

The definition and construction of EGOE(1+2) and BEGOE(1+2) are clearly described in detail in [31]. However, for the sake of completeness we describe these EE here.

Let us consider a system of m fermions (or bosons) without spin and which are to be distributed in N sp states and interacting via (1+2) body interaction. Let these N sp states be denoted by $|i\rangle$ where $i = 1, 2, 3, ..., N$ and the corresponding sp energies are ϵ_i. This distribution of m fermions (or bosons) in N sp states generates a many particle configuration space with dimension $d_F(N, m)$ for fermions (or $d_B(N, m)$ for bosons). For fermions, $d_F(N, m) = \binom{N}{m}$. For example for $(N, m) = (12, 6)$ we have $d_F = 924$, for $(N, m) = (13, 6)$, $d_F = 1716$ and for $(N, m) = (14, 7)$, $d_F = 3432$. On the other hand for bosons, $d_B(N, m) = \binom{N+m-1}{m}$. For example for $(N, m) = (4, 10)$ we have $d_B = 286$, for $(N, m) = (5, 10)$, $d_B = 1001$ and for $(N, m) = (7, 14)$, $d_B = 38760$. A basis state in m-particle space, due to product nature of the states in occupation number representation, is denoted by $\left|\prod_{i=1}^{N} m_i\right\rangle = |m_1, m_2, \ldots, m_N\rangle$, where m_i is the number of fermions (or bosons) in i'th sp state. For fermions, we have $m_i = 0$ or 1 and for bosons m_i can take any value from 0 to m with $\sum_i m_i = m$.

With random two-body interactions $V(2)$, we can define the Hamiltonian of EGOE(1+2) as follows,

$$H = h(1) + \lambda\{V(2)\}. \tag{2.1}$$

In the above Eq. (2.1), $h(1)$ represents the mean field one-body Hamiltonian given by $h(1) = \sum_{i=1}^{N} \epsilon_i n_i$. Here ϵ_i are the sp energies and n_i are number operators acting on sp states. For fermions $n_i = F_i^\dagger F_i$ where $F_i^\dagger$ and F_i are the fermion creation and annihilation operators respectively for the sp state $|i\rangle$. Similarly, for bosons $n_i = B_i^\dagger B_i$ with $B_i^\dagger$ and B_i are the boson creation and annihilation operators respectively for the sp state $|i\rangle$. λ is the two-body interaction strength and notation $\{\ \}$ denotes an ensemble. The dimensionality of m-particle Hamiltonian matrix $H(m)$ is $d(N, m) = d_F(N, m)$ for fermions ($d(N, m) = d_B(N, m)$ for bosons).

For fermion system, one can define the two-body Hamiltonian matrix as follows,

$$V(2) = \sum_{i<j,k<l} \langle kl|\, V(2)|\, ij\rangle\, F_l^{\dagger} F_k^{\dagger} F_i F_j \tag{2.2}$$

These fermion creation and annihilation operators obey the following anticommutation rules, $\{F_i, F_j^{\dagger}\} = \delta_{ij}$ and $\{F_i, F_j\} = \{F_i^{\dagger}, F_j^{\dagger}\} = 0$. In Eq. (2.2), $\langle kl|\, V(2)|\, ij\rangle$ are the two body matrix elements (TBME) which are anti-symmetric for fermion systems with the following symmetries,

$$\begin{gathered} \langle kl|\, V(2)|\, ji\rangle_a = -\langle kl|\, V(2)|\, ij\rangle_a\,, \\ \langle kl|\, V(2)|\, ij\rangle_a = \langle ij|\, V(2)|\, kl\rangle_a\,. \end{gathered} \tag{2.3}$$

For boson system the two-body hamiltonian $V(2)$ is defined in terms of the matrix elements in the two particle basis states $|i,j\rangle \equiv |m_i = 1, m_j = 1\rangle$ if $i \neq j$ or $|m_i = 2\rangle$ if $i = j$,

$$V(2) = \sum_{\substack{i\leq j\\ k\leq l}} \frac{\langle ij|\, V(2)|\, kl\rangle_s}{\sqrt{1+\delta_{ij}}\sqrt{1+\delta_{kl}}} B_i^{\dagger} B_j^{\dagger} B_k B_l, \tag{2.4}$$

For a system of bosons the TBME $\langle ij|\, V(2)|\, kl\rangle$ are symmetrized with the following symmetries,

$$\langle ij|\, V(2)|\, kl\rangle_s = \langle kl|\, V(2)|\, ij\rangle_s = \langle ji|\, V(2)|\, lk\rangle_s = \langle ij|\, V(2)|\, lk\rangle_s. \tag{2.5}$$

These boson creation and annihilation operators obey the following commutation rules,

$\left[B_i, B_j^{\dagger}\right] = \hat{\delta}_{ij}$ and $\left[B_i, B_j\right] = \left[B_i^{\dagger}, B_j^{\dagger}\right] = \hat{0}$.

Action of the two body Hamiltonian operator $V(2)$ on the many particle basis states $|i\rangle$ generates the EGOE(2) (or BEGOE(2)) ensemble in m fermion (or boson) spaces.

The Hamiltonian matrix is a $d(N,m) \times d(N,m)$ matrix and it is symmetric because of the presence of time reversal symmetry with the number of independent matrix elements $ime(N)$ given by,

$$ime(N,2) = \frac{d(N,2)[d(N,2)+1]}{2} \tag{2.6}$$

The m-particle Hamiltonian matrix $H(m)$ is then defined by m particle matrix elements $\langle m'_1, m'_2, \ldots, m'_N|H|m_1, m_2, \ldots, m_N\rangle$. Many of the matrix elements of $H(m)$ for $m >$

2 are zero due to two-body nature of the interaction. The non-zero matrix elements of $H(m)$ are linear combinations of the sp energies and the TBME.

For fermions, the non-zero matrix elements are obtained as follows [31],

$$\begin{aligned}
\langle v_1 v_2 \ldots v_m |\, V(2) |\, v_1 v_2 \ldots v_m \rangle &= \sum_{v_i < v_j \le v_m} \langle v_i v_j |\, V(2) |\, v_i v_j \rangle \,, \\
\langle v_p v_2 v_3 \ldots v_m |\, V(2) |\, v_1 v_2 \ldots v_m \rangle &= \sum_{v_i = v_2}^{v_m} \langle v_p v_i |\, V(2) |\, v_1 v_i \rangle \,, \\
\langle v_p v_q v_3 \ldots v_m |\, V(2) |\, v_1 v_2 v_3 \ldots v_m \rangle &= \langle v_p v_q |\, V(2) |\, v_1 v_2 \rangle \,.
\end{aligned} \tag{2.7}$$

In the above equation Eq. (2.7), $|\, v_1 v_2 \ldots v_m \rangle$ represents the orbits occupied by m fermions.

For bosons, the non-zero matrix elements are obtained as follows,

$$\begin{aligned}
&\langle m'_1, m'_2, \ldots, m'_N | H | m_1, m_2, \ldots, m_N \rangle \\
&= \sum_i \epsilon_i \left\langle \Pi m'_p | B_i^\dagger B_i | \Pi m_p \right\rangle + \lambda \sum_{i \le j,\, k \le l} \frac{\langle ij | V(2) | kl \rangle}{\sqrt{1+\delta_{ij}}\sqrt{1+\delta_{kl}}} \left\langle \Pi m'_p | B_i^\dagger B_j^\dagger B_k B_l | \Pi m_p \right\rangle .
\end{aligned} \tag{2.8}$$

where $B_i^\dagger |m_i\rangle = \sqrt{m_i+1}|m_i+1\rangle$ and $B_i|m_i\rangle = \sqrt{m_i}|m_i-1\rangle$. The ensemble of $H(m)$ is obtained by taking the defining TBME as Gaussian random variables with

$$\overline{\langle kl | V(2) | ij \rangle} = 0 \quad \text{and} \quad \overline{|\langle kl | V(2) | ij \rangle|^2} = v^2 \left(1 + \delta_{(ij),(kl)}\right) \,, \tag{2.9}$$

where the bar indicates an ensemble average.

Thus $V(2)$ is a GOE in two-particle space and $H(m)$ is EGOE of two-body interaction with strength λ plus the single particle hamiltonian $h(1)$.

For realistic finite interacting particle systems like atomic nuclei, quantum dots, nanometallic grains, ultracold spinor gases etc. in addition to particle number m, spin quantum number (S) is important. As group symmetries define various quantum numbers, in general, one has to consider EGOE with group symmetries. The most trivial spin EE are EE with spin $1/2$ degree of freedom. It is very important to study these as they have many applications. They are denoted by EGOE(1+2)-$\mathbf{s}$ (or BEGOE(1+2)-F) for fermions (or bosons) [64,65]. In BEGOE(1+2)-F, F is fictitious spin-1/2 degree of freedom. EGOE(1+2)-$\mathbf{s}$ is used to study universal conductance fluctuations in mesoscopic systems and BEGOE(1+2)-F may be useful in exploring general structures of spinor condensates [31]. For boson systems with spin one degree of freedom, EE model is denoted by BEGOE(1+2)-$S1$ [66]. The definition and construction of EGOE(1+2)-$\mathbf{s}$, BEGOE(1+2)-

F and BEGOE(1+2)- $S1$ are clearly described in detail in [64], [65] and [66] respectively and also in the book [31]. However, for the sake of completeness we describe all these EE here.

2.2 Embedded Fermionic/Bosonic Ensembles with Spin

2.2.1 EGOE for Fermion with Spin 1/2 Degree of Freedom - EGOE(1+2)- s : Definition and Construction

We consider a system of m ($m > 2$) fermions distributed in Ω number of sp orbitals each with spin $\mathrm{s} = \frac{1}{2}$ so that number of sp states is $N = 2\Omega$. The sp states are denoted by $|i, m_s = \pm\frac{1}{2}\rangle$ with $i = 1, 2, \ldots, \Omega$ and similarly two particle antisymmetric states are denoted by $|(ij)s, m_s\rangle_a$ with $s = 0$ or 1 . A complete set of basis states spanning the Hilbert space can be generated by distributing these m fermions into N sp states. As fermions have $\mathrm{s} = \frac{1}{2}$, the two-fermion spin is given by $s = 0$ or 1 and m fermion spin S is given by $S = m/2$, $m/2 - 1$, $\ldots$, 0 or 1/2. The two-body Hamiltonian $V(2)$ is defined by the two-body matrix elements,

$$V^s_{ijkl} = \langle (kl)s, m_s \mid V(2) \mid (ij)s, m_s\rangle \tag{2.10}$$

with the two-particle spins $s = 0, 1$; note that for $s = 1$, only matrix elements with $i \neq j$ and $k \neq l$ can exist. $V(2) = V^{s=0}(2) \oplus V^{s=1}(2)$. As two-particle spins can take two values, the two-body Hamiltonian $V(2)$ is a direct sum matrix of matrices in spin 0 and 1 spaces with dimensions $\Omega(\Omega+1)/2$ and $\Omega(\Omega-1)/2$ respectively. Thus, $V(2)$ is defined by two-body matrix elements that are independent of the m_s quantum number. With $V^{s=0}(2)$ and $V^{s=1}(2)$ being independent GOEs in two-particle spaces, the many fermion Hamiltonian H for EGOE(1+2)- s can be generated by propagating the $\{V(2)\}$ ensemble to the m -particle spaces with a given S by using the direct product structure of the m -particle spaces. Then EGOE(1+2)- s is defined by the Hamiltonian H ,

$$H = h(1) + \lambda_0\{V^{s=0}(2)\} + \lambda_1\{V^{s=1}(2)\} \; . \tag{2.11}$$

where λ_0 and λ_1 are the two-body interaction strengths of the $s = 0$ and $s = 1$ parts respectively. The 1-body Hamiltonian $h(1)$ is defined using sp energies ϵ_i , with unit average level spacing, as $h(1) = \sum_i \epsilon_i \; n_i$. Here, n_i are number operators acting on the sp states $|i, m_s = \pm\frac{1}{2}\rangle$.

For EGOE(1+2)- s , the dimension of H matrix given in Eq. (2.11) is,

$$d(m,S) = \frac{(2S+1)}{(\Omega+1)} \binom{(\Omega+1)}{m/2+S+1} \binom{(\Omega+1)}{m/2-S} . \tag{2.12}$$

They satisfy the sum rule $\sum_S (2S+1)\; d(m,S) = \binom{N}{m}$. For example for $m=6$ and $\Omega=8$, the dimensions are 1176, 1512, 420 and 28 for S = 0,1,2 and 3 respectively.

The many particle Hamiltonian matrix for a given (m,S) can be constructed as follows: first we consider the sp states $\left|i, m_{\mathbf{s}} = \pm\frac{1}{2}\right\rangle$ which are arranged in such a way that the first Ω states have $m_{\mathbf{s}} = \frac{1}{2}$ and the remaining Ω have $m_{\mathbf{s}} = \frac{1}{2}$ so that a state $|r\rangle = \left|i=r, m_{\mathbf{s}} = \frac{1}{2}\right\rangle$ for $r \leq \Omega$ and for $r > \Omega$, $|r\rangle = \left|i = r-\Omega, m_{\mathbf{s}} = -\frac{1}{2}\right\rangle$. Now the m-particle configurations $\mathbf{m}$ due to product nature of the states in occupation number representation, is denoted by

$$\left|\prod_{r=1}^{N=2\Omega} m_r\right\rangle = |m_1, m_2, \ldots, m_\Omega, m_{\Omega+1}, m_{\Omega+2}, \ldots, m_{2\Omega}\rangle , \tag{2.13}$$

where m_r can take values 0 or 1. The corresponding m_S values are,

$$m_S = \frac{1}{2}\left[\sum_{r=1}^{\Omega} m_r - \sum_{r'=\Omega+1}^{2\Omega} m_{r'}\right] . \tag{2.14}$$

Here, for even values of $\mathbf{m}$, the $\mathbf{m}$'s with $m_S = 0$ will include states with all S values and similarly with $m_S = \frac{1}{2}$ for odd $\mathbf{m}$. Therefore, the m-particle Hamiltonian matrix is constructed using the basis defined by $\mathbf{m}$'s with $m_S = 0$ for even values of $\mathbf{m}$ and $m_S = \frac{1}{2}$ for odd values of $\mathbf{m}$.

Moving further, the (1+2)-body Hamiltonian defined by $(\epsilon_i, V(2)^{s=0,1}_{ijkl})$'s is converted into the $\left|i, m_{\mathbf{s}} = \pm\frac{1}{2}\right\rangle$ basis by changing ε_i to ε_r with the index r defined as above and changing $V(2)^{s=0,1}_{ijkl}$ to V_{im_i,jm_j,km_k,lm_l} where

$$V_{i\frac{1}{2},j\frac{1}{2},k\frac{1}{2},l\frac{1}{2}} = V^{s=1}_{ijkl}, V_{i-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2},l-\frac{1}{2}} = V^{s=1}_{ijkl}, V_{i\frac{1}{2},j-\frac{1}{2},k\frac{1}{2},l-\frac{1}{2}} = \frac{\sqrt{(1+\delta_{ij})(1+\delta_{kl})}}{2}\{V^{s=1}_{ijkl}+V^{s=0}_{ijkl}\} \tag{2.15}$$

with all other matrix elements being zero except for the symmetries,

$$V_{im_i,jm_j,km_k,lm_l} = -V_{im_i,jm_j,km_k,lm_l} = V_{im_i,jm_j,km_k,lm_l} = V_{km_k,lm_l,im_i,jm_j} . \tag{2.16}$$

Using $(\epsilon_r, V_{im_i,jm_j,km_k,lm_l})$'s, construction of the m-particle H matrix in the basis defined by Eqs. (2.13) and (2.14) reduces to the problem of EGOE(1+2) for spinless fermion systems. For the S^2 operator, it is easy to recognize that $\epsilon_i = 3/4$ independent of i, $V^{s=0}_{ijij} = -3/2$ and $V^{s=1}_{ijij} = 1/2$ independent of (i,j) and all other V^s's are zero. Using these, for the S^2 operator, the m-particle matrix with $m_S = 0$ for even m (and $m_S = \frac{1}{2}$ for odd m) is constructed and diagonalized. This gives a direct sum of unitary matrices and the unitary matrix, that corresponds to a given S is identified by the eigenvalue $S(S+1)$. Applying the unitary transformation defined by this unitary matrix, the m-particle Hamiltonian matrix with $m_S = 0$ for even m (and $m_S = \frac{1}{2}$ for odd m) is transformed to the basis with good S values.

2.2.2 EGOE for Boson with Spin 1/2 Degree of Freedom - BEGOE(1+2)-F : Definition and Construction

One can also define embedded ensemble for bosons with fictitious spin $\mathbf{f} = \mathbf{1/2}$ degree of freedom denoted as BEGOE(1 + 2)-F with s replaced by $\mathbf{f}$, with two-particle spin s replaced by f and many-particle spin S replaced by F in the previous section. For BEGOE(1+2)-F, the spin algebra remains same as in fermion EGOE(1+2)-s case with $V(2)$ defined by two-body matrix elements given by,

$$V^f_{ijkl} = \langle (kl)f, m_f \mid V(2) \mid (ij)f, m_f \rangle \tag{2.17}$$

and dimensions of spin $f = 0$ and $f = 1$ spaces being $\Omega(\Omega-1)/2$ and $\Omega(\Omega+1)/2$ respectively. For BEGOE(1+2)-F, the dimension of H matrix given in Eq. (2.11) is,

$$d(m,F) = \frac{(2F+1)}{(\Omega-1)} \binom{(\Omega-1)+(m/2+F+1)-1}{m/2+F+1} \binom{(\Omega-1)+(m/2-F)-1}{m/2-F} . \tag{2.18}$$

satisfying the sum rule $\sum_F (2F+1)\, d(m,F) = \binom{N+m-1}{m}$. For example for $m = 10$ and $\Omega = 4$, the dimensions are 196, 540, 750, 770, 594 and 286 for spins S=0, 1, 2, 3, 4 and 5 respectively. In order to construct the many particle Hamiltonian matrix for boson systems for a given (m, F), m_r can take values between 0 to $m/2$ (for even m) and $m/2+1$ (for odd m) with $\sum_r^N m_r = m$.

For boson systems the (1+2)-body Hamiltonian defined by $(\epsilon_i, V(2)^{f=0,1}_{ijkl})$'s is converted into the $\left|i, m_{\mathbf{f}} = \pm\frac{1}{2}\right\rangle$ basis by changing ε_i to ε_r with the index r defined as above and changing $V(2)^{f=0,1}_{ijkl}$ to $V_{im_i,jm_j,km_k,lm_l} = \langle im_i, jm_j|\, V(2)|\, km_k, lm_l\rangle$ using the Eq. (2.15) with all other matrix elements being zero except for the symmetries,

$$V_{im_i,jm_j,km_k,lm_l} = V_{km_k,lm_l,im_i,jm_j} = V_{jm_j,im_i,lm_l,km_k} = V_{im_i,jm_j,lm_l,km_k} . \tag{2.19}$$

Using $(\epsilon_r, V_{im_i,jm_j,km_k,lm_l})$'s, construction of the m-particle H matrix in the basis defined by Eqs. (2.13) and (2.14) reduces to the problem of BEGOE(1+2) for spinless boson systems. The construction of F^2 operator is similar to that of S^2 described in the last paragraph of the previous section. Using F^2 operator, the m-particle matrix with $m_F = 0$ for even m (and $m_F = \frac{1}{2}$ for odd m) is constructed and diagonalized. This gives a direct sum of unitary matrices and the unitary matrix, that corresponds to a given S is identified by the eigenvalue $F(F+1)$. Applying the unitary transformation defined by this unitary matrix, the m-particle Hamiltonian matrix with $m_F = 0$ for even m (and $m_F = \frac{1}{2}$ for odd m) is transformed to the basis with good F values.

2.2.3 EGOE for Boson with Spin 1 Degree of Freedom - BEGOE(1+2)-$S1$: Definition and Construction

Let us consider a system of m ($m > 2$) interacting bosons distributed in Ω number of single particle (sp) orbitals each with spin $\mathbf{s} = 1$. The number of single particle states are $N = 3\Omega$. The single particle states are denoted by $|i, m_{\mathbf{s}} = 0, \pm 1\rangle$ with $i = 1, 2, \ldots, \Omega$ and the two particle symmetric states are denoted by $|(ij)s, m_s\rangle_s$ with $s = 0, 1$ or 2. The total dimensionality of the two-particle space with the matrix dimension for space $s = 0$, $s = 1$ and $s = 2$ are $\Omega(\Omega+1)/2$, $\Omega(\Omega-1)/2$ and $\Omega(\Omega+1)/2$ respectively. The two-body Hamiltonian $V(2)$ preserving m particle spin S is defined by the symmetrized two-body matrix elements $V(2)^s_{ijkl} = {}_s\langle (kl)s, m_s \mid V(2) \mid (ij)s, m_s\rangle_s$ with the two-particle spin $s = 0,\ 1,\ 2$ and they are independent of the m_s quantum number; note that for $s = 1$ only $i \neq j$ and $k \neq l$ matrix elements exist. Thus we have

$$V(2) = V(2)^{s=0} + V(2)^{s=1} + V(2)^{s=2} \tag{2.20}$$

The sum here is a direct sum. Now, by defining the two parts of the two-body Hamiltonian to be independent GOE's in the 2-particle spaces [each one for $V(2)^{s=0}$, $V(2)^{s=1}$ and $V(2)^{s=2}$], BEGOE(2)-$S1$ for a given (m, S) system is generated and then propagating the $V(2)$ ensemble $\{V(2)\} = \{V(2)^{s=0}\} + \{V(2)^{s=1} + \{V(2)^{s=2}\}$ to the m-particle spaces with a given spin S by using the geometry (direct product structure) of the m-particle spaces; here $\{\ \}$ denotes ensemble. The embedding algebra is $U(3) \supset G \supset G1 \otimes SO(3)$ with $SO(3)$ generating spin S. For one plus two-body Hamiltonians preserving m particle spin S, the one-body Hamiltonian is given by,

$$h(1) = \sum_{i=1,2,\ldots,\Omega} \epsilon_i n_i \tag{2.21}$$

where the orbits i are three fold degenerate, n_i are number operators and ϵ_i are sp energies. Then BEGOE(1+2)- $S1$ is defined by

$$\{H\}_{\text{BEGOE(1+2)-S1}} = h(1) + \lambda_0 \{V(2)^{s=0}\} + \lambda_1 \{V(2)^{s=1}\} + \lambda_2 \{V(2)^{s=2}\} \,. \tag{2.22}$$

Here $\{V(2)^{s=0}\}$, $\{V(2)^{s=1}\}$ and $\{V(2)^{s=2}\}$ are GOE's with unit variance and λ_0, λ_1 and λ_2 are the strengths of the $s = 0$, $s = 1$ and $s = 2$ parts of $V(2)$, respectively. The mean-field one body part $h(1)$ in Eq.(2.22) is a fixed one–body operator defined by sp energies with average spacing Δ. Without loss of generality we put $\Delta = 1$ so that λ_0, λ_1 and λ_2 are in the units of Δ.

The H matrix dimension $d(m, S)$ for a given (m, S), i.e. number of states in the (m, S) space [with each of them being $(2S + 1)$-fold degenerate], is given by

$$\begin{aligned}
& d(\Omega, m, S) \\
= \; & {}_4F_3\left[\{\Omega, m^{'}, m^{'} + \frac{1}{2}, \Omega + S\}, \{\frac{1-\Omega}{2} + m^{'}, \frac{2-\Omega}{2} + m^{'}, S+1\}, 1\right] \\
& \times \frac{\Gamma[\Omega - 2m^{'}]\Gamma[\Omega + S]}{\Gamma[\Omega]^2\Gamma[1+2m^{'}]\Gamma[S+1]} \\
& - {}_4F_3\left[\left\{\Omega, m^{'} + \frac{1}{2}, m^{'} + 1, \Omega + S + 1\right\}, \left\{\frac{2-\Omega}{2} + m^{'}, \frac{3-\Omega}{2} + m^{'}, S+2\right\}, 1\right] \\
& \times \frac{\Gamma[\Omega - 2m^{'} - 1]\Gamma[\Omega + S + 1]}{\Gamma[\Omega]^2\Gamma[2m^{'}]\Gamma[S+2]} \,.
\end{aligned} \tag{2.23}$$

where $m^{'} = \frac{S-m}{2}$. They satisfy the sum rule $\sum_S (2S+1)\, d(m, S) = \binom{N+m-1}{m}$. ${}_PF_Q$ and Γ are the hyper geometric function and gamma function respectively. For $S = S_{max} = m$ second term in the above expression is taken to be zero. For example, for $m = 5$ and $\Omega = 5$ the dimensions are 126, 600, 525, 525, 224, 126 for spins S=0-5 respectively. For $m = 10$ and $\Omega = 4$, the dimensions are 714, 1260, 2100, 1855, 1841, 1144, 840, 315 and 165 for spins S=0-10 respectively. Similarly, for $m = 5$ and $\Omega = 6$, the dimensions are 336, 1386,1260,1176,504 and 252 for spins values S=0-5 respectively.

With sp energies ϵ_i and two-body matrix elements $V(2)^s_{ijkl}$, the many particle Hamiltonian matrix for a given (m, S) can be constructed using the M_S representations and a spin projection operator, S^2 as described in [64] or directly in a good S basis using angular-momentum algebra. We have employed the M_S representation for constructing the H matrices and the S^2 operator for projecting states with good S as described in [64]. The dimension of this basis space is $\sum_S d(m, S)$. for example for ($m = 10, \Omega = 4$) we have $\sum_S d(m, S)$=10234, for ($m = 5, \Omega = 5$) we have $\sum_S d(m, S)$=2126, for ($m = 8, \Omega = 6$) we have $\sum_S d(m, S)$=155217, for ($m = 5, \Omega = 6$) we have

$\sum_S d(m,S)$=4914. In order to construct the many particle Hamiltonian matrix for a given (m,S), first the single particle states $|i, m_\mathbf{s}=0, \pm 1\rangle$ are arranged in such a way that the first Ω states have $m_\mathbf{s}=1$, the next Ω states have $m_\mathbf{s}=0$ and the remaining Ω having $m_\mathbf{s}=-1$ so that a state $|r\rangle = |i=\Omega, m_\mathbf{s}=1\rangle$ for $r \leq \Omega$, for $\Omega < r \leq 2\Omega$, $|r\rangle = |i=r-\Omega, m_\mathbf{s}=0\rangle$ and for $r > 2\Omega$ $|r\rangle = |i=r-\Omega, m_\mathbf{s}=-1\rangle$. Now the m-particle configurations $\mathbf{m}$ due to product nature of the states in occupation number representation, is denoted by

$$\left|\prod_{r=1}^{N=3\Omega} m_r\right\rangle = |m_1, m_2, \ldots, m_\Omega, m_{\Omega+1}, m_{\Omega+2}, \ldots, m_{2\Omega}, m_{2\Omega+1}, m_{2\Omega+2}, \ldots, m_{3\Omega}\rangle\,, \quad (2.24)$$

where m_r can take values between 0 to m with $\sum_r^N m_r = m$. The corresponding m_S values are,

$$m_S = \left[\sum_{r=1}^{\Omega} m_r - \sum_{r'=2\Omega+1}^{3\Omega} m_{r'}\right]. \quad (2.25)$$

To proceed further, the (1+2)-body Hamiltonian defined by $(\epsilon_i, V(2)_{ijkl}^{s=0,1,2})$'s is converted into the $|i, m_\mathbf{s}=0, \pm 1\rangle$ basis by changing ϵ_i to ϵ_r with the index r defined as above and changing $V(2)_{ijkl}^{s=0,1,2}$ to $V_{im_i,jm_j,km_k,lm_l} = \langle im_i, jm_j|\, V(2)\,|\, km_k, lm_l\rangle$ using the following equations

$$\begin{aligned}
&V_{i1,j1,k1,l1} = V_{i-1,j-1,k-1,l-1} = V_{ijkl}^{s=2} \\
&V_{i0,j0,k0,l0} = \frac{1}{3}\{V_{ijkl}^{s=0} + 2V_{ijkl}^{s=2}\} \\
&V_{i0,j1,k0,l1} = V_{i0,j-1,k0,l-1} = \frac{\sqrt{(1+\delta_{ij})(1+\delta_{kl})}}{2}\{V_{ijkl}^{s=1} + V_{ijkl}^{s=2}\} \\
&V_{i-1,j1,k-1,l1} = \frac{\sqrt{(1+\delta_{ij})(1+\delta_{kl})}}{6}\{2V_{ijkl}^{s=0} + 3V_{ijkl}^{s=1} + V_{ijkl}^{s=2}\} \\
&V_{i-1,j1,k0,l0} = \frac{\sqrt{(1+\delta_{ij})}}{3}\{V_{ijkl}^{s=2} - V_{ijkl}^{s=0}\}
\end{aligned} \quad (2.26)$$

with all other matrix elements being zero except for the symmetries,

$$V_{im_i,jm_j,km_k,lm_l} = V_{km_k,lm_l,im_i,jm_j} = V_{jm_j,im_i,lm_l,km_k} = V_{im_i,jm_j,lm_l,km_k}\,. \quad (2.27)$$

Using $(\epsilon_r, V_{im_i,jm_j,km_k,lm_l})$'s, construction of the m particle H matrix in the basis defined by Eq. (2.24) reduces to the problem of BEGOE(1+2) for spinless boson systems. For the S^2 operator, it is easy to recognize that $\epsilon_i = 2$ independent of i, $V^{s=0}_{ijij} = -4$, $V^{s=1}_{ijij} = -2$ and $V^{s=0}_{ijij} = 2$ independent of (ij) and all other V^s are zero. Using these, for the S^2 operator, the m particle matrix with $m_S = 0$ is constructed and diagonalized. This gives a direct sum of unitary matrices and the unitary matrix that corresponds to a given S is identified by the eigenvalue $S(S+1)$. Applying the unitary transformation defined by this unitary matrix, the m particle H matrix with $m_S = 0$ is transformed to the basis with good S values.