Chapter 4

Random k-body Ensembles and q-Hermite Polynomials

4.1 Introduction

We have seen in chapter 2 that EGOE(1+2) models are paradigmatic models to study the dynamical transition from integrability to chaos in isolated finite interacting many-body quantum systems. These models initially developed for isolated finite interacting spinless fermion and boson systems, are now studied in detail with spin degree of freedom for both fermion and boson systems. In these investigations, the main focus was on one- plus two-body part of the interaction as inter-particle interaction is known to be only one-body and two-body in nature. However, it is seen that the higher body interactions $k > 2$ play an important role in strongly interacting quantum systems [71,72], nuclear physics [73], quantum black holes [35,74] and wormholes [75] with SYK model and also in quantum transport in disordered networks connected by many-body interactions [76–78]. Also, it is found that the three body interactions play an important role in saturation of nuclear matter [139], super fluidity properties of neutron matter and neutron stars [140],quark dynamics [141] and so on. Therefore, it is necessary to extend the analysis of EE with two-body interactions (discussed in chapter 2) to higher k-body interactions in order to address these problems. They are represented by EGOE(k) (or BEGOE(k)) for fermion (or boson) systems. In the presence of mean-field they are represented by EGOE(1+k) (or BEGOE(1+k)) for fermion (or boson) systems.

Very recently, it is found that q-Hermite polynomials can be used to study spectral densities of the so-called SYK model [80, 81] and quantum spin glasses [82], along with studying the strength functions and fidelity decay (also known as survival or return probability) in EE, both for fermion as well as boson systems [79]. This is because of the fact that the generating function of q-Hermite polynomials exhibits Gaussian to semi-circle transition, which is also exhibited by the spectral densities of these systems. The q-Hermite polynomials were first introduced by L. J. Rogers in Mathematics to prove the Rogers–Ramanujan identities [142] and Szego and Carlitz studied their important properties. They are related with the Chebyshev, Rogers-Szego, Al-Salam-Chihara polynomials and other polynomials [143]. Now, the reason these polynomials are important is that they are very simple as they have only one parameter and other complicated families of orthogonal polynomials (i.e having more than one parameter) can be expressed as linear combinations of q-Hermite polynomials. Over the years lot of developments took place as a result of which q-Hermite polynomials have recently found applications in non-commutative probability, quantum physics, combinatorics and so on [143]. Recently, also the q-calculus has attracted many researchers working in the field of special functions as it is a very powerful tool in quantum computation [144].

In this chapter, firstly we define and describe the construction of EGOE(k) (or BEGOE(k)) for fermions (or bosons) and EGOE(1+k) (or BEGOE(1+k)). Then we introduce q-Hermite polynomials along with their generating function and recurrence relations. Also the so-called q-normal distribution f_{qN}, conditional q-normal distribution f_{CqN} and bivariate q-normal distribution f_{biv-qN} are discussed. The analytical formula of q considering only the one-body part is derived for both fermions and bosons. Also, the formulae of parameter q in terms of m, N and k for both EGOE(k) and BEGOE(k) derived in [79] are given for completeness. Furthermore, the variation of parameter q is studied as the interaction strength λ varies in EGOE(1+k) (or BEGOE(1+k)) for a fixed body rank k. Further, use all this knowledge of q-Hermite polynomials to study the spectral density for EGOE(1+k) and BEGOE(1+k). Lastly we give concluding remarks. The work on bosons presented in this chapter is based on [30] and that of fermions is under preparation to be published.

4.2 Embedded Ensembles with k-body Interactions

In this section we define and describe the construction of k-body embedded ensembles for fermionic and bosonic systems. Throughout this thesis, we consider the orthogonal symmetry i.e. GOE embedding. For fermionic systems these ensembles are called EGOE(k) (and EGOE(1+k) in the presence of mean-field) and for bosonic systems they are called BEGOE(k)(and BEGOE(1+k) in the presence of mean-field). From the previous section we know that $k = 2, 3, 4$ are of physical importance in nuclear reactions and strongly interacting quantum systems [35,71,72]. However for the sake of completeness, to study the generic features of embedded ensembles and the possibility of higher k becoming prominent in future, we address $k = 2$ to $k = m$. Initially, a k-particle Hamiltonian matrix is constructed. Employing the concepts of direct product space and Lie algebra, this k-particle Hamiltonian matrix is further embedded to the m-particle space. Here, the information in k-particle space is propagated to m-particle space using Lie algebra. Now let us see how these ensembles are constructed.

4.2.1 Construction of EGOE(k) and EGOE(1+k)

Now let us see how we can define and construct EGOE(k) and EGOE(1+k). Consider m spinless fermions distributed in N degenerate sp states interacting via k-body ($1 \leq k \leq m$) interactions. Distributing these m fermions in all possible ways in N sp states generates many-particle basis of dimension $d = \binom{N}{m}$. The k-body random Hamiltonian $V(k)$ is defined as,

$$V(k) = \sum_{k_a, k_b} V_{k_a,k_b} F^\dagger(k_a) F(k_b) \; . \tag{4.1}$$

Here V_{k_a,k_b} are the antisymmetrized matrix elements of $V(k)$ in the k-particle space with the matrix dimension being $d_k = \binom{N}{k}$. Here the term V_{k_a,k_b} represents randomly distributed independent Gaussian variables with zero mean and unit variance,

$$\overline{V_{k_a,k_b} V_{k_{a}{'},k_{b}{'}}} = \nu_0^2 (1 + \delta_{k_a,k_{a}{'},k_b,k_{b}{'}}) \tag{4.2}$$

In other words, k-body Hamiltonian is chosen to be a GOE. Here, the overbar denotes the ensemble average and $\nu_0 = 1$ without the loss of generality. For fermions, $F^\dagger(k_a) = f^\dagger_{n_1} f^\dagger_{n_2}$ and $F(k_a) = (F^\dagger(k_a))^\dagger$ $(n_1 < n_2)$. Also, $f^\dagger_{n_i}$ and f_{n_i} are the fermionic creation and annihilation operators respectively. EGOE(k) is generated by action of $V(k)$ on the many-particle basis states. Due to k-body nature of interactions, there will be zero matrix elements in the many-particle Hamiltonian matrix, unlike a GOE. By construction, we have a GOE for the case $k = m$. In realistic systems, fermions also experience mean-field generated by presence of other fermions in the system and hence, it is more appropriate to model these systems by EGOE($1 + k$) defined by,

$$H = h(1) + \lambda V(k) \tag{4.3}$$

Here, the one-body operator $h(1) = \sum_{i=1}^{N} \epsilon_i n_i$ is described by fixed sp energies ϵ_i; n_i is the number operator for the ith sp state. The parameter λ represents the strength of the k-body interaction and it is measured in units of the average mean spacing of the sp energies defining $h(1)$. In this chapter as well as in the next two chapters we have employed fixed sp energies $\epsilon_i = i + 1/i$ in defining the mean-field Hamiltonian $h(1)$.

4.2.2 Construction of BEGOE(k) and BEGOE(1+k)

In the previous section we have described the construction of fermionic k-body embedded ensembles. Now let us define and describe the construction of BEGOE(k) and BEGOE(1+k). Consider a system which contains m spinless bosons interacting via k-body ($1 \leq k \leq m$) interactions, which occupy N degenerate sp states. When we distribute these m bosons in all possible ways in N sp states, it generates a many-particle basis of dimension $d = \binom{N+m-1}{m}$. The k-body random Hamiltonian $V(k)$, for such a system is given as,

$$V(k) = \sum_{k_a,k_b} V_{k_a,k_b} B^\dagger(k_a) B(k_b) \; . \tag{4.4}$$

Here, operators $B^\dagger(k_a)$ and $B(k_b)$ are k-boson creation and annihilation operators respectively. They obey the boson commutation relations. V_{k_a,k_b} are the symmetrized matrix elements of $V(k)$ in the k-particle space and are chosen to be Gaussian random variables with zero mean and unit variance. This means that the k-body Hamiltonian is chosen to be a GOE. The Hamiltonian $V(k)$ in the k-particle space has the matrix dimension $d_k = \binom{N+k-1}{k}$. BEGOE(k) is generated by action of $V(k)$ on the many-particle basis states. The presence of k-body interactions, gives rise to a many-particle Hamiltonian matrix containing zero matrix elements, unlike a GOE. However, the case $k = m$ is a GOE by construction. For further details about these ensembles, their extensions and applications, see [31, 145, 146] and references therein.

From section 4.2.1, we know that realistic systems involve an additional one-body mean-field part and the appropriate model is given in section 4.2.1. Similarly the appropriate model for bosonic systems in the presence of mean field is defined by Eq. (4.3) with $V(k)$ given by Eq. (4.4).

4.3 q-Hermite Polynomials and Conditional q-Normal Distribution

In this section, firstly the q-Hermite polynomials, q-normal distribution f_{qN}, conditional q-normal distribution f_{CqN} and bivariate q-normal distribution f_{biv-qN} are defined and then their basic properties are discussed. These definitions and properties are used in section 4.5 of this chapter to describe the spectral density of EE(k), in chapter 5 to describe the strength functions of EE(k) and in chapter 6 to describe NPC, information entropy and fidelity decay.

Let us begin by defining q-numbers and q-factorials which define the q-Hermite polynomials. The q-numbers $[n]_q$ are defined as $[n]_q = (1-q)^{-1}(1-q^n)$. We have $[n]_{q \to 1} = n$. Using the above definition, we can write down the first few q numbers which are as follows,

$$[0]_q = 0, [1]_q = 1, [2]_q = 1+q, [3]_q = 1+q+q^2, [4]_q = 1+q+q^2+q^3. \tag{4.5}$$

The q-factorials $[n]_q!$ are defined as $[n]_q! = \Pi_{j=1}^{n}[j]_q$. We have $[0]_q! = 1$. Now, using the above definitions of q-numbers and q-factorials, q-Hermite polynomials $H_n(x|q)$ are defined by the recursion relation [142],

$$x\, H_n(x|q) = H_{n+1}(x|q) + [n]_q \; H_{n-1}(x|q) \tag{4.6}$$

with $H_0(x|q) = 1$ and $H_{-1}(x|q) = 0$.

From the above recursion relation we can obtain the first few q-Hermite polynomials which are as follows,

$$\begin{aligned}
&H_1(x|q) = x \\
&H_2(x|q) = x^2 - 1 \\
&H_3(x|q) = x^3 - (2+q)x \\
&H_4(x|q) = x^4 - (3+2q+q^2)x^2 + (1+q+q^2) \\
&H_5(x|q) = x^5 - (4+3q+2q^2+q^3)x^3 + (3+4q+4q^2+3q^3+q^4)x
\end{aligned} \tag{4.7}$$

Note that for $q = 1$, the q-Hermite polynomials reduce to normal Hermite polynomials (related to Gaussian) and for $q = 0$ they will reduce to Chebyshev polynomials (related to semi-circle). Importantly, q-Hermite polynomials are orthogonal within the limits $\pm 2/\sqrt{1-q}$, with the q-normal distribution $f_{qN}(x|q)$ as the weight function. Now let us define the q-normal distribution $f_{qN}(x|q)$ using a standardized variable x (i.e. x is zero centered and has unit variance) [83],

$$f_{qN}(x|q) = \frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}} \prod_{i=0}^{\infty}(1-q^{i+1})[(1+q^i)^2-(1-q)q^i x^2]. \tag{4.8}$$

Here, $-2/\sqrt{1-q} \leq x \leq 2/\sqrt{1-q}$. In principal q can take any value from [-1,1]. Throughout this thesis we take $q \in [0,1]$. Note that $\int_{s(q)} f_{qN}(x|q)\, dx = 1$ over the range $s(q) = (-2/\sqrt{1-q}, 2/\sqrt{1-q})$. It is seen that in the limit $q \to 1$, $f_{qN}(x|q)$ will take Gaussian form given by

$$f_{qN}(x|1) = \frac{1}{\sqrt{2\pi}} exp \frac{-x^2}{2}. \tag{4.9}$$

In the limit $q = 0$ $f_{qN}(x|q)$ will take semi-circle form given by

$$f_{qN}(x|0) = \frac{1}{2\pi}\sqrt{4-x^2}. \tag{4.10}$$

Another important property of $f_{qN}(x|q)$ is that the q-Hermite polynomials are orthogonal with respect to the weight function over the range $s(q)$ which can be inferred from

$$\int_{s(q)} H_n(x|q) H_m(x|q) f_{qN}(x|q)\, dx = [n]_q! \delta_{mn}. \tag{4.11}$$

Now having defined the q-Hermite polynomials and $f_{qN}(x|q)$, let us proceed further by defining the bivariate q-normal distribution $f_{biv-qN}(x,y|\zeta,q)$ using two standardized variables x and y.

Then, $f_{biv-qN}(x,y|\zeta,q)$ is defined as follows [83, 147],

$$\begin{aligned} f_{biv-qN}(x,y|\zeta,q) &= f_{qN}(x|q) f_{CqN}(y|x;\zeta,q) \\ &= f_{qN}(y|q) f_{CqN}(x|y;\zeta,q) \end{aligned} \tag{4.12}$$

where ζ is the bivariate correlation coefficient and the conditional q-normal densities,

f_{CqN} can be given as,

$$\begin{aligned} f_{CqN}(x|y;\zeta,q) &= f_{qN}(x|q)\prod_{i=0}^{\infty}\frac{(1-\zeta^2 q^i)}{h(x,y|\zeta,q)}; \\ f_{CqN}(y|x;\zeta,q) &= f_{qN}(y|q)\prod_{i=0}^{\infty}\frac{(1-\zeta^2 q^i)}{h(x,y|\zeta,q)}; \\ h(x,y|\zeta,q) &= (1-\zeta^2 q^{2i})^2-(1-q)\zeta q^i(1+\zeta^2 q^{2i})xy+(1-q)\zeta^2(x^2+y^2)q^{2i}. \end{aligned} \tag{4.13}$$

In the limit $q \to 1$, $f_{CqN}(x|y;\zeta,q)$ takes the form

$$f_{CqN}(x|y;\zeta,1)=\frac{1}{2\pi\sqrt{(1-\zeta^2)}}exp-\frac{(x-\zeta y)^2}{2(1-\zeta^2)} \tag{4.14}$$

In the limit $q=0$, $f_{CqN}(x|y;\zeta,q)$ takes the form

$$f_{CqN}(x|y;\zeta,0)=\frac{(1-\zeta^2)\sqrt{4-x^2}}{2\pi[(1-\zeta^2)^2-\zeta(1+\zeta^2)xy+\zeta^2(x^2+y^2)]} \tag{4.15}$$

The f_{CqN} and f_{biv-qN} are normalized to 1 over the range $s(q)$, which can be inferred from the following property,

$$\int_{s(q)} H_n(x|q)f_{CqN}(x|y;\zeta,q)\,dx=\zeta^n H_n(y|q). \tag{4.16}$$

Now we have enough knowledge of the conditional q-normal distribution so let us proceed with the first four moments of f_{CqN}. They are the centroid, variance, skewness (γ_1) and excess (γ_2). The generalized formula for moments of all orders of f_{CqN} has been given in [147]. However, the formula given in [147] is a complicated one. Recently, the simplified formulae for the first four moments of f_{CqN} were derived using Eq. (4.16) in [84]. The first four moments of f_{CqN} can be obtained by evaluating

$$\int_{s(q)} y^n f_{CqN}(x|y;\zeta,q)dy. \tag{4.17}$$

The first four moments obtained are as follows,

$$\begin{aligned}
\text{Centroid} &= \zeta y, \\
\text{Variance} &= 1-\zeta^2\,, \\
\text{Skewness, }\gamma_1 &= -\frac{\zeta(1-q)y}{\sqrt{1-\zeta^2}}, \\
\text{Excess, }\gamma_2 &= (q-1)+\frac{\zeta^2(1-q)^2y^2+\zeta^2(1-q^2)}{(1-\zeta^2)}\,.
\end{aligned} \tag{4.18}$$

4.4 Formula of q-parameter

4.4.1 Formula of $q_{h(1)}$

In this section, we derive the analytical formula of q for bosons as well as fermions when the Hamiltonian consists of only the one-body part $h(1)$ i.e. considering $\lambda = 0$ in Eq.(4.3). This analytical formula is derived based on the trace propagation method introduced in [148].

We start with the reduced fourth moment of one-body part denoted by $\langle h(1)^4\rangle^m$ and which is defined as [148],

$$\langle h(1)^4\rangle^m = \frac{\langle h(1)^4\rangle^m}{(\langle h(1)^2\rangle^m)^2} \tag{4.19}$$

Here, $\langle h(1)^2\rangle^m$ is the second moment of one-body part and $\langle h(1)^4\rangle^m$ is the fourth moment of one-body part.

4.4.1.1 For Fermion System:

First let us derive the analytical formula of $q_{h(1)}$ for fermions from the reduced fourth moment of one-body part. For fermions $\langle h(1)^2\rangle^m$ and $\langle h(1)^4\rangle^m$ are expressed as follows,

$$\langle h(1)^2\rangle^m = \frac{m(N-m)}{N(N-1)}\sum_{i=1}^{N}\tilde{\epsilon_i}^2$$

$$\begin{aligned}\langle h(1)^4\rangle^m &= \frac{m(N-m)}{N(N-1)}\sum_{i=1}^{N}\tilde{\epsilon_i}^4 \\ &+\frac{m(m-1)(N-m)(N-m-1)}{N(N-1)(N-2)(N-3)} \\ &\times[3(\textstyle\sum_{i=1}^{N}\tilde{\epsilon_i}^2)^2 - 6\sum_{i=1}^{N}\tilde{\epsilon_i}^4].\end{aligned} \tag{4.20}$$

Using Eq. (4.20), we can obtain the formula of $q_{h(1)}$ as follows,

$$\begin{aligned} q_{h(1)} &= \langle h(1)^4\rangle^m - 2 \\ &= \left\{\frac{3(m-1)N(N-1)(N-m-1)}{m(N-2)(N-3)(N-m)} - 2\right\} \\ &+\frac{N(N-1)[N^2+N-6mN+6m^2]}{m(N-m)(N-2)(N-3)}\frac{\sum_{i=1}^{N}\tilde{\epsilon_i}^4}{(\sum_{i=1}^{N}\tilde{\epsilon_i}^2)^2}.\end{aligned} \tag{4.21}$$

Here $\tilde{\epsilon_i}$ are the traceless sp energies of i'th state. Figs. 4.1 and 4.2 represent $q_{h(1)}$ results as a function of m for fermion systems. The results are obtained by considering different values of m/N using Eq. (4.21). In Fig. 4.1 sp energies $\epsilon_i = i$ are used and in Fig. 4.2 sp energies $\epsilon_i = i + 1/i$ are used. In both these figures the dilute limit curve represents results for $m/N = 0.001$.

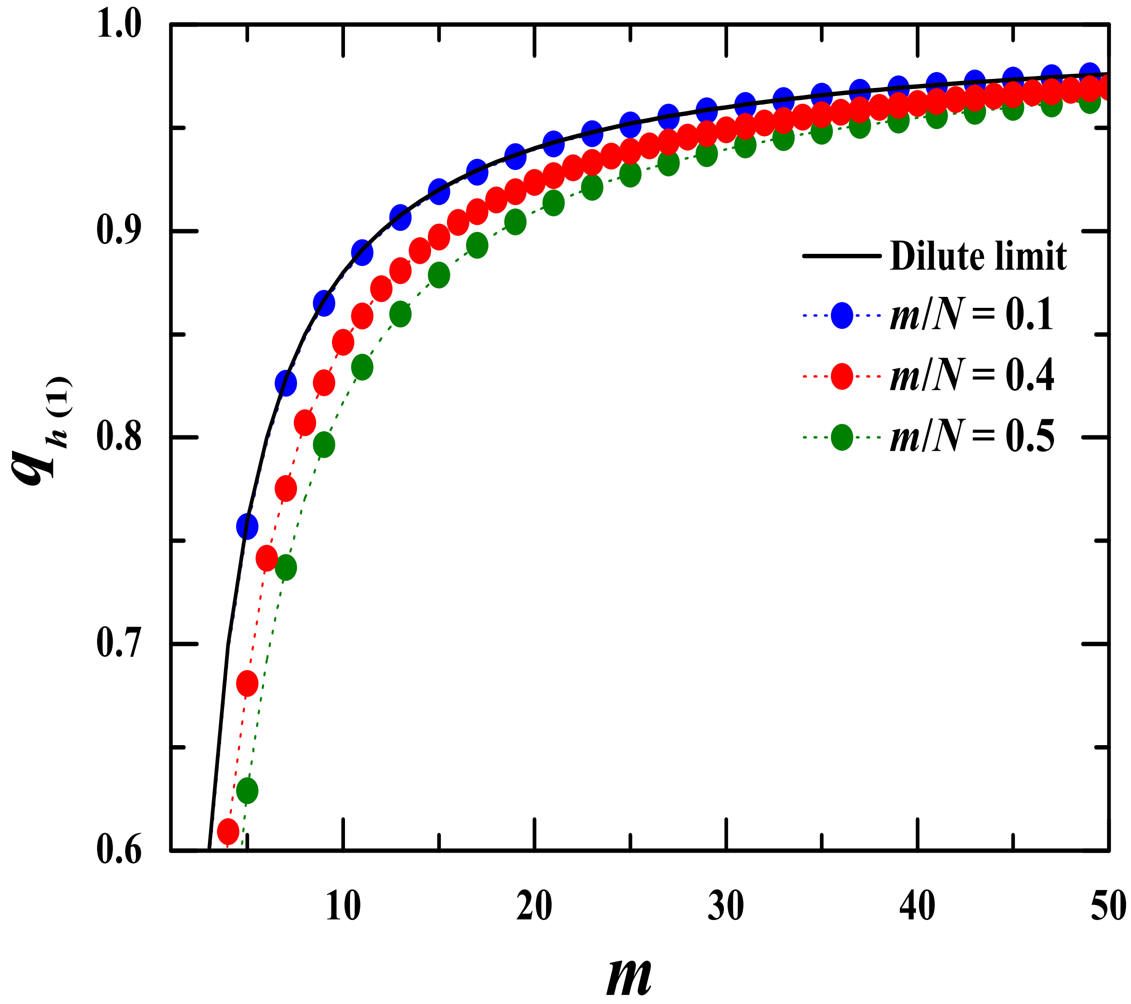

Figure 4.1: Variation of $q_{h(1)}$ as a function of m for various values of m/N for fermion system. Here, sp energies $\epsilon_i = i$ are used.

4.4.1.2 For Boson System:

Moving further let us derive the analytical formula of $q_{h(1)}$ for bosons. For bosons, $\langle h(1)^2\rangle^m$ and $\langle h(1)^4\rangle^m$ are expressed as follows [148],

$$\langle h(1)^2\rangle^m = \frac{m(N+m)}{N(N+1)} \sum_{i=1}^{N} \tilde{\epsilon}_i^{\,2}$$

$$\langle h(1)^4\rangle^m = \frac{m(N+m)}{N(N+1)} \sum_{i=1}^{N} \tilde{\epsilon}_i^{\,4} \tag{4.22}$$

$$+\frac{m(m-1)(N+m)(N+m+1)}{N(N+1)(N+2)(N+3)}$$

$$\times[3(\textstyle\sum_{i=1}^{N} \tilde{\epsilon}_i^{\,2})^2 + 6\sum_{i=1}^{N} \tilde{\epsilon}_i^{\,4}].$$

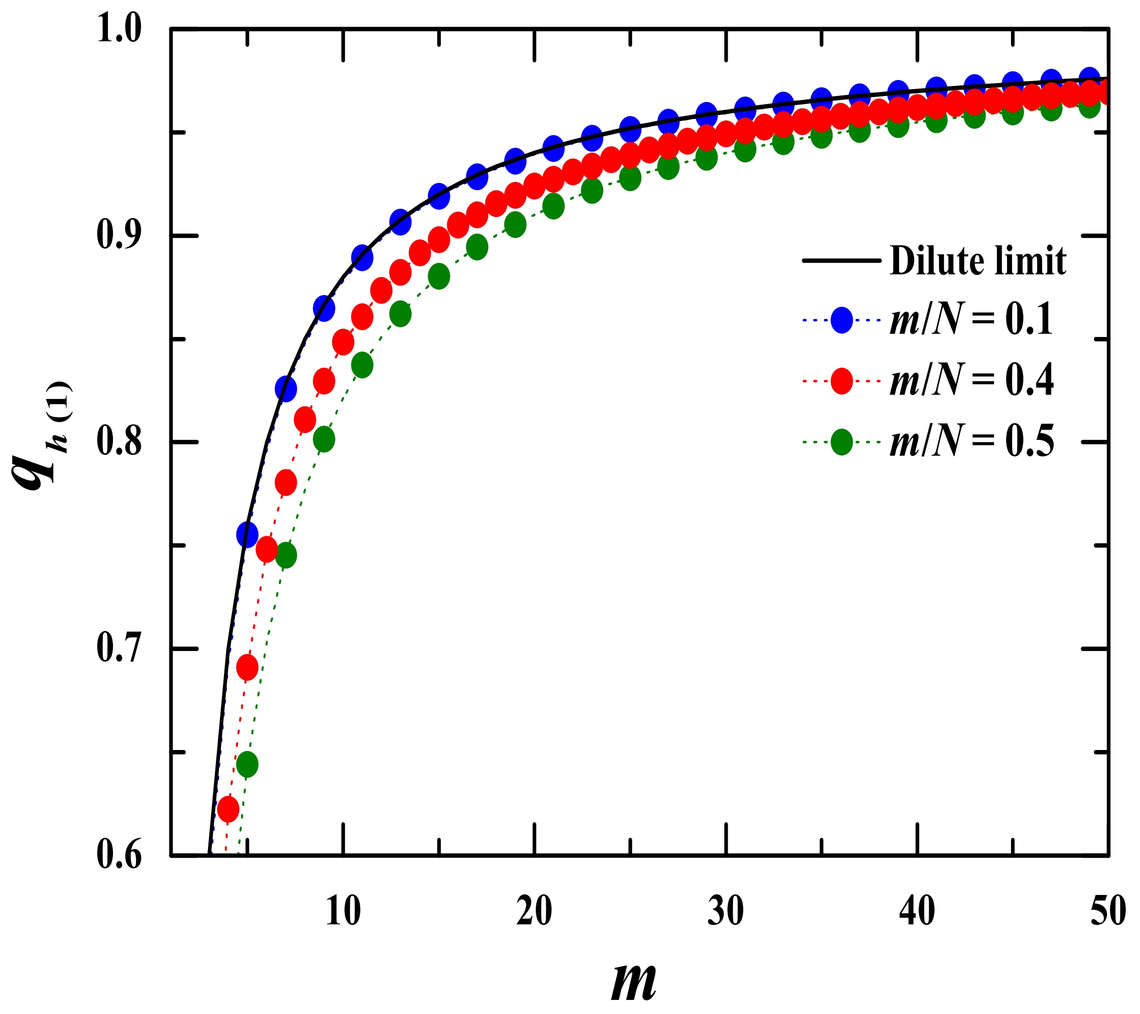

Figure 4.2: Variation of $q_{h(1)}$ as a function of m for various values of m/N for fermion system. Here, sp energies $\epsilon_i = i + 1/i$ are used.

Using Eq.(4.22), we can obtain the formula of $q_{h(1)}$ as follows,

$$\begin{aligned} q_{h(1)} &= \langle h(1)^4 \rangle^m - 2 \\ &= \left\{ \frac{3(m-1)N(1+N)(1+m+N)}{m(2+N)(3+N)(m+N)} - 2 \right\} \\ &+ \frac{m^2+(N+m)^2+(N+2m)^2}{m(N+m)} \frac{\sum_{i=1}^{N} \tilde{\epsilon_i}^4}{(\sum_{i=1}^{N} \tilde{\epsilon_i}^2)^2}. \end{aligned} \tag{4.23}$$

Here $\tilde{\epsilon_i}$ are the traceless sp energies of i'th state. Considering $H = h(1)$ and taking the sp energies to be uniform i.e. $\epsilon_i = i$, for example ($m = 5, N = 10$) Eq.(4.23) gives $q = 0.71$ and for example ($m = 10, N = 5$) it gives $q = 0.68$. While with sp energies used in the present study i.e. $\epsilon_i = i + 1/i$, one obtains $q = 0.68$ for example ($m = 5, N = 10$) and $q = 0.63$ for example ($m = 10, N = 5$). In Fig. 4.3 we present the results of variation of $q_{h(1)}$ as a function of N for various values of m/N for bosonic systems. Here, sp energies

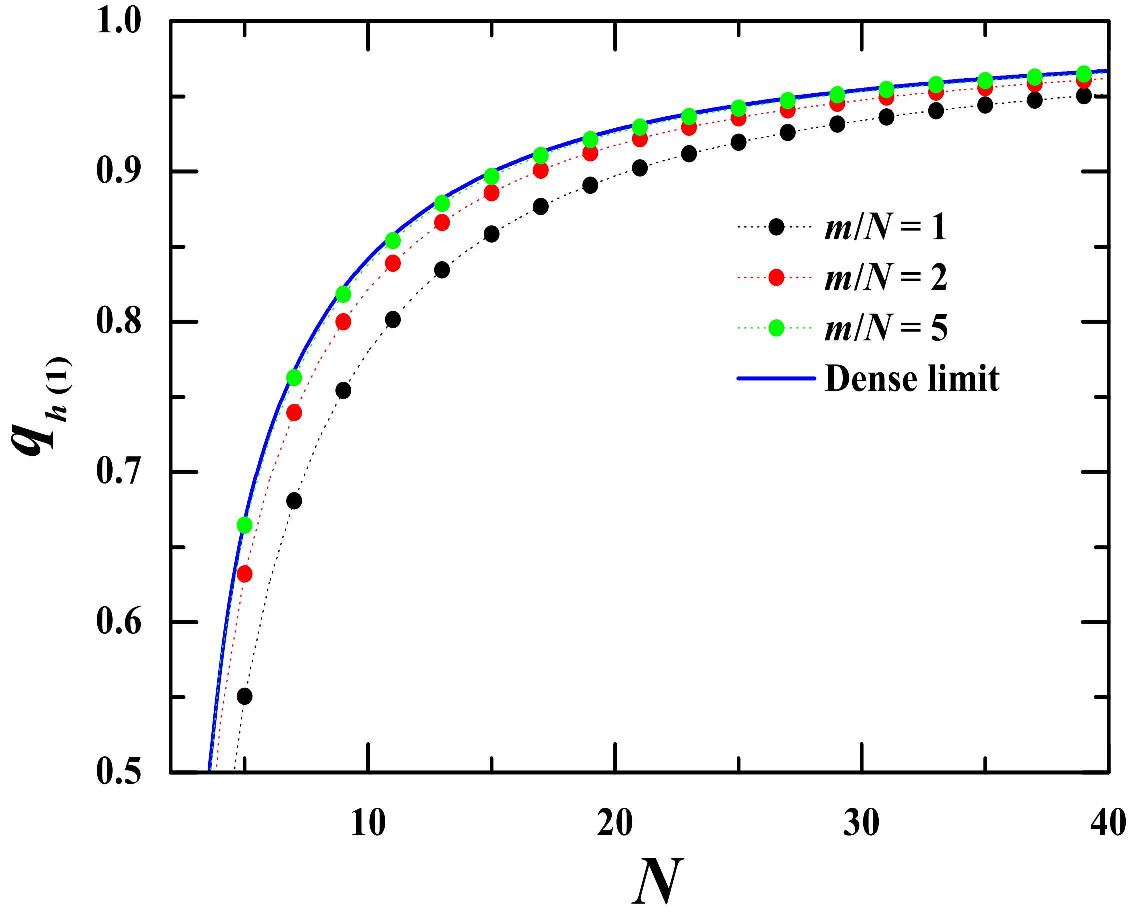

Figure 4.3: $q_{h(1)}$ vs. N for various values of m/N for boson system. The values of $q_{h(1)}$ are obtained using Eq. (4.23).

$\epsilon_i = i + 1/i$ are used. The dense limit curve represents results for $m/N = 1000$. It can be clearly seen that in the dense limit ($m \to \infty$, $N \to \infty$ and $m/N \to \infty$), $q_{h(1)} \to 1$. In the dilute limit ($m \to \infty$, $N \to \infty$ and $m/N \to 0$), similar variation in $q_{h(1)}$ can be observed due to $m \leftrightarrow N$ symmetry between the dense limit and the dilute limit as identified in [59, 148].

4.4.2 Formula of $q_{V(k)}$

In the previous section, we derived the formula of q for bosons as well as fermions considering only the one-body part in Eq.(4.3). In this section we discuss the formulae of q valid in strong interaction domain, $q_{V(k)}$ derived in [79] only for the sake of completeness. Here strong interaction means λ in Eq.(4.3) is sufficiently high and hence the k-body part of the Hamiltonian dominates over the one-body part.

4.4.2.1 For Fermion Systems:

The formula for q using finite N corrections to the fourth order moment for EGOE(k) given in [79] is as follows,

$$
\begin{aligned}
& q_{V(k)} \sim F(N,m,k)/[T(N,m,k)]^2 \\
& T(N,m,k) = \binom{m}{k}\left[\binom{N-m+k}{k}+1\right], \\
& F(N,m,k) = \binom{m}{k}^2 + \sum_{s=0}^{k}\binom{m-s}{k-s}^2\binom{N-m+k-s}{k} \times \binom{m-s}{k}\binom{N-m}{s}\binom{m}{s} \\
& \times\left[\frac{N-2s+1}{N-s+1}\right]\binom{N-s}{k}^{-1}\binom{k}{s}^{-1}\left\{2+\binom{N+1}{s}\right\}
\end{aligned}
\tag{4.24}
$$

4.4.2.2 For Boson Systems:

The formula of q for BEGUE(k) can be used for BEGOE(k) as well to a good approximation which is given by [79]

$$
\begin{aligned}
& q_{V(k)} \sim \binom{N+m-1}{m}^{-1} \sum_{\nu=0}^{\nu_{max}=\min[k,m-k]} \frac{X(N,m,k,\nu)\; d(g_\nu)}{\left[\Lambda^0(N,m,k)\right]^2}\;; \\
& X(N,m,k,\nu) = \Lambda^\nu(N,m,m-k)\;\Lambda^\nu(N,m,k)\;; \\
& \Lambda^\nu(N,m,r) = \binom{m-\nu}{r}\binom{N+m+\nu-1}{r}, \\
& d(g_\nu) = \binom{N+\nu-1}{\nu}^2 - \binom{N+\nu-2}{\nu-1}^2.
\end{aligned}
\tag{4.25}
$$

Further, in the strong interaction domain, one can also apply Eqs.(4.24) and (4.25) to EGOE(1+k) and BEGOE(1+k) respectively. This is due to the fact that the k-body part of the interaction is expected to dominate over one-body part.

4.4.3 Variation of q as a Function of λ

Up till now we have discussed about the formula of q by considering only the one-body part of the Hamiltonian and also the formula of q in strong interaction domain. Now let us see how the value of parameter q varies as a function of the interaction strength λ.

4.4.3.1 Results for Fermion Systems:

First let us study how the parameter q varies with λ in EGOE(1+k) for a particular body rank of interaction k. For this study we consider the following two examples: (i) a 100 member EGOE(1+k) ensemble with $m = 6$ fermions distributed in $N = 12$ sp states and (ii) a 20 member EGOE(1+k) ensemble with $m = 7$ fermions distributed in $N = 14$ sp states. We construct EGOE(1+k) ensemble for these examples and use its eigenvalues to compute the ensemble averaged value of q for various values of k. Fig. 4.4 and Fig. 4.5 represent these results. In these figures the horizontal marks on the left correspond to value of parameter q for the case $H = h(1)$ and that on the right correspond to the case $H = V(k)$ respectively. One can observe that when the value of λ is very small, the ensemble averaged values of q are found very close to the values of $q_{h(1)}$. Now as we gradually increase λ and reach a sufficiently large value of λ, the ensemble averaged q values approach the corresponding $q_{V(k)}$ values given by Eq.(4.24). In the case when λ is sufficiently large, the k-body part of the Hamiltonian dominates over the one-body part. This trend is true for all body rank k.

4.4.3.2 Results for Boson Systems:

Moving further, let us see the variation of the parameter q as a function of λ in BEGOE(1+k) for a fixed body rank k. For this study we consider the example of $m = 10$ bosons occupying $N = 5$ sp states. The ensemble averaged value of q is computed for this example using 100 member BEGOE(1+k) ensemble. The results are presented in Fig. 4.6. q estimates are also shown in the figure by horizontal marks for $H = h(1)$ and $H = V(k)$ on left and right vertical axes respectively. One can see that for very small values of λ, ensemble averaged q values are found very close to $q_{h(1)}$ for all body rank k. While for a sufficiently large λ, where k-body part dominates over one-body part, the ensemble averaged q values approach the corresponding $q_{V(k)}$ values given by Eq.(4.25).

From the variation of ensemble averaged q values in Figs. 4.4, 4.5 and 4.6, one can see that the shape of the state density takes intermediate form between Gaussian to semi-circle as λ changes in both EGOE(1+k) and BEGOE(1+k) for a fixed k. Therefore, the q-normal distribution f_{qN} formula can be used to describe the transition in the state density with any value of λ and k in both EGOE(1+k) and BEGOE(1+k).

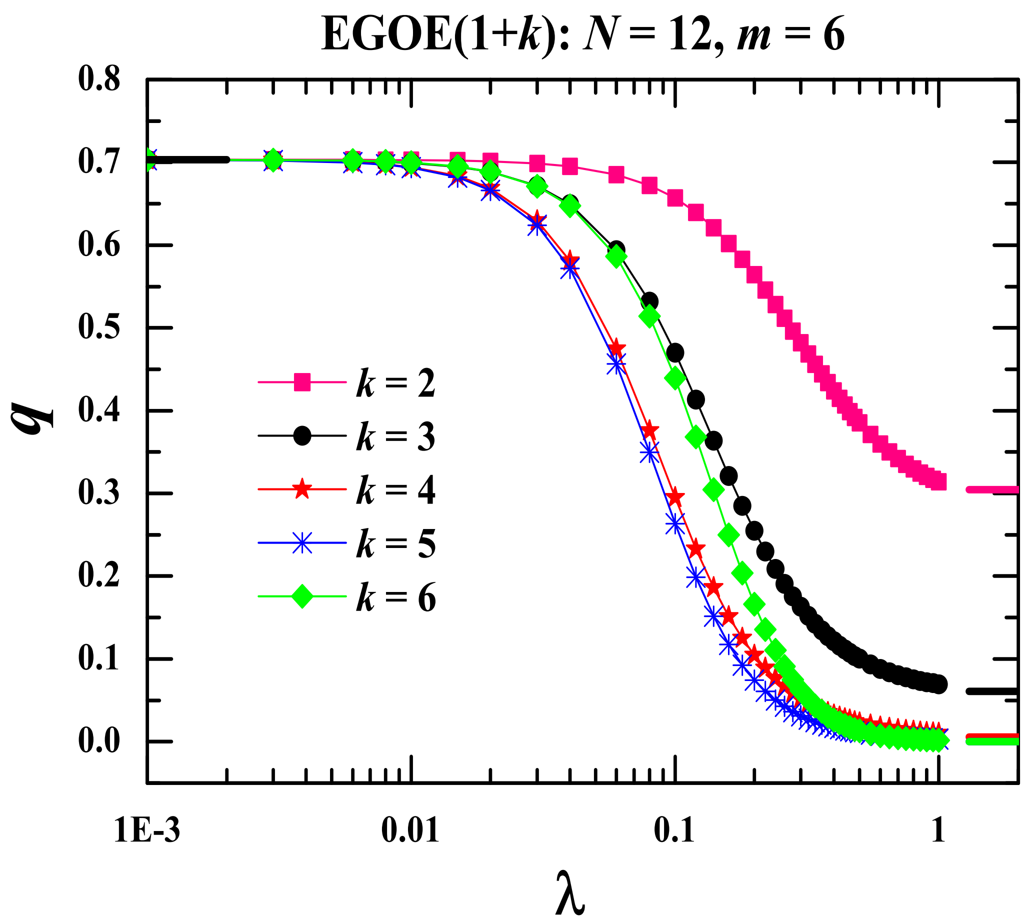

Figure 4.4: Variation of ensemble averaged q as a function of λ for EGOE(1+k) ensemble by distributing $m = 6$ fermions in $N = 12$ sp states for $k = 2$ to $k = m = 6$. An ensemble of 100 members is considered. See text for more details.

4.5 Spectral Density

In order to know the distribution of energy between identical particles in a quantum system, it is important to have the knowledge of number of available states in a given energy interval. The spectral density (or eigenvalue density or state density) gives us this information.

From the past studies we know that the spectral density for EE(k)(and also EE(1+k)) in general exhibits Gaussian to semi-circle transition as k increases from 1 to m [58]. This is now well verified in many numerical calculations and analytical proofs obtained via lower order moments [10, 31, 34, 61, 145, 149]. This is known for both EGOE(k)(also for EGOE(1+k)) and BEGOE(k)(also for BEGOE(1+k)) for a system of m fermions/bosons distributed in N sp states [31, 34, 55, 79].

Very recently it was shown that the smooth form of spectral density can be described

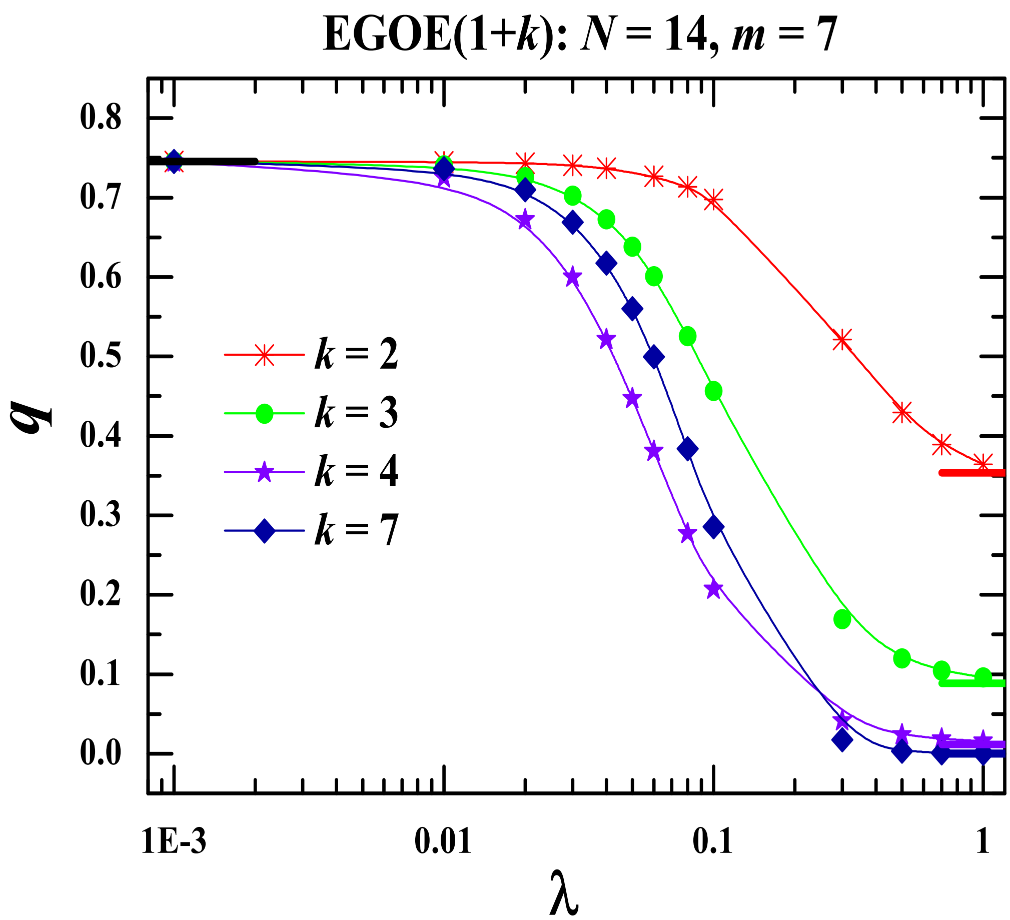

Figure 4.5: Variation of ensemble averaged q as a function of λ for EGOE(1+k) ensemble by distributing $m = 7$ fermions in $N = 14$ sp states for $k = 2, 3, 4$ and $k = m = 7$. An ensemble of 20 members is considered. See text for more details.

by the so-called q-normal distribution f_{qN}. The formulas for parameter q in terms of m, N and k are derived for fermionic and bosonic EE(k) which explain the Gaussian to semi-circle transition in spectral densities, in many-body quantum systems as k changes from 1 to m. This is shown both for EGOE(k) and their Unitary variants EGUE(k), both for fermion and boson systems [79]. In this section we present the spectral density results for both fermion and boson systems.

4.5.1 Results for Fermion Systems

We present the spectral density results by taking two different examples: (i) a 100 member EGOE(1+k) ensemble with $m = 6$ fermions occupying $N = 12$ sp states and (ii) a 20 member EGOE(1+k) ensemble with $m = 7$ fermions occupying $N = 14$ sp states. We take different values of λ. For various ranks of interaction (i.e k = 2,3,4 and $k = m = 6$), we first obtain the eigenvalue spectrum which is then zero centered (with ϵ_H as centroid) and scaled to unit width σ_H for each member. Histograms representing

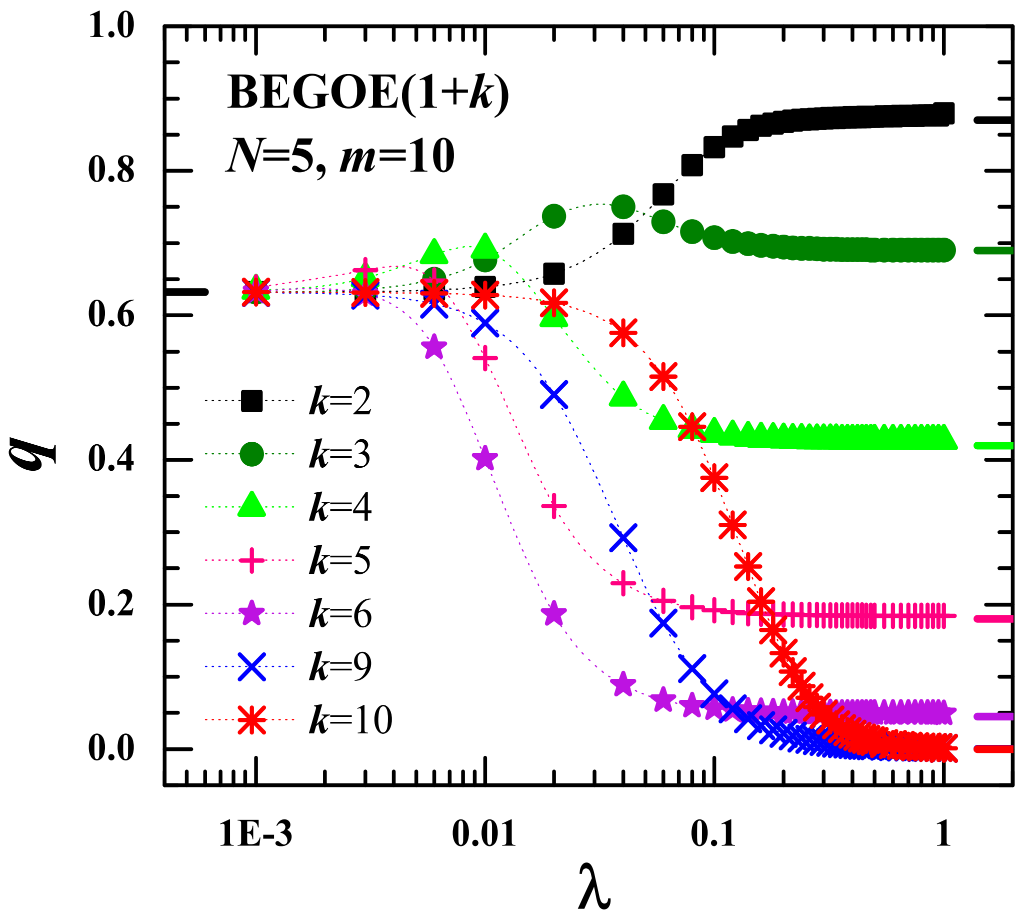

Figure 4.6: Ensemble averaged q vs. λ for a 100 member BEGOE(1+k) ensemble with $m = 10$ bosons in $N = 5$ sp states for different k values. The horizontal black mark on left q-axis indicates q estimate for $H = h(1)$ given by Eq. (4.23), while the colored marks on right q-axis represent the q values, given by Eq. (4.25), for corresponding k-body rank with $H = V(k)$. See text for more details.

the ensemble averaged spectral density are constructed using these normalized eigenvalues. Here bin size is 0.2. These results are presented in Fig. 4.7 and Fig. 4.8. The Gaussian to semi-circle transition in the spectral density can be clearly observed from the histograms as k changes from 2 to $k = m$ for both the systems. The numerical results obtained from the histograms are compared with the normalized spectral density $\rho(E) = d\, f_{qN}(x|q)$ with $\epsilon_H - \frac{2\sigma_H}{\sqrt{1-q}} \leq E \leq \epsilon_H + \frac{2\sigma_H}{\sqrt{1-q}}$. Here the corresponding ensemble averaged values of parameter q are taken.

4.5.2 Results for Boson Systems

Now moving further, we demonstrate this Gaussian to semi-circle transition in spectral densities for boson systems. We consider a system of $m = 10$ bosons distributed in $N = 5$ sp states in Fig. 4.9. Histograms represent the numerical results for a 100 member

BEGOE($1+k$) ensemble with λ = 0.04, 0.1 and 0.5. The continuous black curves are obtained using the theoretical form of f_{qN} with corresponding ensemble averaged q values. As in the case of fermions, here also the eigenvalue spectrum for each member of the ensemble is zero centred and scaled to unit width prior to the construction of the histograms. A bin size of 0.2 is considered to plot the histograms. In the strong coupling domain, one can also apply Eq.(4.25) to BEGOE(1+ k) [79]. The numerical results in Fig. 4.9 clearly display the Gaussian to semi-circle transition in the spectral density as k changes from 2 to $k=m=10$ and are in excellent agreement with the theoretical curves of f_{qN} .

One can see that the numerical results in Figs. 4.7, 4.8 and 4.9 are well described by the theoretical forms of q -normal distribution f_{qN} . These results mainly show that the q -normal distribution f_{qN} describes the transition in spectral density of EGOE(1+ k) and BEGOE(1+ k) not only in strong interaction domain but in all domains of interaction strength.

4.6 Conclusion

Taking motivation from recently known importance of higher k -body ranks of interactions in interacting many-particle quantum systems, we extend the embedded random matrix ensembles of two-body interactions to k -body interactions in presence of a mean-field. The definition and construction of embedded random matrix ensembles of k -body interactions in presence of a mean-field is given for fermions (denoted by EGOE(1+ k)) and bosons (denoted by BEGOE(1+ k)) in the beginning of this chapter. Moving further, the q -Hermite polynomials along with their generating function and recurrence relations, q -normal distribution f_{qN} , conditional q -normal distribution f_{CqN} and bivariate q -normal distribution f_{biv-qN} are introduced. The analytical formula of q considering only the one-body part is derived for both fermions and bosons. We proceeed further with the variation of parameter q as a function of the interaction strength λ in EGOE(1+ k) and BEGOE(1+ k) for a fixed body rank k . Lastly, we have used all this preliminary knowledge of q -Hermite polynomials to study the spectral density for EGOE(1+ k) and BEGOE(1+ k). It is shown that the Gaussian to semi-circle transition exhibited by the spectral density as body rank k of the interaction increases is described by the q -normal density f_{qN} for any value of interaction strength λ .

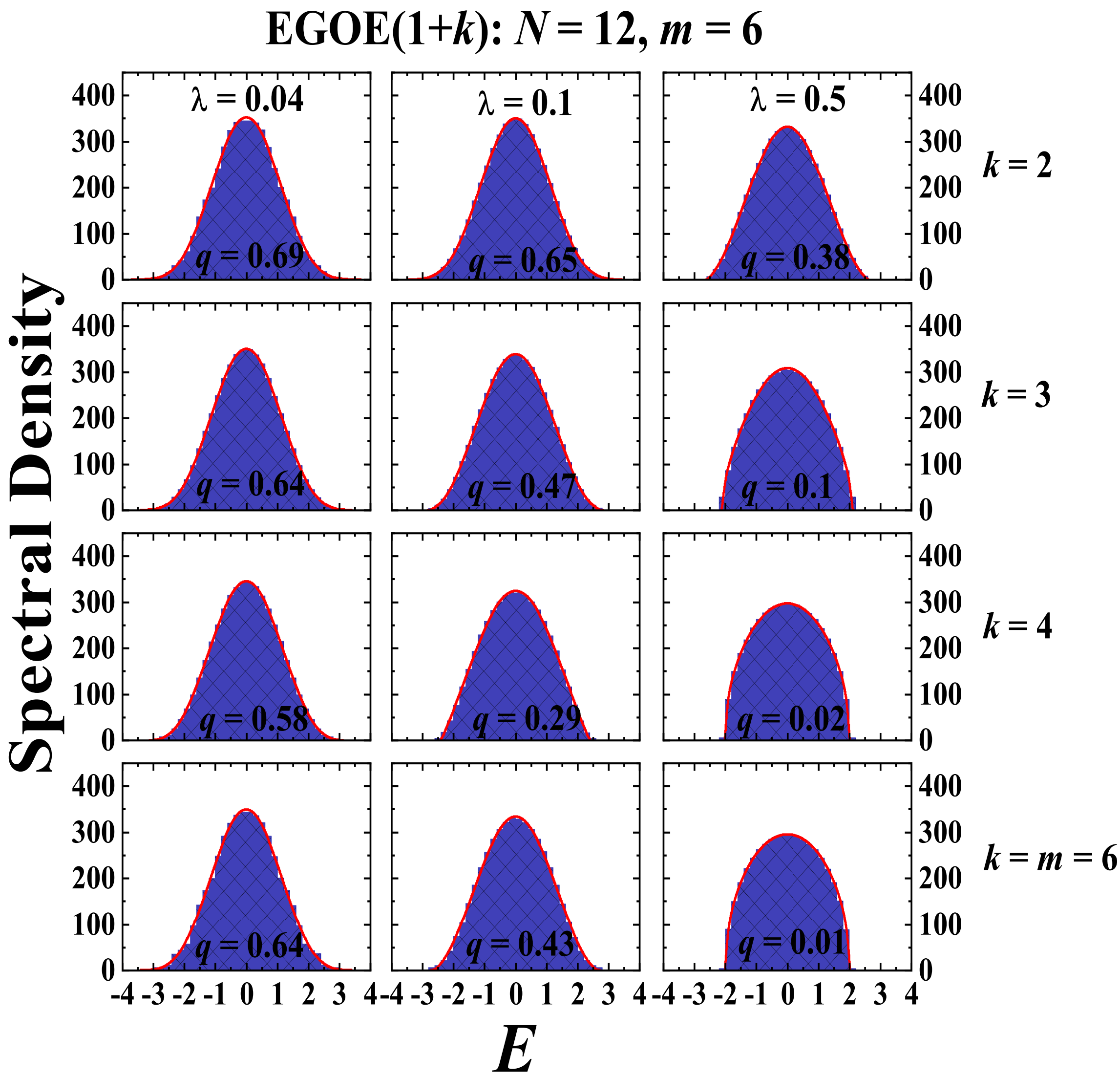

Figure 4.7: Histograms represent the spectral density vs. normalized energy E results of the spectra of a 100 member EGOE($1+k$) ensemble with $m = 6$ fermions in $N = 12$ sp states. Results are shown for different values of interaction strength λ = 0.04, 0.1 and 0.5 for $k = 2, 3, 4$ and $k = m = 6$. In all the plots $\int \rho(E)dE = d = 924$. Ensemble averaged spectral density histogram is compared with q-normal distribution (continuous curves) given by $f_{qN}(x|q)$ with the corresponding ensemble averaged q values.

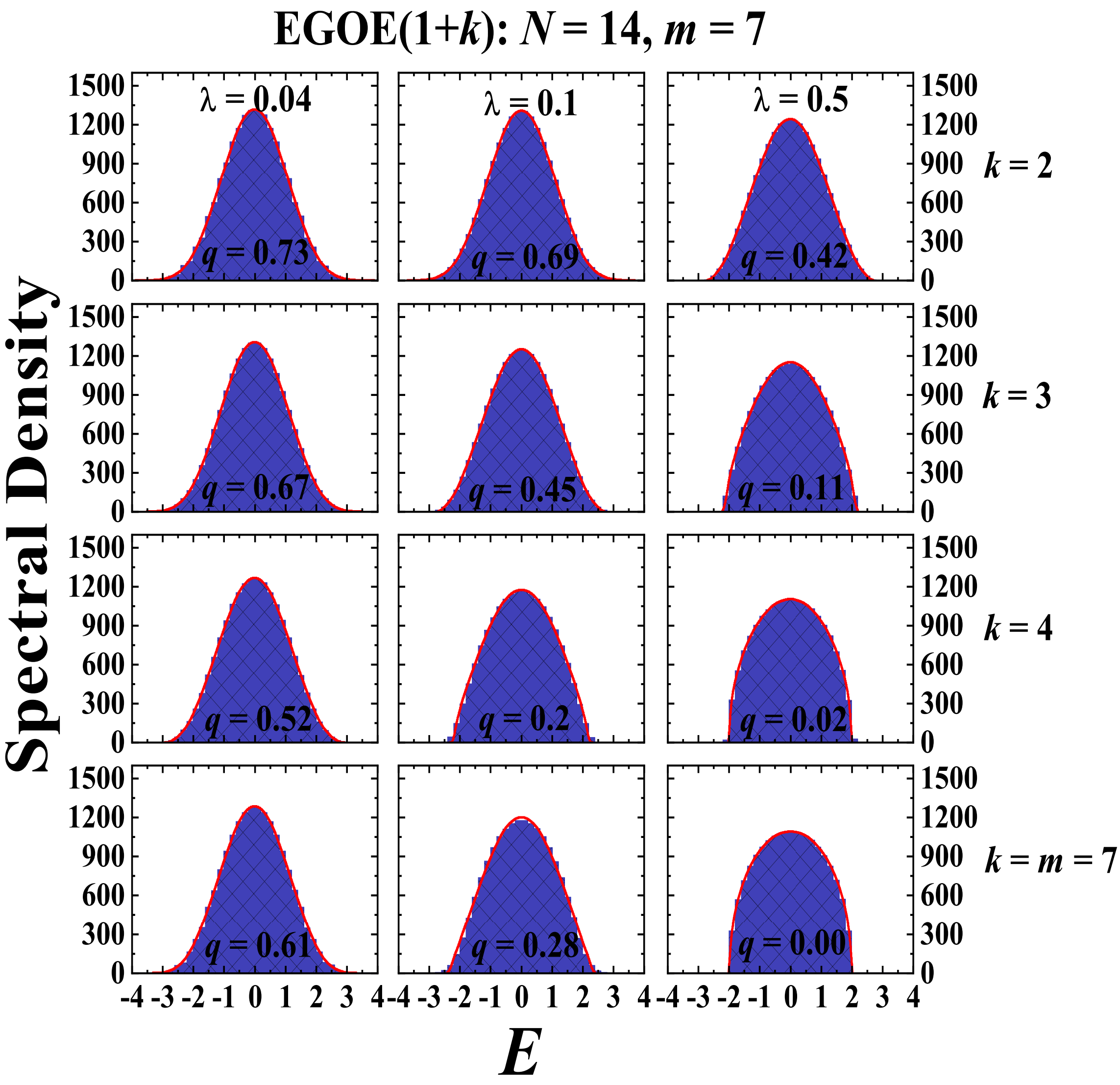

Figure 4.8: Histograms represent the spectral density vs. normalized energy E results of the spectra of a 20 member EGOE($1+k$) ensemble with $m = 7$ fermions in $N = 14$ sp states. Results are shown for different values of interaction strength λ = 0.04, 0.1 and 0.5 for $k = 2, 3, 4$ and $k = m = 7$. In all the plots $\int \rho(E)dE = d = 3432$. Ensemble averaged spectral density histogram is compared with q-normal distribution (continuous curves) given by $f_{qN}(x|q)$ with the corresponding ensemble averaged q values.

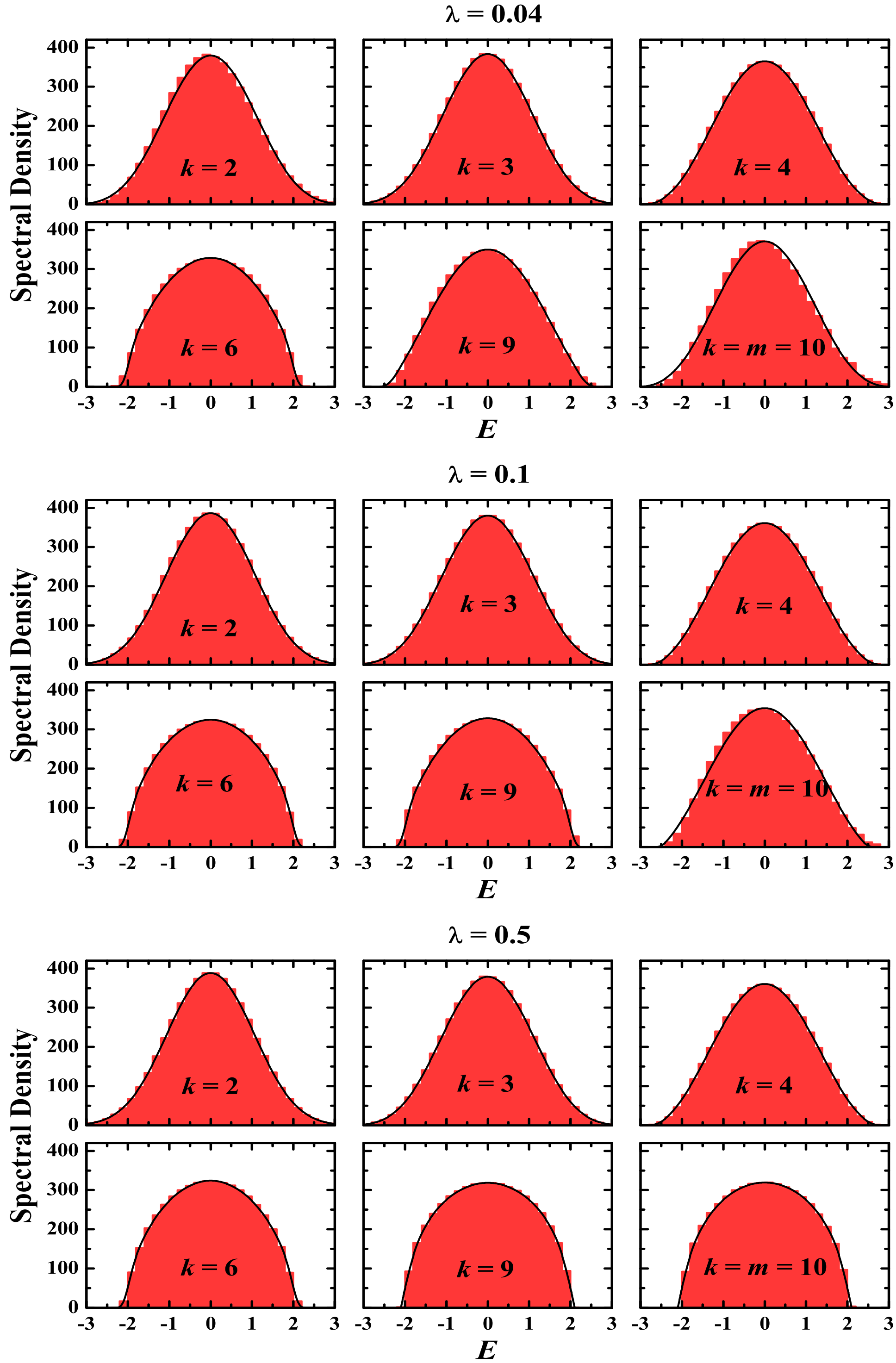

Figure 4.9: Ensemble averaged spectral density $\rho(E)$ vs. normalized energy E results for a 100 member BEGOE($1+k$) ensemble with $m=10$ bosons in $N=5$ sp states for different k values. Results are shown for λ = 0.04, 0.1 and 0.5. In all the plots $\int \rho(E)dE = 1001$. The continuous black curves are obtained using q-normal distribution given by $f_{qN}(x|q)$ with the corresponding ensemble average q values.