CHAPTER 4

RATE OF CONVERGENCE IN LOCAL LIMIT THEOREM: DOMAIN OF NON-NORMAL ATTRACTION OF A STABLE LAW WITH INDEX $\alpha \neq 1$, $\alpha \neq 2$

4.1 INTRODUCTION:

Let $\{X_n\}$ be a sequence of independent r.v.s each having a common d.f. F_1. Suppose that F_1 belongs to the domain of attraction of the stable law F_o with index α, $o< \alpha< 2$, $\alpha \neq 1$. That is, there exist real sequences $\{A_n\}$ and $\{B_n,\ B_n> 0\}$ such that $Z_n = (S_n-A_n)/B_n$ converges in law to the stable r.v. with d.f. F_o.

In this chapter, we obtain an uniform rate of convergence in (3.1.1) with $\phi(x)$ replaced by $v_o(x)$, the p.d.f. corresponding to stable law F_o. We state below the main result of this chapter.

THEOREM 4.1.1: *Under the assumptions* **[A1]-[A5]**, *stated in Section 4.2 below, for large* n,

$$\sup_{x\in R} \Delta_n(x) \leq C \left\{ (1/\log n) + \vartheta_n + |\zeta_n| \right\}$$

where $\vartheta_n = |nB_n^{-2}H(B_n)-1|$ *and*

$$\zeta_n = n \int_{|u|>B_n} \cos(tuB_n^{-1})\, dF_1(u) .$$

Remark 4.1.1: The uniform convergence rate in Theorem 4.1.1 has three components. The first component viz. (1/log n), depends on the size of the sample only, whereas the remaining two components depend on the properties of the d.f. F_1 (which is in the domain of non-normal attraction of stable law). The effect of behaviour of truncated variance function $H(B_n)$ for large values of n, is reflected in the convergence rate through the term ϑ_n. The tail behaviour of the c.f. of corresponding F_1 is reflected is in $|\zeta_n|$ term.

We prove Theorem 4.1.1 in Section 4.4. The notations and assumptions are introduced in Section 4.2. In Section 4.3, we prove some lemmas which will be useful in proving Theorem 4.1.1.

4.2 NOTATIONS AND ASSUMPTIONS:

Throughout this chapter, we will use notation define Section 3.2. While using these notations, d.f.s F and Φ introduced in Section 3.1 (of Chapter 3) will be replaced by d.f.s F_1 and F_0 introduced in the previous section respectively.

Suppose r.v. $Y_i \sim F_i$ for i = 0, 1.

Without loss of generality, we assume that $EX_1 = 0$, whenever it exists. Since A_n can be taken as 0, under our assumptions, for all t,

$$\lim_{n\to\infty}\{f_1(tB_n^{-1})\}^n = \exp\{-c|t|^{\alpha}\} \equiv f_o(t). \qquad \ldots(4.2.1)$$

Further, for $k = 0, 1,$

$$R_k(x) = P(|Y_k| > x) = x^{-\alpha} s_k(x) r_k(x) \qquad \ldots(4.2.2)$$

where the function $s_k(x)$ is such that $s_k(x) \to c_k$ as $x \to \infty$, c_k being a positive constant and the function $r_k(x)$ is slowly varying in the sense of Karamata. In fact, $r_o(x)$ asymptotically equals a constant and

$$\lim_{n\to\infty} nB_n^{-\alpha} r_1(B_n) = \text{constant}. \qquad \ldots(4.2.3)$$

We now make the following assumptions:

[A1] The d.f. F_1 is symmetric and absolutely continuous;

[A2] There exists an integer $r \geq 1$ such that $\int_{-\infty}^{\infty} |f_1(t)|^r dt < \infty$;

[A3] The d.f. F_1 belongs to the domain of non-normal attraction of the stable law F_o with index α, $0 < \alpha < 2$, $\alpha \neq 1$;

[A4] $\sum_{k=[n/\log n]+1}^{[n-n/\log n]} \left| \int_{|u|\leq B_n} \cos(tuB_n^{-1}) dF_1(u) \right|^{-k} = o(n^{-(\theta-\alpha)\gamma})$

for every $\alpha < \theta < \alpha+1$. Here $\gamma = 1/\alpha$; and

[A5] $\zeta_n = n \int_{|u|>B_n} \cos(tuB_n^{-1}) dF_1(u) \to 0$ as $n \to \infty$.

4.3 PRELIMINARY RESULTS:

We shall need the following lemmas.

Lemma 4.3.1: Under the assumptions **[A1]** and **[A3]**, there exist positive constants, ε, $\lambda < \alpha$, C_1 and c such that for all large n,

$$|f_1(tB_n^{-1})|^n \le C_1 \exp\{-c|t|^{\alpha} P_{\lambda}(t)\} \qquad \ldots(4.3.1)$$

for all t with $|t| \le \varepsilon B_n$.

Proof: We notice, as a consequence of Karamata Indices Theorem, that a positive measurable function h is slowly varying iff $c(h) = d(h) = 0$ (see Definition 3.3.1). In the light of this, Proposition 2.2.3 of Bingham et al.(1987, p.73) becomes:

(i) Let h be a slowly varying function. Then for every $\delta_1 > 0$ and $A_1 > 1$ there exists $X_1 = X_1(A_1, \delta_1)$ such that

$h(y)/h(x) \le A_1 (y/x)^{\delta_1} \quad (y \ge x \ge X_1)$.

(ii) Let h be a slowly varying function. Then for every $\delta_1 < 0$ and $A_2 \in (0,1)$ there exists $X_2 = X_2(A_2, \delta_1)$ such that

$h(y)/h(x) \ge A_2 (y/x)^{\delta_1} \quad (y \ge x \ge X_2)$.

Recall that the function $r_1(x)$ defined through (4.2.2) is a slowly varying function at infinity. Applying both lower bound and upper bound type results for $r_1(x)$, we observe that for every $\lambda > 0$, $c_1 < 1$ and $c_2 > 1$, there exists positive constants ε_1, such that, for all large n,

$$r_1(B_n/|t|) \geq c_1|t|^{\lambda} r_1(B_n) \qquad \ldots(4.3.2a)$$

whenever $|t| \leq 1$;

$$r_1(B_n) \leq c_2|t|^{\lambda} r_1(B_n/|t|) \qquad \ldots(4.3.2b)$$

whenever $1 \leq |t| \leq \varepsilon_1 B_n$.

Also, by a Lemma in Gnedenko and Kolmogorov (1968, p.238-240), there exists a positive constant ε_2, such that for $|t| \leq \varepsilon_2 B_n$,

$$|f_1(tB_n^{-1})|^n \leq \exp\{-cnR_1(B_n/|t|)\}. \qquad \ldots(4.3.3)$$

Using (4.3.2a), (4.3.2b) in (4.3.3), we observe that there exist an $\varepsilon > 0$, such that for all large n,

$|f_1(tB_n^{-1})|^n \leq C_1 \exp\{-c|t|^{\alpha} P_{\lambda}(t)\}$, for all t with $|t| \leq \varepsilon B_n$.□

Lemma 4.3.2: Under the assumptions **[A1]** and **[A3]**, there exist constants ε, $\lambda < \alpha$, both sufficiently small, c and C such that for all large n,

$$|A_n(t,1)| \leq C\exp\{-c|t|^{\alpha} P_{\lambda}(t)\} \qquad \ldots(4.3.4)$$

for all t with $|t| \leq \varepsilon B_n$.

Remark 4.3.1 : This Lemma can be proved along the lines of Lemma 3.3.2.

Next, for every integer n and real x, define

$$d_n(t,x) = n\{\alpha_n(tB_n^{-1},x)-\exp\{-c|t|^{\alpha}B_n^{-2}H(B_n)\}\}, \qquad \ldots(4.3.5)$$

$S_n(t,x) =$

$$n^{-1}\sum_{k=1}^{n}\{\alpha_n(tB_n^{-1},x)\}^{n-k}\exp\{-c|t|^{\alpha}B_n^{-2}H(B_n)(k-1)\} \qquad \ldots(4.3.6)$$

Property of the function $d_n(t,1)$

Lemma 4.3.3: Under the assumptions of Lemma 4.3.2, for all values of t, we have, for sufficiently large n,

$$|d_n(t,1)| \le P_1(|t|)n^{-2}. \qquad \ldots(4.3.7)$$

Proof: Consider $0<\alpha<1$ case.

$$n^{-1}|d_n(t,1)| = |\alpha_n(tB_n^{-1},1)-\exp\{-c|t|^{\alpha}B_n^{-2}H(B_n)\}|$$

$$= |\int_{|u|\le B_n}\cos(tuB_n^{-1})dF_1(u)-\exp\{-c|t|^{\alpha}B_n^{-2}H(B_n)\}|$$

$$= |\int_{|u|\le B_n}(\cos(tuB_n^{-1})-1)dF_1(u) + \int_{|u|\le B_n}dF_1(u)$$

$$-\{\exp\{-c|t|^{\alpha}B_n^{-2}H(B_n)\} -1\} -1|$$

$$\le \int_{|u|\le B_n}|\cos(tuB_n^{-1})-1|dF_1(u)+R_1(B_n)$$

$$+|\{\exp\{-c|t|^{\alpha}B_n^{-2}H(B_n)\} -1\}|$$

$$\le \int_{|u|\le B_n}(t^2u^2B_n^{-2}/2))dF_1(u)$$

$$+ R_1(B_n) +c|t|^{\alpha}B_n^{-2}H(B_n)$$

$$\le \max\{t^2/2,c|t|^{\alpha}\}\{B_n^{-2}H(B_n)+R_1(B_n)\}$$

$$\le \max\{t^2/2,c|t|,c\}\ \{B_n^{-2}H(B_n)+R_1(B_n)\}$$

$\leq P(|t|)\{B_n^{-2}H(B_n)+R_1(B_n)\}$

$\leq P(|t|)n^{-1}$, because $nR_1(B_n) \to$ constant.

Now, consider $1< \alpha< 2$ case.

$$n^{-1}|d_n(t,1)| = |\alpha_n(tB_n^{-1},1)-\exp\{-c|t|^{\alpha}B_n^{-2}H(B_n)\}|$$

$$= |\int_{|u|\leq B_n} \cos(tuB_n^{-1})dF_1(u)-\exp\{-c|t|^{\alpha}B_n^{-2}H(B_n)\}|$$

$$= |\int_{|u|\leq B_n} (\cos(tuB_n^{-1})-1+(t^2/2)u^2B_n^{-2})dF_1(u)$$

$$+ \int_{|u|\leq B_n} dF_1(u) - \int_{|u|\leq B_n} (t^2/2)u^2B_n^{-2}dF_1(u)$$

$$-\{\exp\{-c|t|^{\alpha}B_n^{-2}H(B_n)\}-1+c|t|^{\alpha}B_n^{-2}H(B_n)\}$$

$$-1+c|t|^{\alpha}B_n^{-2}H(B_n)|$$

$$\leq \int_{|u|\leq B_n} |\cos(tuB_n^{-1})-1+(t^2/2)u^2B_n^{-2}|dF_1(u)+R_1(B_n)$$

$$+|\exp\{-c|t|^{\alpha}B_n^{-2}H(B_n)\}-1+c|t|^{\alpha}u^2B_n^{-2}|$$

$$+B_n^{-2}H(B_n)\{|c|t|^{\alpha}-t^2/2|\}$$

$$\leq \int_{|u|\leq B_n} (t^4u^4B_n^{-4}/2)dF_1(u)$$

$$+ R_1(B_n) +(c|t|^{\alpha}B_n^{-2}H(B_n))^2/2+P(|t|)B_n^{-2}H(B_n),$$

because $|\cos(x)-1+x^2/2|\leq x^4/2$.

$$\leq P(|t|)\{B_n^{-2}H(B_n)+R_1(B_n)+B_n^{-2}H(B_n)\}$$

$$\leq P(|t|)\{B_n^{-2}H(B_n)+R_1(B_n)\}$$

$$\leq P(|t|)n^{-1}.$$

This proves (4.3.7).□

Properties of functions $\alpha_n(t,x)$ and $\beta_n(t,x)$

Lemma 4.3.4: For all $x \neq 0$, every large integer s, there exists a constant C such that

$$\int_{-\infty}^{\infty} |\alpha_n(t,x)|^n dt = O(B_n^{-1}), \quad \ldots(4.3.8)$$

$$\int_{-\infty}^{\infty} |\alpha_n(t,x)|^{2s} dt \leq C, \quad \ldots(4.3.9)$$

$$\int_{-\infty}^{\infty} |\beta_n(t,x)|^{2s} dt \leq C. \quad \ldots(4.3.10)$$

Remark 4.3.2: The proof of this lemma follows from Basu et al. (1980; (3.3) - (3.5)).□

Property of function $S_n(t,1)$

Lemma 4.3.5: Under the assumptions **[A1]**, **[A3]** and **[A4]**, we have, for all t with $|t| \leq \varepsilon B_n$ and all large n,

$$|S_n(t,1)| \leq (C/\log B_n) e^{-C|t|^{\alpha} P_\lambda(t)}. \quad \ldots(4.3.11)$$

Proof: Write $S_n(t,1)$ as

$$nS_n(t,1) = \left\{ \sum_{k=1}^{[n/\log B_n]} + \sum_{k=[n/\log B_n]+1}^{[n-(n/\log B_n)]} + \sum_{k=[n-n/\log B_n]+1}^{n} \right\}$$

$$\{\alpha_n(tB_n^{-1},1)\}^{n-k} e^{-C|t|^{\alpha} B_n^{-2} H(B_n)(k-1)}$$

$$= S_{n1}(t,1) + S_{n2}(t,1) + S_{n3}(t,1), \text{ say.} \quad \ldots(4.3.12)$$

Consider first $S_{n1}(t,1)$. Using Lemma 4.3.3, we have, for sufficiently large n,

$$|S_{n1}(t,1)|$$

$$= |\sum_{k=1}^{[n/\log B_n]} \{\alpha_n(tB_n^{-1},1)\}^{n-k} e^{-C|t|^{\alpha} B_n^{-2} H(B_n)(k-1)}|$$

$$\leq \sum_{k=1}^{[n/\log B_n]} |\alpha_n(tB_n^{-1},1)|^{n(1-k/n)}$$

$$\leq Ce^{-C|t|^{\alpha} P_\lambda(t)} \sum_{k=1}^{[n/\log B_n]} C^{-(k/n)} \{e^{C|t|^{\alpha} P_\lambda(t)}\}^{(k/n)}$$

$$\leq Ce^{-C|t|^{\alpha} P_\lambda(t)} \sum_{k=1}^{[n/\log B_n]} \max\{C^{-(1/n)}, C^{-(1/\log B_n)}\}$$

$$\{e^{C|t|^{\alpha} P_\lambda(t)}\}^{(1/\log B_n)}$$

$$\leq (n/\log B_n) C^{*} e^{-C|t|^{\alpha} P_\lambda(t)} \{e^{C|t|^{\alpha} P_\lambda(t)}\}^{\varepsilon}$$

$$\leq (n/\log B_n)\ Ce^{-C|t|^{\alpha} P_\lambda(t) + C|t|^{\alpha} P_\lambda(t)\varepsilon}$$

$$\leq (n/\log B_n)\ Ce^{-C|t|^{\alpha} P_\lambda(t)(1-\varepsilon)}$$

$$\leq (n/\log B_n)\ Ce^{-C|t|^{\alpha} P_\lambda(t)}. \qquad \ldots(4.3.13)$$

Next, we consider $S_{n2}(t,1)$. In view of the assumption **[A4]** and Lemma 4.3.2, we have, for sufficiently large n,

$$S_{n2}(t,1)$$

$$\leq \sum_{k=[n/\log B_n]+1}^{[n-(n/\log B_n)]} |\alpha_n(tB_n^{-1},1)|^{n-k} e^{-C|t|^{\alpha} B_n^{-2} H(B_n)(k-1)}$$

$$\leq Ce^{-C|t|^{\alpha} P_\lambda(t)} \sum_{k=[n/\log B_n]+1}^{[n-(n/\log B_n)]} |\alpha_n(tB_n^{-1},1)|^{-k}$$

$$\leq Ce^{-C|t|^{\alpha} P_\lambda(t)} o(n^{-(\theta-\alpha)\gamma})$$

$$\leq Cn^{-(\theta-\alpha)\gamma} e^{-C|t|^{\alpha}},\quad \alpha< \theta< \alpha+1. \qquad \ldots(4.3.14)$$

Finally, consider $S_{n3}(t,1)$.

$$|S_{n3}(t,1)|$$

$$\leq \sum_{k=[n-n/\log B_n]+1}^{n} |\alpha_n(tB_n^{-1},1)|^{n-k} e^{-C|t|^{\alpha} B_n^{-2} H(B_n)(k-1)}$$

$$\leq \sum_{k=[n-n/\log B_n]+1}^{n} e^{-C|t|^{\alpha} B_n^{-2} H(B_n)(n-n/\log B_n)}$$

$$\leq (n/\log B_n) e^{-C|t|^{\alpha} nB_n^{-2} H(B_n)(1-1/\log B_n)}$$

$$\leq (n/\log B_n) e^{-C|t|^{\alpha}(1-\varepsilon)^2}, \text{ as } nB_n^{-2}H(B_n) \to 1$$

$$\leq C(n/\log B_n) e^{-C|t|^{\alpha}}, \qquad \ldots(4.3.15)$$

Thus combining the results of (4.3.13), (4.3.14) and (4.3.15) at (4.3.12), we obtain, for sufficiently large n,

$$S_n(t,1) \leq (C/\log B_n) e^{-C|t|^{\alpha} P_{\lambda}(t)}. \square$$

Lemma 4.3.6: Let $\varepsilon > 0$, $0 < \lambda < \alpha$ and c be as in Lemma 4.3.2. Then there exists a polynomial $P(|t|)$ in $|t|$ with non-negative coefficients independent of n such that for all t with $|t| \leq \varepsilon B_n$ and all large n,

$$|A_n(t,1) - e^{-c|t|^{\alpha}}|$$

$$\leq e^{-C|t|^{\alpha} P_{\lambda}(t)} P(|t|) \{(1/\log n) + \vartheta_n\}, \qquad \ldots(4.3.16)$$

where $\vartheta_n = |nB_n^{-2}H(B_n) - 1|$.

Proof: We write

$$A_n(t,1) - e^{-c|t|^{\alpha}} = D_{1n}(t,1) + D_{2n}(t) \qquad \ldots(4.3.17)$$

where

$$D_{1n}(t,1) = A_n(t,1) - e^{-c|t|^{\alpha} n B_n^{-2} H(B_n)} \qquad \ldots(4.3.18)$$

$$D_{2n}(t) = e^{-c|t|^{\alpha} n B_n^{-2} H(B_n)} - e^{-c|t|^{\alpha}}. \qquad \ldots(4.3.19)$$

Note that in view of Lemma 4.3.3 and Lemma 4.3.5, for all t with $|t| \le \varepsilon B_n$ and all large n, there exists a polynomial $P(|t|)$ in $|t|$ with positive coefficients such that

$|D_{1n}(t,1)|$

$$\le e^{-c|t|^{\alpha} p_{\lambda}(t)} (n/\log B_n)\ P(|t|)\ \{B_n^{-2} H(B_n) + R_1(B_n)\}$$

$$\le e^{-c|t|^{\alpha} p_{\lambda}(t)}\ P(|t|)\ (n/\log B_n)\ \{B_n^{-2} H(B_n) + R_1(B_n)\}$$

$$\le e^{-c|t|^{\alpha} p_{\lambda}(t)}\ P(|t|)\ (1/\log B_n)\ \{n B_n^{-2} H(B_n) + n R_1(B_n)\}$$

$$\le e^{-c|t|^{\alpha} p_{\lambda}(t)}\ P(|t|)\ (1/\log B_n),$$

because $n B_n^{-2} H(B_n) \to 1$ and $n R_1(B_n) \to C > 0$.

$$\le e^{-c|t|^{\alpha} p_{\lambda}(t)}\ P(|t|)\ (1/\log n).$$

Thus, for sufficiently large n,

$$|D_{1n}(t,1)| \le e^{-c|t|^{\alpha} p_{\lambda}(t)}\ P(|t|)(1/\log n). \qquad \ldots(4.3.20)$$

On the other hand, proceeding exactly similar to the proof of (3.3.32), we obtain

$$|D_{2n}(t)| \le \vartheta_n e^{-c|t|^{\alpha}} P(|t|). \qquad \ldots(4.3.21)$$

Equations (4.3.17), (4.3.20) and (4.3.21) now prove the desired result.□

We quote in the following lemma which can be proved along the lines of Lemma 3.3.8.

Lemma 4.3.7: Let $\varepsilon > 0$ be as same as in Lemma 4.3.2 and let n_o be a positive integer. Let $\Theta = \{|(t, x, n)|: |x| \geq 1, |t| \geq \varepsilon, n \geq n_o\}$. Then, under the assumptions of Lemma 4.3.2, $\mu = \sup_{\Theta} |\alpha_n(t,x)|$ satisfies $0 \leq \mu < 1$.

4.4 PROOF OF THE THEOREM:

Proof of Theorem 4.1.1:

By inversion formula for absolutely continuous density, we have

$$2\pi |v_n(x) - v_o(x)|$$

$$= \left| \int_{-\infty}^{\infty} e^{-itx} \{\{f_1(tB_n^{-1})\}^n - e^{-c|t|^\alpha}\} dt \right|$$

$$\leq \int_{-\infty}^{\infty} |\{f_1(tB_n^{-1})\}^n - e^{-c|t|^\alpha}| dt$$

$$\leq \int_{-\infty}^{\infty} |A_n(t,1) + B_n(t,1) - e^{-c|t|^\alpha}| dt, \text{ using (3.2.12)}$$

$$\leq \int_{-\infty}^{\infty} |A_n(t,1) - e^{-c|t|^\alpha}| dt + \int_{-\infty}^{\infty} |B_n(t,1)| dt$$

$$= I_{1n} + I_{2n}, \text{ say.} \qquad \ldots (4.4.1)$$

Consider I_{1n} first.

$$I_{1n} = \int_{-\infty}^{\infty} |A_n(t,1) - e^{-c|t|^\alpha}| dt$$

$$\leq \int_{|t| \leq \varepsilon B_n} |A_n(t,1) - e^{-c|t|^\alpha}| dt + \int_{|t| > \varepsilon B_n} |A_n(t,1)| dt$$

$$+ \int_{|t| > \varepsilon B_n} e^{-c|t|^\alpha} dt$$

$$= I_{1n1}+I_{1n2}+I_{1n3}, \text{ say.} \qquad \ldots(4.4.2)$$

In view of Lemma 4.3.6, we have

$$I_{1n1} = \int_{|t|\le\varepsilon B_n} |A_n(t,1)-e^{-c|t|^{\alpha}}|dt$$

$$\le C\{(1/\log n)+\vartheta_n\} \int_{|t|\le\varepsilon B_n} e^{-c|t|^{\alpha}p_{\lambda(t)}}P(|t|)dt$$

$$\le C\{(1/\log n)+\vartheta_n\}, \qquad \ldots(4.4.3)$$

for all large n, where $\vartheta_n = |nB_n^{-2}H(B_n)-1|$.

Now, for sufficiently large n, we have

$$I_{1n2} = \int_{|t|>\varepsilon B_n} |A_n(t,1)|dt$$

$$= \int_{|t|>\varepsilon B_n} |\alpha_n(tB_n^{-1},1)|^n dt$$

$$= B_n \int_{|t|>\varepsilon} |\alpha_n(t,1)|^n dt$$

$$= B_n \int_{|t|>\varepsilon} |\alpha_n(t,1)|^{n-2S}|\alpha_n(t,1)|^{2S}dt$$

$$\le B_n \int_{|t|>\varepsilon} \{\sup_{\Theta} |\alpha_n(t,1)|\}^{n-2S}|\alpha_n(t,1)|^{2S}dt$$

$$= B_n\, \mu^{n-2S} \int_{-\infty}^{\infty}|\alpha_n(t,1)|^{2s}dt$$

$$\le CB_n\mu^{n-2S}, \text{ using (4.3.9).} \qquad \ldots(4.4.4)$$

Finally, we consider I_{1n3}. For sufficiently large n,

$$I_{1n3} = \int_{|t|>\varepsilon B_n} e^{-c|t|^{\alpha}}dt$$

$$= 2 \int_{\varepsilon B_n}^{\infty} e^{-ct^{\alpha}}dt$$

$$\le CB_n^{\gamma-1}\, e^{-cB_n}. \qquad \ldots(4.4.5)$$

Combining (4.4.3), (4.4.4), (4.4.5) and using Lemma 4.3.8 we get from (4.4.2),

$$I_{1n} \le C\{(1/\log n)+\vartheta_n\} + CB_n\mu^{n-2S} + CB_n^{\gamma-1} e^{-cB_n},$$

$$\le C\{(1/\log n)+\vartheta_n + B_n\mu^{n-r} + B_n^{\gamma-1} e^{-cB_n}\}$$

$$\le C\{(1/\log n)+\vartheta_n\}. \qquad \ldots(4.4.6)$$

Now we consider the estimation of I_{2n}.

From the definition of $B_n(t,1)$ at (3.2.11), we find for sufficiently large but fixed s,

$$I_{2n} = \int_{-\infty}^{\infty}|B_n(t,1)|dt$$

$$\le \int_{-\infty}^{\infty} \sum_{j=1}^{n}\binom{n}{j}|\alpha_n(tB_n^{-1},1)|^{n-j}|\beta(tB_n^{-1},1)|^{j}dt$$

$$= \sum_{j=1}^{n}\binom{n}{j} \int_{-\infty}^{\infty}|\alpha_n(tB_n^{-1},1)|^{n-j}|\beta(tB_n^{-1},1)|^{j}dt$$

$$= \left\{\sum_{j=1}^{[n/2]} + \sum_{j=[n/2]+1}^{n-2s} + \sum_{j=n-2s+1}^{n}\right\}\binom{n}{j}\int_{-\infty}^{\infty}|\alpha_n(tB_n^{-1},1)|^{n-j} |\beta_n(tB_n^{-1},1)|^{j}dt$$

$$= J_1(n)+J_2(n)+J_3(n), \text{ say.} \qquad \ldots(4.4.7)$$

Firstly, let us consider $J_1(n)$. In view of assumption **[A5]**, we have

$J_1(n)$

$$= \sum_{j=1}^{[n/2]}\binom{n}{j}\int_{-\infty}^{\infty}|\alpha_n(tB_n^{-1},1)|^{n-j}|\beta_n(tB_n^{-1},1)|^{j}dt$$

$$\le \sum_{j=1}^{[n/2]}(n^{j}/j!)\int_{-\infty}^{\infty}|\alpha_n(tB_n^{-1},1)|^{n-j} \left|\int_{|u|>B_n}\cos(tuB_n^{-1})dF_1(u)\right\}^{j}|dt$$

$$\leq \sum_{j=1}^{[n/2]} |(|\zeta_n|^j/j!) B_n \int_{-\infty}^{\infty} |\alpha_{[n/2]}(t, 1/\sqrt{2})|^{n/2} dt$$

$$\leq \sum_{j=1}^{[n/2]} |(|\zeta_n|^j/j!) B_n O(B_n^{-1}), \text{ using } (4.3.8)$$

$$\leq C(e^{|\zeta_n|}-1)$$

$$\leq C|\zeta_n|, \text{ because } |\zeta_n| \to 0. \qquad \ldots(4.4.8)$$

Next we estimate $J_2(n)$. In view of Lemma 4.3.5, we find, for sufficiently large but fixed s, that

$J_2(n)$

$$= \sum_{j=[n/2]+1}^{n-2s} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1}, 1)|^{n-j} |\beta_n(tB_n^{-1}, 1)|^j dt$$

$$\leq \sum_{j=[n/2]+1}^{n-2s} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1}, 1)|^{n-j} \{P(|X_1| > B_n)\}^j dt$$

$$\leq \{P|X_1| > B_n)\}^{n/2+1} \sum_{j=[n/2]+1}^{n-2s} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1}, 1)|^{n-j} dt$$

$$\leq \{P|X_1| > B_n)\}^{n/2+1} B_n \sum_{j=[n/2]+1}^{n-2s} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(t, 1)|^{2s} dt$$

$$\leq \{P|X_1| > B_n)\}^{n/2+1} B_n\ C\ 2^n, \text{ using } (4.3.9)$$

$$= CB_nR_1(B_n)\ [4R_1(B_n)]^{n/2}$$

$$\cong C\ B_nR_1(B_n)\ [4C_1/n]^{n/2}$$

$$= o(1/\log n). \qquad \ldots(4.4.9)$$

Finally, we estimate $J_3(n)$. Using Lemma 4.3.4, for sufficiently large but fixed s and n large, we find that

$$J_3(n) = \sum_{j=n-2s+1}^{n} \binom{n}{j} \int_{-\infty}^{\infty} |\alpha_n(tB_n^{-1},1)|^{n-j} |\beta_n(tB_n^{-1},1)|^{j} dt$$

$$\leq \sum_{j=n-2s+1}^{n} \binom{n}{j} \int_{-\infty}^{\infty} |\beta_n(tB_n^{-1},1)|^{j-2s+2s} dt$$

$$\leq \sum_{j=n-2s+1}^{n} (n^{n-j}/(n-j)!) \{P(|X_1|> B_n)\}^{j-2s}$$

$$B_n \int_{-\infty}^{\infty} |\beta_n(t,1)|^{2s} dt$$

$$\leq B_n \sum_{j=n-2s+1}^{n} (n^{n-j}/(n-j)!) \ \{P(|X_1|> B_n)\}^{j-2s}$$

$$\leq B_n\ n^{2s-1} \ \{ P(|X_1|> B_n)\}^{n-4s+1} \ (n-n+2s-1)$$

$$\leq C\ B_n\ n^{2s-1} \ \{C_1/n\}^{n-4s+1}$$

$$\leq C\ B_n\ n^{2s-1-(n-4s+1)} C_1^{\ n-4s+1}$$

$$\leq C\ n^{-n+6s-2} B_n C_1^{n-4s+1}$$

$$= o(1/\log n). \qquad \ldots(4.4.10)$$

Combining the results of (4.4.7), (4.4.8), (4.4.9) and (4.4.10), we obtain,

$$I_{2n} \leq J_1(n)+J_2(n)+J_3(n)$$

$$\leq C_1|\zeta_n|+C_2(1/\log n)+C_3(1/\log n)$$

$$\leq C_1|\zeta_n|+C_2(1/\log n). \qquad \ldots(4.4.11)$$

Therefore, (4.4.1), (4.4.6) and (4.4.11) prove the theorem.□

CONCLUDING REMARKS:

In this chapter we proved uniform rate of convergence type results under the assumption that the d.f. of the summands is in the domain of non-normal attraction of a stable law (with $\alpha \neq 1$, $\alpha \neq 2$). In the next chapter we shall consider a kind of non-identically distributed summands but all in the domain of normal attraction of the same non-normal stable law. We shall prove both uniform and non-uniform rates of convergence type results.