CHAPTER 6

RATES OF CONVERGENCE IN LOCAL LIMIT THEOREM FOR INDEPENDENT SUMMANDS - II

6.1 INTRODUCTION:

Let $\{X_n\}$ be a sequence of mutually independent r.v.s with corresponding sequence of absolutely continuous d.f.s $\{G_n\}$. Suppose, for each n, $G_n \in \{F_1, F_2\}$. Let the d.f. F_i belong to the domain of normal attraction of a stable law with index α_i, $0< \alpha_1< \alpha_2< 2$.
Let $S_{j,\tau_j(n)}$ denote the sum of the r.v.s among X_1, X_2, ... , X_n which have F_j as their d.f., j = 1, 2. Here $\tau_j(n)$ is the number of X_i's having F_j as their d.f. for a specified n. Therefore, $S_n = S_{1,\tau_1(n)} + S_{2,\tau_2(n)}$.

It is known that the limit distribution of the normalized sums $(S_n-A_n)/B_n$ need not exist (see: Sreehari (1970), Theorem 4.1). When the limit distribution of the normalized sums exists, for an appropriate choice of A_n and $B_n> 0$, Kruglov (1968) proved the local limit theorem (uniform version) in this set up. However, Kruglov assumed that the limit distribution of normalized S_n exist via necessary and sufficient conditions given by Zinger (1965).

In this chapter, we obtain the uniform as well as the non-uniform rates of convergence in the local limit theorem, under the assumption that the limit distribution of the normalized sums exists.

We state our theorems first:

Theorem 6.1.1: *Under the assumptions* [A1]-[A5], *stated below,*

$$\sup_{x \in R}|v_n(x)-v_o(x)| = O\left(\sum_{i=1}^{2}\tau_i^{1-([\alpha_i]+1)\gamma_i} + \kappa_n\right)$$

as $n \to \infty$, *where* $\kappa_n = |\{C_2(\tau_2)/C_1(\tau_1)\}^{\alpha_2} - \lambda^{\alpha_2}|$.

Theorem 6.1.2: *Under the assumptions* [A1]-[A5], *stated below,*

$$\sup_{x \in R}(1+|x^*|^{\alpha_1})|v_n(x)-v_o(x)| = O\left(\sum_{i=1}^{2}\tau_i^{1-([\alpha_i]+1)\gamma_i} + \kappa_n\right)$$

as $n \to \infty$, *where* $x^* = \max(|x|, 1)$.

We prove the theorems in Section 6.4. We introduce notations and assumptions in Section 6.2. Preliminary results required to prove these theorems are proved in the Section 6.3. These results can be proved on the lines of corresponding results of Section 2.3 of Chapter 2.

6.2 NOTATIONS AND ASSUMPTIONS:

Let Y_i denote a stable or strictly stable r.v. with index α_i, according as $1< \alpha_i< 2$ (with $EY_i = 0$) or $0< \alpha_i\leq 1$, respectively, having d.f. $F_{i,0}$, and let $w_{io}(.)$ denote its c.f., i = 1, 2. Assume that $0< \alpha_1< \alpha_2< 2$. Let $v_{io}^{\bullet}(.)$

and $v^*_{2o}(.)$ denote the p.d.f. corresponding to the c.f.s $w_{1o}(.)$ and $w_{2o}(.)$ respectively.

We assume $EX_n = 0$, whenever it exists.

It is known from Theorem 4.1 of Sreehari (1970) that the limit distribution of normalized sums $(S_n-A_n)/B_n$ exists and is a mixture of the two stable d.f.s $F_{1,0}$ and $F_{2,0}$ iff $\lim_{n\to\infty} C_j(\tau_j)/C_k(\tau_k) = \lambda_{j,k} > 0$ exists and is finite; and in this case, we may take $B_n = C_k(\tau_k)$, $j \neq k = 1, 2$. Here $C_j(n) = C_j\ n^{\gamma_j}$, C_j being the constant of proportionality depending on d.f. F_j, $\gamma_j = 1/\alpha_j$, $j = 1, 2$. Thus, the limit distribution of S_n, properly normalized, need not always exist.

We shall prove the results under the assumption of the existence of the limit distribution of S_n, properly normalized, or equivalently under the assumption that $\lim_{n\to\infty} C_2(\tau_2)/C_1(\tau_1) = \lambda > 0$ exists and is finite. We take $B_n = C_1(\tau_1)$, without any loss of generality.

Let $w_1(.)$ and $w_2(.)$ denote c.f.s corresponding to d.f.s F_1 and F_2 respectively. Let $Z_n = S_n/B_n$.

We write $\phi_n(t) = E[\exp(itZ_n)]$

$= \{w_1(tB_n^{-1})\}^{\tau_1}\{w_2(tB_n^{-1})\}^{\tau_2}$, $n = 1, 2, \ldots$.

Note that we have from canonical representation of stable c.f.

$$w_o(t) = w_{1o}(t)w_{2o}(\lambda t), \text{ for all } t. \qquad \ldots(6.2.1)$$

Here $w_o(t)$ represents c.f. corresponding to a mixture of stable d.f.s $F_{1,o}$ and $F_{2,o}$.

Thus, in view of the discussions in Section 6.1,

$$\lim_{n \to \infty} \phi_n(t) = w_o(t) \text{ for all } t. \qquad \ldots(6.2.2)$$

Let $\gamma_i = 1/\alpha_i$, $i = 1, 2$.

Define, for each integer n and real x,

$$\alpha_{o,\tau_1}(t,x) = \int_{|u|\le|x|\tau_1^{\gamma_1}} e^{itu}\, dF_{1,o}(u), \qquad \ldots(6.2.3)$$

$$\alpha_{o,\tau_2}(t,x) = \int_{|u|\le|x|\tau_2^{\gamma_2}C_1/C_2} e^{ituC_1^{-1}}\, dF_{2,o}(u), \qquad \ldots(6.2.4)$$

$$\alpha_{1,\tau_1}(t,x) = \int_{|u|\le|x|C_1\tau_1^{\gamma_1}} e^{ituC_1^{-1}}\, dF_1(u), \qquad \ldots(6.2.5)$$

$$\alpha_{2,\tau_2}(t,x) = \int_{|u|\le|x|C_1\tau_2^{\gamma_2}} e^{ituC_1^{-1}}\, dF_2(u), \qquad \ldots(6.2.6)$$

$$\beta_{o,\tau_1}(t,x) = \int_{|u|>|x|\tau_1^{\gamma_1}} e^{itu}\, dF_{1,o}(u), \qquad \ldots(6.2.7)$$

$$\beta_{o,\tau_2}(t,x) = \int_{|u|>|x|\tau_2^{\gamma_2} C_1/C_2} e^{ituC_1^{-1}C_2} dF_{2,o}(u), \qquad \ldots(6.2.8)$$

$$\beta_{1,\tau_1}(t,x) = \int_{|u|>|x|C_1\tau_1^{\gamma_1}} e^{ituC_1^{-1}}\, dF_1(u), \qquad \ldots(6.2.9)$$

$$\beta_{2,\tau_2}(t,x) = \int_{|u|>|x|C_1\tau_2^{\gamma_2}} e^{ituC_1^{-1}}\, dF_2(u), \qquad \ldots(6.2.10)$$

$$A_{o,\tau_1}(t,x) = \{\alpha_{o,\tau_1}(t\tau_1^{-\gamma_1},x)\}^{\tau_1}, \qquad \ldots(6.2.11)$$

$$A_{o,\tau_2}(t,x) = \{\alpha_{o,\tau_2}(t\tau_1^{-\gamma_1},x)\}^{\tau_2}, \qquad \ldots(6.2.12)$$

$$A_{1,\tau_1}(t,x) = \{\alpha_{1,\tau_1}(t\tau_1^{-\gamma_1},x)\}^{\tau_1}, \qquad \ldots(6.2.13)$$

$$A_{2,\tau_2}(t,x) = \{\alpha_{2,\tau_2}(t\tau_1^{-\gamma_1},x)\}^{\tau_2}, \qquad \ldots(6.2.14)$$

$$B_{0,\tau_1}(t,x) = \{w_{10}(t\tau_1^{-\gamma_1})\}^{\tau_1} - A_{0,\tau_1}(t,x)$$

$$= \sum_{j=1}^{\tau_1} \binom{\tau_1}{j} \{\alpha_{0,\tau_1}(t\tau_1^{-\gamma_1},x)\}^{\tau_1-j}\{\beta_{0,\tau_1}(t\tau_1^{-\gamma_1},x)\}^j, \qquad (6.2.15)$$

$$B_{0,\tau_2}(t,x) = \{w_{20}(tC_2C_1^{-1}\tau_1^{-\gamma_1})\}^{\tau_2} - A_{0,\tau_2}(t,x)$$

$$= \sum_{j=1}^{\tau_2} \binom{\tau_2}{j} \{\alpha_{0,\tau_2}(t\tau_1^{-\gamma_1},x)\}^{\tau_2-j}\{\beta_{0,\tau_2}(t\tau_1^{-\gamma_1},x)\}^j, \qquad (6.2.16)$$

$$B_{1,\tau_1}(t,x) = \{w_1(tC_1^{-1}\tau_1^{-\gamma_1})\}^{\tau_1} - A_{1,\tau_1}(t,x)$$

$$= \sum_{j=1}^{\tau_1} \binom{\tau_1}{j} \{\alpha_{1,\tau_1}(t\tau_1^{-\gamma_1},x)\}^{\tau_1-j}\{\beta_{1,\tau_1}(t\tau_1^{-\gamma_1},x)\}^j, \qquad (6.2.17)$$

$$B_{2,\tau_2}(t,x) = \{w_2(tC_1^{-1}\tau_1^{-\gamma_1})\}^{\tau_2} - A_{2,\tau_2}(t,x)$$

$$= \sum_{j=1}^{\tau_2} \binom{\tau_2}{j} \{\alpha_{2,\tau_2}(t\tau_1^{-\gamma_1},x)\}^{\tau_2-j}\{\beta_{2,\tau_2}(t\tau_1^{-\gamma_1},x)\}^j. \qquad (6.2.18)$$

Note that for

$$f_n(u) = (1/2\pi) \int_{-\infty}^{\infty} \phi_n(t)e^{-itu}dt, \qquad \ldots(6.2.19)$$

the inversion integral on the right hand side is absolutely convergent. The absolutely convergent integrals provide the continuous p.d.f. that we shall use in our theorems.

Let $\Xi = \{(t, n, x): |t| \le \varepsilon\tau_1^{\gamma_1}, n \ge n_0, |x| \ge 1\}$, where ε will be same as in Lemma 6.3.1 and n_0 is large positive integer.

We now make the following assumptions.

[A1] The d.f. F_j, $j = 1, 2$ is absolutely continuous;

[A2] The d.f. F_i, $i = 1, 2$ belongs to the domain of normal attraction of a stable law $F_{i,o}$ with index α_i or of a strictly stable law $F_{i,o}$ with index α_i, according as $1< \alpha_i< 2$ or $0< \alpha_i\leq 1$. Also we assume that $\alpha_1< \alpha_2$. If $F_i \in D_{NA}(\alpha_i)$, $\alpha_i> 1$ then the mean of the corresponding d.f. will be assumed zero.

[A3] $\lim_{n \to \infty} C_2(\tau_2)/C_1(\tau_1) = \lambda> 0$ exists and is finite;

[A4] For some integer $p\geq 1$, $\int_{-\infty}^{\infty}|w_j(t)|^p dt< \infty$, $j = 1, 2$;

[A5] $\int_{-\infty}^{\infty}u^{[\alpha_1]+1}|v_1^*(u) - (1/C_1)v_{1o}^*(u/C_1)|du< \infty$
and $\int_{-\infty}^{\infty}u^{[\alpha_2]+1}|v_2^*(u) - (1/C_1)v_{2o}^*(u/C_1)|du< \infty$;

Remark 6.2.1: It may be noted that assumption **[A5]** becomes $\int_{-\infty}^{\infty}u^{[\alpha_i]+1}|v_i^*(u) - (1/C_2)v_{io}^*(u/C_2)|du < \infty$, $i = 1, 2$, when B_n is taken to be $C_2(\tau_2)$.

6.3 PRELIMINARY RESULTS:

Now we mention some preliminary lemmas required to prove the theorems of Section 6.1.

Lemma 6.3.1: Under the assumptions **[A1]**, **[A2]** and **[A3]**, there exist positive constants ε, d and d_1 such that, for all $(t, n, x) \in \Xi$, we have

$$|A_{O,\tau_1}(t,x)| \le d_1 e^{-d|t|^{\alpha_1}}, \qquad \ldots(6.3.1)$$

$$|A_{1,\tau_1}(t,x)| \le d_1 e^{-d|t|^{\alpha_1}}, \qquad \ldots(6.3.2)$$

$$|A_{O,\tau_2}(t,x)| \le d_1 e^{-d|t|^{\alpha_2}}, \qquad \ldots(6.3.3)$$

$$|A_{2,\tau_2}(t,x)| \le d_1 e^{-d|t|^{\alpha_2}}. \qquad \ldots(6.3.4)$$

Proof: The proofs of (6.3.1) and (6.3.2) can be obtained on the lines of the Lemma 2.3.2. We shall prove equation (6.3.4); equation (6.3.3) can be proved similarly.

By a lemma in Gnedenko and Kolmogorov (1968, p.238), there exist positive constants d and d_1 such that in a neighbourhood of origin, we have $|w_2(t)| \le e^{-d|t|^{\alpha_2}}$. Choose $0 < \varepsilon < (\lambda/C_2)^{\alpha_2}$ such that $|w_2(t)| \le e^{-d|t|^{\alpha_2}}$ whenever $|t| \le \varepsilon/C_1$. Then

$$|w_2(tB_n^{-1})|^{\tau_2} \le e^{-d|t|^{\alpha_2} B_n^{-\alpha_2} \tau_2}$$

$$\le e^{-d|t|^{\alpha_2}((\lambda/C_2)^{\alpha_2}-\varepsilon)},$$

because under **[A3]**, $B_n^{-\alpha_2}\tau_2 \to (\lambda/C_2)^{\alpha_2}$.

$$\le e^{-d|t|^{\alpha_2}} \text{ for } |t| \le \varepsilon\tau_1^{\gamma_1}.$$

Now consider,

$$|A_{2,\tau_2}(t,x)| \le |\alpha_{2,\tau_2}(t\tau_1^{-\gamma_1},x)|^{\tau_2}$$

$$\le |w_2(tB_n^{-1}) - \beta_{2,\tau_2}(t\tau_1^{-\gamma_1},x)|^{\tau_2}$$

$$\leq \sum_{j=0}^{\tau_2} \binom{\tau_2}{j} |w_2(tB_n^{-1})|^{\tau_2-j} |\beta_{2,\tau_2}(t\tau_1^{-\gamma_1}, x)|^j$$

$$\leq \sum_{j=0}^{\tau_2} (\tau_2^j / j!) \quad (e^{-d|t|^{\alpha_2}})^{(\tau_2-j)/\tau_2}$$

$$\{1-F_2(|x|\tau_2^{\gamma_2}C_1)+F_2(|x|\tau_2^{\gamma_2}C_1)\}^j$$

$$\leq e^{-d|t|^{\alpha_2}} \sum_{j=0}^{\tau_2} (\tau_2^j / j!) \quad (e^{d|t|^{\alpha_2}})^{j/\tau_2} \{d_2|x|^{-\alpha_2}\tau_2^{-1}C_1^{-\alpha_2}\}^j$$

$$\leq e^{-d|t|^{\alpha_2}} \sum_{j=0}^{\tau_2} (\tau_2^j / j!) \quad e^{(d|t|/\tau_1^{\gamma_1})^{\alpha_2}(\tau_1^{\gamma_1}/\tau_2^{\gamma_2})^{\alpha_2 j}}$$

$$\{d_2|x|^{-\alpha_2}\tau_2^{-1}C_1^{-\alpha_2}\}^j$$

$$\leq e^{-d|t|^{\alpha_2}} \sum_{j=0}^{\infty} [\exp\{(d\varepsilon)^{\alpha_2}\{(\lambda C_2/C_1)^{\alpha_2}+\varepsilon\}d_2C_1^{-\alpha_2}\}]^j/j!$$

$$\leq d_1 e^{-d|t|^{\alpha_2}}.$$

Hence the result. □

Throughout the rest of the chapter, ε will be taken as same as that of Lemma 6.3.1.

Lemma 6.3.2: Under the assumptions **[A1]**, **[A2]**, **[A3]** and **[A4]**, there exist polynomials $P_1(.)$ and $P_2(.)$ such that, for all $(t, n, x) \in \Xi$, we have

(i) $|\{\alpha_{1,\tau_1}(t\tau_1^{-\gamma_1}, x)\}^{\tau_1-j} - \{\alpha_{0,\tau_1}(t\tau_1^{-\gamma_1}, x)\}^{\tau_1-j}|$

$$\leq \tau_1^{1-([\alpha_1]+1)\gamma_1} P_1(|t|) e^{-d|t|^{\alpha_1}}, \quad 1 \leq j \leq \tau_1 \qquad \ldots(6.3.5)$$

and

(ii) $|\{\alpha_{2,\tau_2}(t\tau_1^{-\gamma_1},x)\}^{\tau_2-j} - \{\alpha_{0,\tau_2}(t\tau_1^{-\gamma_1},x)\}^{\tau_2-j}|$

$$\leq \tau_2^{1-([\alpha_2]+1)\gamma_2} P_2(|t|)e^{-d|t|^{\alpha_2}}, \quad 1\leq j\leq \tau_2. \qquad \ldots(6.3.6)$$

Proof: We shall prove equation (6.3.6) only. Equation (6.3.5) can be proved on the lines of proof of Lemma 2.3.3.

As regards the proof of (ii), the LHS can be rewritten as

$$|\{\alpha_{2,\tau_2}(t^*\tau_2^{-\gamma_2},x)\}^{\tau_2-j} - \{\alpha_{0,\tau_2}(t^*\tau_2^{-\gamma_2},x)\}^{\tau_2-j}| \qquad ..(6.3.7)$$

where $t^* = t\tau_1^{-\gamma_1}\tau_2^{-\gamma_2}$.

Now by Lemma 2.3.3 with n and γ replaced by τ_2 and γ_2 we get

$$|\{\alpha_{2,\tau_2}(t^*\tau_2^{-\gamma_2},x)\}^{\tau_2-j} - \{\alpha_{0,\tau_2}(t^*\tau_2^{-\gamma_2},x)\}^{\tau_2-j}|$$

$$\leq \tau_2^{1-([\alpha_2]+1)\gamma_2} P_2(|t|)e^{-d|t|^{\alpha_2}}, \qquad \ldots(6.3.8)$$

for $|t^*|\leq \varepsilon\tau_2^{\gamma_2}$. $\qquad \ldots(6.3.9)$

But $|t^*|\leq \varepsilon\tau_2^{\gamma_2}$ is same as $|t|\leq \varepsilon\tau_1^{\gamma_1}$ so that (6.3.6) follows for $|t|\leq \varepsilon\tau_1^{\gamma_1}$.□

Now we define some functions, similar to equations (2.3.18) and (2.3.19), which will be useful in the proofs of the theorems and some of the lemmas.

Let

$$d_{1,\tau_1}(t,x) \equiv d_{\tau_1}(t,x)$$

$$= \tau_1\left\{\{\alpha_{1,\tau_1}(t\tau_1^{-\gamma_1},x)\} - \{\alpha_{0,\tau_1}(t\tau_1^{-\gamma_1},x)\}\right\}, \qquad \ldots(6.3.10)$$

$$d_{2,\tau_2}(t,x) \equiv d_{\tau_2}(t,x)$$

$$= \tau_2\left\{\{\alpha_{2,\tau_2}(t\tau_1^{-\gamma_1},x)\} - \{\alpha_{0,\tau_2}(t\tau_1^{-\gamma_1},x)\}\right\}, \qquad \ldots(6.3.11)$$

$$S_{1,\tau_1}(t,x) \equiv S_{\tau_1}(t,x)$$

$$= \tau_1^{-1}\left\{\sum_{m=1}^{\tau_1}\{\alpha_{1,\tau_1}(t\tau_1^{-\gamma_1},x)\}^{\tau_1-m}\{\alpha_{0,\tau_1}(t\tau_1^{-\gamma_1},x)\}^{m-1}\right\}, \qquad \ldots(6.3.12)$$

$$S_{2,\tau_2}(t,x) \equiv S_{\tau_2}(t,x)$$

$$= \tau_2^{-1}\left\{\sum_{m=1}^{\tau_2}\{\alpha_{2,\tau_2}(t\tau_1^{-\gamma_1},x)\}^{\tau_2-m}\{\alpha_{0,\tau_2}(t\tau_1^{-\gamma_1},x)\}^{m-1}\right\}, \qquad \ldots(6.3.13)$$

Properties of the function $d_{\tau_j}(t,x)$

Lemma 6.3.3: Under the assumptions **[A1]**, **[A2]**, **[A3]** and **[A5]**,. for all x with $|x| \geq 1$ and all values of t, following holds truen:

(i) $|d_{\tau_1}(t,x)| \leq \tau_1^{1-([\alpha_1]+1)\gamma_1} P_1(|t|)$ ...(6.3.14)

(ii) $|d_{\tau_1}^{(1)}(t,x)| \leq \tau_1^{1-([\alpha_1]+1)\gamma_1} P_2(|t|)$ ...(6.3.15)

(ii) $|d_{\tau_1}^{(2)}(t,x)| \leq \tau_1^{1-([\alpha_1]+1)\gamma_1} P_3(|t|)$ ...(6.3.16)

(iv) $|d_{\tau_2}(t,x)| \leq \tau_2^{1-([\alpha_2]+1)\gamma_2} P_4(|t|)$ ...(6.3.17)

(v) $|d_{\tau_2}^{(1)}(t,x)| \leq \tau_2^{1-([\alpha_2]+1)\gamma_2} P_5(|t|)$ ...(6.3.18)

(vi) $|d_{\tau_2}^{(2)}(t,x)| \leq \tau_2^{1-([\alpha_2]+1)\gamma_2} P_6(|t|)$ ...(6.3.19)

Remark 6.3.1: This lemma follows from Lemma 2.3.4. It may be noted that the expressions at (2.3.20) and (2.3.22) (also other similar expressions could be combined as done here. Further the second derivative was not required in the case of $0< \alpha< 1$ although it is also obtainable as stated here.

Properties of the functions $\alpha_{k,\tau_j}(t,x)$

Lemma 6.3.4: Assume that **[A1]**, **[A2]** and **[A3]** hold. For each fixed n and x, $\alpha_{k,\tau_j}(t\tau_1^{-\gamma_1},x)$ is differentiable any number of times under the integral sign, k = 0, 1, 2; j = 1, 2. For all values of t and x with $|x|\geq 1$ we have

$$\text{(i)}\ |\alpha^{(1)}_{0,\tau_1}(t\tau_1^{-\gamma_1},x)| \leq \begin{cases} d|x|^{1-\alpha_1}\tau_1^{\gamma_1-1}, & \text{if } 0<\alpha_1<1 \\ \tau_1^{\gamma_1-1}\ P_1(|t|), & \text{if } 1\leq\alpha_1<2, \end{cases} \qquad \ldots(6.3.20)$$

$$|\alpha^{(1)}_{1,\tau_1}(t\tau_1^{-\gamma_1},x)| \leq \begin{cases} d|x|^{1-\alpha_1}\tau_1^{\gamma_1-1}, & \text{if } 0<\alpha_1<1 \\ \tau_1^{\gamma_1-1}\ P_2(|t|), & \text{if } 1\leq\alpha_1<2, \end{cases} \qquad \ldots(6.3.21)$$

$$\text{(ii)}\ |\alpha^{(1)}_{0,\tau_2}(t\tau_1^{-\gamma_1},x)| \leq \begin{cases} d|x|^{1-\alpha_2}\tau_2^{\gamma_2-1}, & \text{if } 0<\alpha_2<1 \\ \tau_2^{\gamma_2-1}\ P_3(|t|), & \text{if } 1\leq\alpha_2<2, \end{cases} \qquad \ldots(6.3.22)$$

$$|\alpha^{(1)}_{2,\tau_2}(t\tau_1^{-\gamma_1},x)| \leq \begin{cases} d|x|^{1-\alpha_2}\tau_2^{\gamma_2-1}, & \text{if } 0<\alpha_2<1 \\ \tau_2^{\gamma_1-1}\ P_4(|t|), & \text{if } 1\leq\alpha_2<2, \end{cases} \qquad \ldots(6.3.23)$$

(iii) $|\alpha^{(2)}_{0,\tau_1}(t\tau_1^{-\gamma_1},x)| \le |x|^{2-\alpha_1}\tau_1^{2\gamma_1-1}$ ...(6.3.24)

$|\alpha^{(2)}_{1,\tau_1}(t\tau_1^{-\gamma_1},x)| \le |x|^{2-\alpha_1}\tau_1^{2\gamma_1-1}$ ...(6.3.25)

(iv) $|\alpha^{(2)}_{0,\tau_2}(t\tau_1^{-\gamma_1},x)|$

$\le |x|^{2-\alpha_2}\tau_2^{2\gamma_2-1}$, (6.3.26)

$|\alpha^{(2)}_{2,\tau_2}(t\tau_1^{-\gamma_1},x)|$

$\le |x|^{2-\alpha_2}\tau_2^{2\gamma_2-1}$ ...(6.3.27)

Also, for all $x \neq 0$, $0< \alpha_k< 2$, every large but fixed integer s,

(v) $\int_{-\infty}^{\infty} |\alpha_{0,\tau_j}(t,x)|^{\tau_j}\, dt = 0(\tau_j^{-\gamma_j})$, j1, 2; ...(6.3.28)

$\int_{-\infty}^{\infty} |\alpha_{j,\tau_j}(t,x)|^{\tau_j}\, dt = 0(\tau_j^{-\gamma_j})$, j1, 2; ...(6.3.29)

(vi) $\int_{-\infty}^{\infty} |\alpha_{0,\tau_j}(t,x)|^{2s}\, dt < \infty$, j = 1, 2; ...(6.3.30)

$\int_{-\infty}^{\infty} |\alpha_{j,\tau_j}(t,x)|^{2s}\, dt < \infty$, j = 1, 2. ...(6.3.31)

(vii) $\int_{-\infty}^{\infty} |\beta_{0,\tau_j}(t,x)|^{2s}\, dt < \infty$, j = 1, 2; ...(6.3.30a)

$\int_{-\infty}^{\infty} |\beta_{j,\tau_j}(t,x)|^{2s}\, dt < \infty$, j = 1, 2. ...(6.3.31b)

Remark 6.3.2 This Lemma can be proved on the lines of Lemma 2.3.5.

Properties of $S_{\tau_j}(t,x)$ function

Lemma 6.3.5: Under the assumptions **[A1]**, **[A2]** and **[A3]**, for all (t, n, x) $\in \Xi$, we have

(i) $|S_{\tau_1}(t,x)| \le d_1 e^{-d|t|^{\alpha_1}}$, ...(6.3.32)

(ii) $|S^{(1)}_{\tau_1}(t,x)|\,|x|^{([\alpha_1]+1)-\alpha_1} e^{-d|t|^{\alpha_1}} P_1(|t|)$, ...(6.3.33)

(iii) $|S^{(2)}_{\tau_1}(t,x)|\,|x|^{([\alpha_1]+1)-\alpha_1} e^{-d|t|^{\alpha_1}} P_2(|t|)$, ...(6.3.34)

(iv) $|S_{\tau_2}(t,x)| \le d_1 e^{-d|t|^{\alpha_2}}$, ...(6.3.35)

(v) $|S^{(1)}_{\tau_2}(t,x)| \le |x|^{([\alpha_2]+1)-\alpha_2} e^{-d|t|^{\alpha_2}} P_3(|t|)$, ...(6.3.36)

(vi) $|S^{(2)}_{\tau_2}(t,x)|\,|x|^{([\alpha_2]+1)-\alpha_2} e^{-d|t|^{\alpha_2}} P_4(|t|)$. ...(6.3.37)

Remark 6.3.3. This lemma follows from Lemma 2.3.6. It may be noted that we have combined the two cases $0< \alpha <1$ and $1\le \alpha < 2$ and all the results hold for $|t|\le \varepsilon\tau_1^{\gamma_1}$ for reasons explained in the proof of Lemma 6.3.1.

Lemma 6.3.6· Under the assumptions **[A1]**, **[A2]**, **[A3]** and **[A5]**, there exist polynomials $P_1(.)$ and $P_2(.)$ such that for all (t, n, x) $\in \Xi$, we have,

(i) $|A_{1,\tau_1}(t,x) - A_{O,\tau_1}(t,x)|$

$\le e^{-d|t|^{\alpha_1}} P_1(|t|)\,\tau_1^{1-([\alpha_1]+1)\gamma_1}$, ...(6.3.38)

(ii) $|A^{(1)}_{1,\tau_1}(t,x) - A^{(1)}_{O,\tau_1}(t,x)|$

$\le |x|^{([\alpha_1]+1)-\alpha_1} e^{-d|t|^{\alpha_1}} P_2(|t|)\,\tau_1^{1-([\alpha_1]+1)\gamma_1}$, ...(6.3.39)

(iii) $|A^{(2)}_{1,\tau_1}(t,x) - A^{(2)}_{0,\tau_1}(t,x)|$

$\le |x|^{([\alpha_1]+1)-\alpha_1} e^{-d|t|^{\alpha_1}} P_3(|t|)\tau_1^{1-([\alpha_1]+1)\gamma_1}$, ...(6.3.40)

(iv) $|A_{2,\tau_2}(t,x) - A_{0,\tau_2}(t,x)|$

$\le e^{-d|t|^{\alpha_2}} P_4(|t|)\tau_2^{1-([\alpha_2]+1)\gamma_2}$, ...(6.3.41)

(v) $|A^{(1)}_{2,\tau_2}(t,x) - A^{(1)}_{0,\tau_2}(t,x)|$

$\le |x|^{([\alpha_2]+1)-\alpha_2} e^{-d|t|^{\alpha_2}} P_5(|t|)\tau_2^{1-([\alpha_2]+1)\gamma_2}$, ...(6.3.42)

(vi) $|A^{(2)}_{2,\tau_2}(t,x) - A^{(2)}_{0,\tau_2}(t,x)|$

$\le |x|^{([\alpha_2]+1)-\alpha_2} e^{-d|t|^{\alpha_2}} P_6(|t|)\tau_2^{1-([\alpha_2]+1)\gamma_2}$. ...(6.3.43)

Proof: The results follow from Lemma 2.3.7 as in the previous lemmas.□

Lemma 6.3.7: Under the assumptions **[A1]**, **[A2] and [A3]**, there exist polynomials $P_1(.)$, $P_2(.)$, $P_3(.)$ and $P_4(.)$ such that for all $(t, n, x) \in \Xi$, we have,

(i) $$|A^{(1)}_{0,\tau_1}(t,x)| \le \begin{cases} d_1|x|^{1-\alpha_1} e^{-d|t|^{\alpha_1}}, & \text{if } 0<\alpha_1<1 \\ P_1(|t|)\ e^{-d|t|^{\alpha_1}}, & \text{if } 1\le\alpha_1<2 \end{cases}$$ ...(6.3.44)

(ii) $$|A^{(1)}_{1,\tau_1}(t,x)| \le \begin{cases} d_1|x|^{1-\alpha_1} e^{-d|t|^{\alpha_1}}, & \text{if } 0<\alpha_1<1 \\ P_1(|t|)\ e^{-d|t|^{\alpha_1}}, & \text{if } 1\le\alpha_1<2 \end{cases}$$ ...(6.3.45)

(iii) $|A^{(2)}_{0,\tau_1}(t,x)| \le P_2(|t|) e^{-d|t|^{\alpha_1}} |x|^{2-\alpha_1}$ ...(6.3.46)

(iv) $|A^{(2)}_{1,\tau_1}(t,x)| \le P_2(|t|) e^{-d|t|^{\alpha_1}} |x|^{2-\alpha_1}$ ...(6.3.47)

(v) $|A^{(1)}_{0,\tau_2}(t,x)| \le \begin{cases} d_1|x|^{1-\alpha_2} e^{-d|t|^{\alpha_1}}, & \text{if } 0<\alpha_2<1 \\ P_3(|t|) \quad e^{-d|t|^{\alpha_1}}, & \text{if } 1\le\alpha_2<2 \end{cases}$

...(6.3.48)

(vi) $|A^{(1)}_{2,\tau_2}(t,x)| \le \begin{cases} d_1|x|^{1-\alpha_2} e^{-d|t|^{\alpha_1}}, & \text{if } 0<\alpha_2<1 \\ P_3(|t|) \quad e^{-d|t|^{\alpha_1}}, & \text{if } 1\le\alpha_2<2 \end{cases}$

...(6.3.49)

(vii) $|A^{(2)}_{0,\tau_2}(t,x)| \le P_3(|t|)e^{-d|t|^{\alpha_1}}|x|^{2-\alpha_2}$ ...(6.3.50)

(viii) $|A^{(2)}_{2,\tau_2}(t,x)| \le P_3(|t|)e^{-d|t|^{\alpha_1}}|x|^{2-\alpha_2}$ ...(6.3.51)

Remark 6.3.4: This lemma can be proved on the lines of Lemma 2.3.8.

Lemma 6.3.8: Assume that **[A1]** and **[A2]** hold. Let $\varepsilon > o$ and an integer n_o be fixed. Let $\Theta = \{(t, n, x) \mid |t| > \varepsilon, n \ge n_o, |x| \ge 1\}$, where ε is same as in Lemma 6.3.1 and n_o is large positive integer. Let

$\mu_{(0,1)} = \sup_{\Theta} |\alpha_{0,\tau_1}(t,x)|,$ ...(6.3.52)

$\mu_{(0,2)} = \sup_{\Theta} |\alpha_{0,\tau_2}(t,x)|,$ ...(6.3.53)

$\mu_{(1,1)} = \sup_{\Theta} |\alpha_{1,\tau_1}(t,x)|,$ ...(6.3.53)

$\mu_{(2,2)} = \sup_{\Theta} |\alpha_{2,\tau_2}(t,x)|.$ ...(6.3.54)

Then, it follows that $0 \le \mu_{(0,k)} < 1$ and $0 \le \mu_{(k,k)} < 1$ for k = 1, 2.

Remark 6.3.5 This lemma is a simple modification of Lemma 2.3.9.

Lemma 6.3.9 Assume that **[A1]**, **[A2]**, **[A3]** and **[A5]** hold. Let $g_i(t,k)$ be a complex-valued function such that $|g_i(t,x)| \le \max(1, d|x|^{-\alpha_1})$, $i = 1, 2$, for $|x| \ge 1$ and for all t. Then, under assumption **[A1]** - **[A5]**, we have

(i) $\left| \int_{-\infty}^{\infty} \{B_{1,\tau_1}(t,x) - B_{0,\tau_1}(t,x)\} g_1(t,x) e^{-itx}\, dt \right|$
$= |x|^{-\alpha_1} O(\tau_1^{1-([\alpha_1]+1)\gamma_1})$

and

(ii) $\left| \int_{-\infty}^{\infty} \{B_{2,\tau_2}(t,x) - B_{0,\tau_2}(t,x)\} g_2(t,x) e^{-itx}\, dt \right|$
$= |x|^{-\alpha_2} O(\tau_2^{1-([\alpha_2]+1)\gamma_2})$.

Remark 6.3.6. These results can be proved on the lines of proof of Lemma 2.3.10.

Lemma 6.3.10 For all values of t, all x with $|x| \ge 1$ and large n,

$$B_{0,\tau_1}(t,x) \le d|x|^{-\alpha_1}, \quad \ldots(6.3.58)$$

$$B_{1,\tau_1}(t,x) \le d|x|^{-\alpha_1}, \quad \ldots(6.3.59)$$

$$B_{0,\tau_2}(t,x) \le d|x|^{-\alpha_2}, \quad \ldots(6.3.60)$$

$$B_{2,\tau_2}(t,x) \le d|x|^{-\alpha_2}. \quad \ldots(6.3.61)$$

Proof: We prove (6.3.61) only. Other equations follow similarly.

$$|B_{2,\tau_2}(t,x)|$$

$$\le \sum_{j=1}^{\tau_2} (\tau_2^j/j!)\ |\alpha_{2,\tau_2}(t\tau_1^{-\gamma_1},x)|^{\tau_2-j}\ |\beta_{2,\tau_2}(t\tau_1^{-\gamma_1},x)|^j$$

$$\le \sum_{j=1}^{\tau_2} (\tau_2^j/j!)\ \{1-F_2(|x|C_1\tau_2^{\gamma_2})+F_2(-|x|C_1\tau_2^{\gamma_2})\}^j$$

$$\le \sum_{j=1}^{\tau_2} (\tau_2^j/j!)\ d(|x|^{-\alpha_2}C_1^{-\alpha_2}\tau_2^{-1})\}^j$$

$$\le d|x|^{-\alpha_2}e^{C_1^{-\alpha_2}}$$

$$\le d|x|^{-\alpha_2}.\square$$

We shall now prove a result which is an extension of Lemma 2.3.12 to non-identically distributed case under consideration.

Lemma 6.3.11: Assume that **[A1]**, **[A2]**, **[A3]** and **[A5]** hold. Then, for all $(t,\ n,\ x) \in \Xi$, there exists a positive constant $d>0$ such that

$$|\phi_n(t) - w_o(t)| \le \{\sum_{i=1}^{2} \tau_i^{1-([\alpha_1]+1)\gamma_1} + \kappa_n\}\ P(|t|)e^{-d|t|^{\alpha_1}}.$$

where $\kappa_n = |\{C_2(\tau_2)/C_1(\tau_1)\}^{\alpha_2} - \lambda^{\alpha_2}|$.

Proof In view of the discussions in the begining of Section 6.2 of this chapter, we have

$$|\phi_n(t) - w_o(t)|$$

$$\le |\{w_1(tB_n^{-1})\}^{\tau_1}\{w_2(tB_n^{-1})\}^{\tau_2}-\{w_{1o}(t)\}\ \{w_{2o}(\lambda t)\}|$$

$$\le |\{w_1(tB_n^{-1})\}^{\tau_1}\{w_2(tB_n^{-1})\}^{\tau_2}$$

$$-\{w_{1o}(t\tau_1^{-\gamma_1})\}^{\tau_1}\ \{w_{2o}(\lambda t\tau_2^{-\gamma_2})\}^{\tau_2}|$$

$$\leq |\{w_1(tB_n^{-1})\}^{\tau_1}\{w_2(tB_n^{-1})\}^{\tau_2}$$

$$-\{w_{10}(t\tau_1^{-\gamma_1})\}^{\tau_1}\{w_2(tB_n^{-1})\}^{\tau_2}|$$

$$+ |\{w_{10}(t\tau_1^{-\gamma_1})\}^{\tau_1}\{w_2(tB_n^{-1})\}^{\tau_2}$$

$$-\{w_{10}(t\tau_1^{-\gamma_1})\}^{\tau_1}\{w_{20}(tC_2B_n^{-1})\}^{\tau_2}|$$

$$+ |\{w_{10}(t\tau_1^{-\gamma_1})\}^{\tau_1}\{w_{20}(tC_2B_n^{-1})\}^{\tau_2}$$

$$-\{w_{10}(t\tau_1^{-\gamma_1})\}^{\tau_1}\{w_{20}(\lambda t\tau_2^{-\gamma_2})\}^{\tau_2}|$$

$$= I_{1n} + I_{2n} + |\{w_{10}(t\tau_1^{-\gamma_1})\}^{\tau_1}\{w_{20}(tC_2B_n^{-1})\}^{\tau_2}$$

$$-\{w_{10}(t\tau_1^{-\gamma_1})\}^{\tau_1}\{w_{20}(\lambda t\tau_2^{-\gamma_2})\}^{\tau_2}|$$

$$= I_{1n} + I_{2n}$$

$$+|e^{-d|tc_2c_1^{-1}\tau_1^{-\gamma_1}|^{\alpha_2}\tau_2} - e^{-d|t\lambda\tau_2^{-\gamma_2}|^{\alpha_2}\tau_2}||w_{10}(t)|$$

$$= I_{1n} + I_{2n}$$

$$+|e^{-d|t|^{\alpha_2}\{c_2\tau_2^{\gamma_2}c_1^{-1}\tau_1^{-\gamma_1}\}^{\alpha_2}} - e^{-d|t|^{\alpha_2}\lambda^{\alpha_2}}|$$

$$= I_{1n} + I_{2n} + de^{-d|t|^{\alpha_2}}\kappa_n$$

$$\leq \{\sum_{i=1}^{2}\tau_i^{1-([\alpha_i]+1)\gamma_i} + \kappa_n\}\, P(|t|)e^{-d|t|^{\alpha_i}},$$

using Lemma 2.3.12.□

6.4 PROOFS OF THE RESULTS:

Proof of Theorem 6.1.1:

We shall prove the relation

$$\sup_{x \in R} |v_n(x) - v_o(x)| = O\left(\sum_{i=1}^{2} \tau_i^{1-([\alpha_i]+1)\gamma_i} + \kappa_n\right). \qquad \ldots(6.4.1)$$

Note that the inversion formula for continuous density gives that

$$2\pi\, |f_n(x) - v_o(x)| \le I_{1n} + I_{2n} + I_{3n} \qquad \ldots(6.4.2)$$

where

$$I_{1n} = \int_{|t| \le \varepsilon\tau_1^{\gamma_1}} |\phi_n(t) - w_o(t)|\, dt$$

$$I_{2n} = \int_{|t| > \varepsilon\tau_1^{\gamma_1}} |\phi_n(t)|\, dt$$

$$I_{3n} = \int_{|t| > \varepsilon\tau_1^{\gamma_1}} |w_o(t)|\, dt$$

$\varepsilon > 0$ being as in Lemma 6.3.1.

By Lemma 6.3.11, it now follows that

$$I_{1n} = \int_{|t| \le \varepsilon\tau_1^{\gamma_1}} |\phi_n(t) - w_o(t)|\, dt$$

$$\le \sum_{i=1}^{2} d_i \tau_i^{1-([\alpha_i]+1)\gamma_i}$$

$$\le O\left(\sum_{i=1}^{2} \tau_i^{1-([\alpha_i]+1)\gamma_i} + \kappa_n\right). \qquad \ldots(6.4.3)$$

There exists, for any $\varepsilon > 0$, a $c(\varepsilon) > 0$ such that

$|w_i(t)| \le e^{-c(\varepsilon)}$ and

$|w_{io}(t)| \le e^{-c(\varepsilon)}$, for all t with $|t| > \varepsilon$ and for $i = 1, 2$.

Therefore, using assumption **[A4]**, we have

$$I_{2n} = \int_{|t|>\varepsilon\tau_1^{\gamma_1}} |\phi_n(t)|dt$$

$$= \int_{|t|>\varepsilon\tau_1^{\gamma_1}} |w_1(tB_n^{-1})|^{\tau_1}|w_1(tB_n^{-1})|^{\tau_2}dt$$

$$\le \int_{|t|>\varepsilon\tau_1^{\gamma_1}} |w_1(tB_n^{-1})|^{\tau_1-p}|w_1(tB_n^{-1})|^{p}dt$$

$$\le C_1\tau_1^{\gamma_1}e^{-C(\varepsilon)(\tau_1-p)} \int_{|t|>\varepsilon/C_1} |w_1(t)|^{p}dt$$

$$\le C_1\tau_1^{\gamma_1}e^{-C(\varepsilon)(\tau_1-p)}$$

$$\le o(\sum_{i=1}^{2}\tau_i^{1-([\alpha_i]+1)\gamma_i}). \qquad ...(6.4.4)$$

Similarly, we obtain

$$I_{3n} \le C_1\tau_1^{\gamma_1}e^{-C(\varepsilon)(\tau_1-p)}$$

$$\le o(\sum_{i=1}^{2}\tau_i^{1-([\alpha_i]+1)\gamma_i}). \qquad ...(6.4.5)$$

Thus, (6.4.1) follows from the relation (6.4.2) through (6.4.5).

Proof of Theorem 6.1.2:

Note that in view of Theorem 6.1.1, it is sufficient for us to prove

$$\sup_{|x|\ge 1} |x|^{\alpha_1}|v_n(x)-v_o(x)|=O(\sum_{i=1}^{2}\tau_i^{1-([\alpha_i]+1)\gamma_i}+\kappa_n) \qquad ..(6.4.6)$$

consider, for $|x| \geq 1$,

$$|x|^{\alpha_1} |v_n(x) - v_o(x)|$$

$$\leq |x|^{\alpha_1} (2\pi)^{-1} \left| \int_{-\infty}^{\infty} e^{-itx} [\{w_1(tB_n^{-1})\}^{\tau_1} \{w_2(tB_n^{-1})\}^{\tau_2} - w_{10}(t) w_{20}(t\lambda)] dt \right|$$

$$\leq |x|^{\alpha_1} \left| \int_{-\infty}^{\infty} e^{-itx} [\{w_1(tB_n^{-1})\}^{\tau_1} - \{w_{10}(t\tau_1^{-\gamma_1})\}^{\tau_1}] \{w_2(tB_n^{-1})\}^{\tau_2} dt \right|$$

$$+ |x|^{\alpha_1} \left| \int_{-\infty}^{\infty} e^{-itx} [\{w_2(tB_n^{-1})\}^{\tau_2} - \{w_{20}(tC_2\tau_2^{-\gamma_2})\}^{\tau_2}] \{w_{10}(t)\} dt \right|$$

$$+ |x|^{\alpha_1} \left| \int_{-\infty}^{\infty} e^{-itx} [\{w_{20}(tC_2B_n^{-1})\}^{\tau_2} - \{w_{2O}(\lambda t\tau_2^{-\gamma_2})\}^{\tau_2} \{w_{1O}(t\tau_1^{-\gamma_1})\}^{\tau_1} dt \right|$$

$$= T_{1n} + T_{2n} + O(\kappa_n), \text{ say,} \qquad \ldots(6.4.7)$$

using argument of Lemma 6.3.11.

In order to establish (6.4.6), it is sufficient to prove

$$T_{1n} = O\left(\sum_{i=1}^{2} \tau_i^{1-([\alpha_1]+1)\gamma_1}\right), \qquad \ldots(6.4.8)$$

$$T_{2n} = O\left(\sum_{i=1}^{2} \tau_i^{1-([\alpha_1]+1)\gamma_1}\right). \qquad \ldots(6.4.9)$$

We shall prove equation (6.4.9) only. Equation (6.4.8) can be proved similarly. Observe that using (6.2.12), (6.2.14), (6.2.16) and (6.2.18), we can write

$$\int_{-\infty}^{\infty} e^{-itx}[\{w_2(tB_n^{-1})\}^{\tau_2} - \{w_{20}(tC_2\tau_2^{-\gamma_2})\}^{\tau_2}]\{w_{10}(t)\}dt$$

$$= \int_{-\infty}^{\infty} e^{-itx}[A_{2,\tau_2}(t,x) - A_{O,\tau_2}(t,x)]\{w_{10}(t)\}dt$$

$$+ \int_{-\infty}^{\infty} e^{-itx}[B_{2,\tau_2}(t,x) - B_{O,\tau_2}(t,x)]\{w_{10}(t)\}dt$$

$$= T_{2n}(A) + T_{2n}(B), \text{ say.} \qquad \ldots(6.4.10)$$

Estimation of $T_{2n}(A)$:

We shall prove that

$$|T_{2n}(A)| = |x|^{-\alpha_1} O\left(\sum_{i=1}^{2} \tau_i^{1-([\alpha_1]+1)\gamma_1}\right). \qquad \ldots(6.4.11)$$

Now,

$$|T_{2n}(A)| = \left|\int_{-\infty}^{\infty} e^{-itx}[A_{2,\tau_2}(t,x) - A_{O,\tau_2}(t,x)]\{w_{10}(t)\}dt\right|$$

$$= \left|\int_{-\infty}^{\infty} e^{-itx}[A_{2,\tau_2}(t,x) - A_{O,\tau_2}(t,x)]\right.$$

$$\left.[A_{o,\tau_1}(t,x) + B_{O,\tau_1}(t,x)]dt\right|$$

$$\leq \left|\int_{-\infty}^{\infty} e^{-itx}[A_{2,\tau_2}(t,x) - A_{O,\tau_2}(t,x)]A_{O,\tau_1}(t,x)dt\right|$$

$$+ \left|\int_{-\infty}^{\infty} e^{-itx}[A_{2,\tau_2}(t,x) - A_{O,\tau_2}(t,x)]B_{O,\tau_1}(t,x)dt\right|$$

$$= T_{2n1}(A) + T_{2n2}(A), \text{ say.} \qquad .(6.4.12)$$

Estimate of $T_{2n1}(A)$:

First of all we consider the integral

$$\int_{-\infty}^{\infty} e^{-itx} A_{o,\tau_1}(t,x)\ A_{2,\tau_2}(t,x)dt.$$

Because $A_{O,\tau_1}(t,x)$, $A^{(1)}_{O,\tau_1}(t,x)$, $A^{(2)}_{O,\tau_1}(t,x)$, $A_{2,\tau_2}(t,x)$, $A^{(1)}_{2,\tau_2}(t,x)$ and $A^{(2)}_{2,\tau_2}(t,x)$ are absolutely integrable, simple techniques involving integration by parts give us,

$$\int_{-\infty}^{\infty} e^{-itx} A_{o,\tau_1}(t,x)\ A_{2,\tau_2}(t,x)\,dt$$

$$= -x^{-2}\ [\ \int_{-\infty}^{\infty} e^{-itx} A^{(2)}_{o,\tau_1}(t,x)\ A_{2,\tau_2}(t,x)\,dt$$

$$+\ 2\int_{-\infty}^{\infty} e^{-itx} A^{(1)}_{o,\tau_1}(t,x)\ A^{(1)}_{2,\tau_2}(t,x)\,dt$$

$$+\ \int_{-\infty}^{\infty} e^{-itx} A_{o,\tau_1}(t,x)\ A^{(2)}_{2,\tau_2}(t,x)\,dt\ ]. \qquad \ldots(6.4.13)$$

On evaluating $\int_{-\infty}^{\infty} e^{-itx} A_{o,\tau_1}(t,x) A_{o,\tau_2}(t,x)\,dt$ on the lines of (6.4.13), we have then

$T_{2n1}(A)$

$$= \left| \int_{-\infty}^{\infty} e^{-itx} [A_{2,\tau_2}(t,x) - A_{O,\tau_2}(t,x)] A_{o,\tau_1}(t,x)\,dt \right|$$

$$\le x^{-2}\left(\int_{|t|\le\varepsilon\tau_1^{\gamma_1}} + \int_{|t|>\varepsilon\tau_1^{\gamma_1}} \right) |A_{2,\tau_2}(t,x) - A_{O,\tau_2}(t,x)|\ |A^{(2)}_{o,\tau_1}(t,x)|\,dt$$

$$+2x^{-2}\left(\int_{|t|\le\varepsilon\tau_1^{\gamma_1}} + \int_{|t|>\varepsilon\tau_1^{\gamma_1}} \right) |A^{(1)}_{2,\tau_2}(t,x) - A^{(1)}_{O,\tau_2}(t,x)|\ |A^{(1)}_{o,\tau_1}(t,x)|\,dt$$

$$+\ x^{-2}\left(\int_{|t|\le\varepsilon\tau_1^{\gamma_1}} + \int_{|t|>\varepsilon\tau_1^{\gamma_1}} \right) |A^{(2)}_{2,\tau_2}(t,x) - A^{(2)}_{O,\tau_2}(t,x)|\ |A_{o,\tau_1}(t,x)|\,dt$$

$$= M_1(x) + \ldots + M_6(x),\ \text{say}. \qquad \ldots(6.4.14)$$

Observe that (6.3.41), (6.3.42), (6.3.43), (6.3.44), (6.3.46) together with (6.3.1) imply that

$$M_i(x) = |x|^{-\alpha_1}\ O\Big(\sum_{i=1}^{2} \tau_i^{\,1-([\alpha_1]+1)\gamma_1}\Big), \qquad \ldots(6.4.15)$$

for $i = 1, 3, 5$.

Finally, as a consequence of inequalities (i), (ii), (iii) and (iv) of Lemma 6.3.4, Lemma 6.3.8 and assumption **[A2]**, we get for i = 2, 4, 6, as $n\to\infty$,

$$M_i(x) = |x|^{-\alpha_1} O(\sum_{i=1}^{2} \tau_i^{1-([\alpha_i]+1)\gamma_i}), \quad \ldots(6.4.16)$$

for i = 2, 4, 6.

Thus from equations (6.4.12) to (6.4.16) it follows that,

$$|T_{2n1}(A)| = |x|^{-\alpha_1} O(\sum_{i=1}^{2} \tau_i^{1-([\alpha_i]+1)\gamma_i}). \quad \ldots(6.4.17)$$

Next we consider the estimation of $T_{2n2}(A)$.

Estimate of $T_{2n2}(A)$:

Write $T_{2n2}(A)$ as

$$T_{2n2}(A) = \left(\int_{|t|\le\varepsilon\tau_1^{\gamma_1}} + \int_{|t|>\varepsilon\tau_1^{\gamma_1}} \right) e^{-itx}\{A_{2,\tau_2}(t,x) - A_{O,\tau_2}(t,x)\} B_{o,\tau_1}(t,x)\,dt$$

$$= T_{2n2}(A1) + T_{2n2}(A2), \text{ say.} \quad \ldots(6.4.18)$$

Now,

$$|T_{2n2}(A1)| = |x|^{-\alpha_2} O(\tau_2^{1-([\alpha_2]+1)\gamma_2}) \le |x|^{-\alpha_1} O(\sum_{i=1}^{2} \tau_i^{1-([\alpha_i]+1)\gamma_i}) \quad \ldots(6.4.19)$$

is evident from (6.3.41) and (6.3.58); whereas, as a consequence of Lemma 6.3.8 and assumption **(A2)**, we get

$|T_{2n2}(A2)|$

$$\le d_1|x|^{-\alpha_1}[\int_{|t|>\varepsilon\tau_1^{\gamma_1}}\{|A_{2,\tau_2}(t,x)|+|A_{o,\tau_2}(t,x)|\}dt]$$

$$\le d_1|x|^{-\alpha_1}\tau_1^{\gamma_1}\ \{\mu_{(2,2)}^{\tau_2-p}\int_{|t|>\varepsilon}\{|\alpha_{2,\tau_2}(t,x)|^p dt$$

$$+\mu_{(0,2)}^{\tau_2-p}\int_{|t|>\varepsilon}\{|\alpha_{O,\tau_2}(t,x)|^p dt\}$$

$$\le d_1|x|^{-\alpha_1}\tau_1^{\gamma_1}\ \{\mu_{(2,2)}^{\tau_2-p}+\ \mu_{(0,2)}^{\tau_2-p}\}. \qquad \ldots(6.4.20)$$

Therefore, it follows that,

$$|T_{2n2}(AA2)| = |x|^{-\alpha_1}\ o(\sum_{i=1}^{2}\tau_i^{1-([\alpha_1]+1)\gamma_1}). \qquad \ldots(6.4.21)$$

Thus, (6.4.11) is proved.

Estimation of $T_{2n}(B)$:

We have $T_{2n}(B)$

$$= \int_{-\infty}^{\infty}e^{-itx}[B_{2,\tau_2}(t,x) - B_{O,\tau_2}(t,x)]\{w_{10}(t)\}dt$$

$$= \int_{-\infty}^{\infty}e^{-itx}[B_{2,\tau_2}(t,x) - B_{O,\tau_2}(t,x)]A_{0,\tau_1}(t,x)dt$$

$$+ \int_{-\infty}^{\infty}e^{-itx}[B_{2,\tau_2}(t,x) - B_{O,\tau_2}(t,x)]B_{0,\tau_1}(t,x)dt.$$

Note that $A_{O,\tau_1}(t,x)$ and $B_{O,\tau_1}(t,x)$ are complex valued functions with absolute value of each of them being less than or equal to $\max(1, d|x|^{-\alpha_1})$ by Lemma 6.3.10. Each of the terms $A_{O,\tau_1}(t,x)$ and $B_{O,\tau_1}(t,x)$ satisfies all the properties of the function $g_1(t,x)$ introduced in Lemma 6.3.9. We take $g_{11}(t,x) = A_{O,\tau_1}(t,x)$ and $g_{12}(t,x) = B_{O,\tau_1}(t,x)$ as $g_1(t,x)$ of Lemma 6.3.9 and apply Lemma 6.3.9. Therefore,

$$|T_{2n}(BB)| = |x|^{-\alpha_2} O(\tau_2^{1-([\alpha_2]+1)\gamma_2})$$

$$\leq |x|^{-\alpha_1} O(\sum_{i=1}^{2} \tau_i^{1-([\alpha_i]+1)\gamma_i}). \qquad \ldots(6.4.22)$$

Equation (6.4.9) now follows from equations (6.4.10), (6,4,11), (6.4.22).

In view of the remarks before equations (6.4.6) and (6.4.8) the proof of the Theorem 6.1.2 is complete.□

CONCLUDING REMARKS:

In this chapter the basic structure of non-identical nature of the summands is that the observations come from different populations according to a fixed pattern (sampling scheme). In real life problems this may be over simplification and rather unlikely. In reality the number of observations coming from a particular population in the first n observations may itself be a r.v.; that is, the observations come from different populations according to a random mechanism. Under this assumption, we shall prove a central limit theorem type result.