CHAPTER 8

A LOCAL LIMIT THEOREM WHEN SUMMANDS COME RANDOMLY FROM r POPULATIONS

8.1 INTRODUCTION AND STATEMENT OF THE MAIN RESULT:

Let $\{X_n\}$ be a sequence of mutually independent r.v.s with respective d.f.s $\{G_n\}$, all of which belong to the domain of normal attraction of a symmetric stable law G with index α, $0< \alpha< 2$. Suppose further that at most two of the d.f.s $\{G_n\}$ are distinct, i.e. $G_n \in \{F_1, F_2\}$.

Without loss of generality assume that $EX_i = 0$, i = 1, 2, ... whenever it exists.

For each n, let $\tau_1(n)$ be the number of r.v.s among $X_1, X_2, \ldots, X_n$ which have F_1 as their d.f. Assume that $\tau_1(n)$ and $\{X_n\}$ are independent.

If $\tau_1(n)$ is a non-random function of n such that $0< \lim_{n \to \infty} \tau_1(n)/n = \lambda< 1$, then from the Theorem 5.1.1 it follows that

$$\sup_{x \in R}|v_n(x)-v_o(x)| = o(1) \qquad \ldots(8.1.1)$$

where v_n and v_o denote the p.d.f.s of S_n, properly normalized, and symmetric stable law with index α respectively.

In case $\tau_1(n)$ is not random, say $\tau_1(n) = k(n)$, as pointed out earlier the limit distribution of S_n, properly normalized always exists. In case the original r.v.s have an absolutely continuous distribution, the p.d.f. $v_{n,k(n)}$ of S_n, properly normalized, will converge to v_o.

In this chapter, we take $\tau_1(n)$ as r.v. satisfying the conditions:

[A1] $\tau_1(n)/n \to \lambda$ in probability, $0< \lambda< 1$, λ constant.

[A2] G_1 and G_2 are absolutely continuous d.f. belonging to the domain of normal attraction of a symmetric stable law with index α, $0< \alpha< 2$.

[A3] for every n and $k(n) = k$, $|v_{n,k}(x)|\leq M$ for some positive constant M and for all $x \in R$.

[A4] $\int_{-\infty}^{\infty} u^{[\alpha]+1} |v_k^*(u) - v_o(u)| du< \infty$, v_k^* being the p.d.f. corresponding to the d.f. F_k, $k = 1, 2$.

[A5] If $w_j(t)$ represents the c.f. corresponding to the d.f. F_j then for some integer $p\geq 1$, $\int_{-\infty}^{\infty} |w_j(t)|^p dt< \infty$, $j = 1, 2$.

We prove the following theorem:

THEOREM 8.1.1: *Under the assumption* **[A1]**-**[A5]**,

$$\sup_{x \in R}|v_n(x)-v_o(x)| = o(1) \text{ as } n \to \infty.$$

We prove this theorem in the Section 8.2.

8.2 PROOF OF THE THEOREM:

Let $0 < \lambda < 1$. Let $\varepsilon < \min(\lambda, 1-\lambda)$.

then given $\delta > 0$, there exists positive integer $N(\varepsilon)$ such that, for all $n \geq N(\varepsilon)$, we have $P\{|\tau_1(n)/n - \lambda| \leq \varepsilon\} > 1-\delta$.

That is,

$P\{|\tau_1(n)/n - \lambda| > \varepsilon\} < \delta$ or

$$P\{\tau_1(n) < (\lambda-\varepsilon)n \text{ or } \tau_1(n) > (\lambda+\varepsilon)n\} \leq \delta. \qquad \ldots(8.2.1)$$

Consider $|v_n(x) - v_o(x)|$

$$= \sum_{k=1}^{n} P(\tau_1(n) = k)\,|v_{n,k}(x) - v_o(x)|$$

$$\leq 2M\delta + \sum_{k=[(\lambda-\varepsilon)n]}^{[(\lambda+\varepsilon)n]} P(\tau_1(n) = k)\,|v_{n,k}(x) - v_o(x)|,$$

using (8.2.1), where M is the bound on the p.d.f.

$$\leq 2M\delta + \sum_{k=[(\lambda-\varepsilon)n]}^{[(\lambda+\varepsilon)n]} P(\tau_1(n) = k)\ \sup_{x\in R}|v_{n,k}(x) - v_o(x)|$$

Here last but second inequality follows from the assumption **[A3]** and the fact that all stable densities are bounded; whereas the last inequality follows from Theorem 5.1.1.

Thus we have,

$$\sup_{x\in R}|v_n(x) - v_o(x)| \leq 2M\delta + C_1 n^{1-([\alpha]+1)\gamma} = o(1).\square$$

Remark 8.2.1: This result can be extended, in general, to r (finite positive integer) populations.