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MISCELLANEOUS NOTES

A CLT WHEN SUMMANDS COME RANDOMLY FROM *r* POPULATIONS

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ABSTRACT: Let $\{X_n\}$ be a sequence of mutually independent random variables (r.v.s.) defined on a probability space (Ω, β, P) with respective distribution functions (d.f.s.) $\{F_n\}$, all of which belong to the domain of normal attraction of a symmetric stable law with characteristic exponent α, $0<\alpha\leqslant 2$. Suppose further that at most r of the d.f.s. $\{F_n\}$ are distinct, i.e. $F_n \in \{G_1, G_2, \ldots G_r\}$.

A Central Limit Theorem type result is proved when the number of variables among $\{X_1, X_2, \ldots, X_n\}$ that follow a G_j is a r.v. satisfying certain conditions.

Key Words : Central Limit Theorem, Independent r.v.s., Domain of attraction, Symmetric stable law, Stable process

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1. Introduction and statement of the main result

Let $\{X_n\}$ be a sequence of mutually independent random variables (r.v.s.) defined on a probability space (Ω, β, P) with respective distribution functions (d.f s.) $\{F_n\}$, all of which belong to the domain of normal attraction of a symmetric stable law G with characteristic exponent α, $0<\alpha\leqslant 2$. Suppose further that at most r of the d.f.s. $\{F_n\}$ are distinct i.e. $F_n \in \{G_1, G_2, \ldots G_r\}$.

Without loss of generality assume that $EX_i=0$, $i=1, 2, \ldots$ whenever it exists.

For each n, let $\tau_j(n)$ be the number of r.v.s. among $X_1, X_2, \ldots, X_n$ which have $G_j(x)$ as their d.f.

Suppose that $\tau_j(n)$, for fixed j, is a r.v. possibly depending upon $\{X_n\}$.

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Define $S_n = \sum_{i=1}^{n} X_i$.

If $\tau_j(n)$, $j=1, 2, \ldots, r$ are constants dependent on n, Sreehari [4] and Mason [2] have proved that

$$P\{S_n \leqslant x B_n\} \to G(x),$$

with $B_n = \left[\sum_{j=1}^{n} C_j^{\alpha}(\tau_j(n)) \right]^{1/\alpha}$. Here $C_j(n)$ is proportional to $n^{1/\alpha}$, the constant of proportionality changing with j.

In this paper, we take $\tau_j(n)$, $j=1, 2, \ldots, r$ as r.v.s. satisfying the condition: For $j=1, 2, \quad, r$

$$\tau_j(n)/n \to N_j \text{ in probability, where } P(N_j > 0) = 1.$$

We prove the following theorem.

Theorem *$P\{S_n \leqslant x\, v(n)\} \to G(x)$, where $v(n)$ is such that*

$$v^{\alpha}_{(n)} = \sum_{i=1}^{r} C_i^{\alpha}(\tau_i(n))$$

The proof of this theorem is given in section 3.

2. Preliminary results

This section is devoted to some preliminary results required for the proof of the theorem. For $A, B \varepsilon \beta$, denote the conditional probability of A given B by $P(A|B)$. If $P(B)=0$, then we use the convention $P(A|B)=P(A)$.

Definition 1 : A sequence $\{A_n\}$ of events is said to be P-mixing if $\lim_{n\to\infty} [P(A_n|A) - P(A_n)] = 0$ for every $A \varepsilon \beta$.

Definition 2 : Let $\xi(t)$ be an independent separable homogeneous process with independent increments defined on $[0, 1]$ such that $\xi(0)=0$ and $\xi(t)$

(1) is stochastically continuous on the right,

(2) has at most a denumerable number of discontinuities, all of the first kind, and

(3) is defined by $E \exp\{i\, u\, \xi(t)\} = \exp\{-t\, \theta\, |u|^{\alpha}\}$ $0<\alpha\leqslant 2$, $\theta>0$.

Then the process $\{\xi(t)\}$ is called a symmetric stable process with exponent α.

Lemma 1. (Barndorff-Neilson Lemma)

Let $\{k_n\}$ and $\{m_n\}$ with $k_n < m_n$ be two increasing sequences of positive integers with $k_n \to \infty$ and let $\{A_n\}$ be a sequence of events of β such that A_n depends only on $X_{k_n}, \ldots, X_{m_n}$. Then $\{A_n\}$ is P-mixing.

Whenever the observations do not come randomly from r populations (i e. $\tau_j(n)$ are not random variables) but positive integer valued function of n such that $\sum_{j=1}^{r} \tau_j(n) = n$, then $\tau_j(n)$ will be, in the remaining part of this section, denoted by $t_j(n)$, $j = 1, 2, \cdots, r$. Let us define

$$\psi^{\alpha}(n) = \sum_{i=1}^{r} C_i^{\alpha}(t_i(n)), \quad \sum_{i=1}^{r} t_i(n) = n.$$

Lemma 2. *The sequence $\{A_n\}$ defined by $A_n = \{S_n \leqslant x\psi(n)\}$ is P-mixing.*

Proof: Let $\epsilon > 0$ be an arbitrary constant and let A be any event.

Define $E_n = \{|S_{[\log n]}| > \epsilon\ \psi(n)\}$

where $[x]$ is n if $n \leqslant x < n+1$, n is an integer.

Denote $\delta_n = P(E_n|A)$.

Writing

$$P[S_n \leqslant x\ \psi(n)|A]$$
$$= P[S_n - S_{[\log n]} + S_{[\log n]} \leqslant x\psi(n)|A]$$

and intersecting with the event E_n we get after usual manipulations

$$P[S_n - S_{[\log n]} \leqslant (x-\epsilon)\psi(n)|A] - \delta_n$$
$$\leqslant P[S_n \leqslant x\ \psi(n)|A]$$
$$\leqslant P[S_n - S_{[\log n]} \leqslant (x+\epsilon)\psi(n)|A] + \delta_n. \quad \ldots \quad (2.1)$$

Note that $\delta_n = P[E_n|A]$

$$= P[|S_{[\log n]}| > \epsilon(\psi(n)/\psi(\log n))\ \psi(\log n)]/P(A)$$

$\to 0$ because $\psi(n)/\psi(\log n) \to \infty$ as $n \to \infty$.

Similarly, it can be proved that

$$P[S_n - S_{[\log n]} \leqslant (x-\epsilon)\ \psi(n)] - \delta_n^*$$
$$\leqslant [S_n \leqslant x\ \psi(n)]$$
$$\leqslant P[S_n - S_{[\log n]} \leqslant (x+\epsilon)\ \psi(n)] + \delta_n^* \quad \ldots \quad (2.2)$$

where $\delta_n^* = P(E_n)$.

Inequalities (1) and (2) imply that

$$\begin{aligned} & P[S_n - S_{[\log n]} \leqslant (x-\epsilon)\, \psi(n)] \mid A] \\ & \qquad - P[S_n - S_{[\log n]} \leqslant (x+\epsilon)\, \psi(n)] - \delta_n - \delta_n^* \\ & \leqslant P[S_n \leqslant x\, \psi(n)|A] - P[S_n \leqslant x\, \psi(n)] \\ & \leqslant P[S_n - S_{[\log n]} \leqslant (x+\epsilon)\, \psi(n)|A] \\ & \qquad - P[S_n - S_{[\log n]} \leqslant (x-\epsilon)\, \psi(n)] + \delta_n + \delta_n^*. \qquad \ldots (2.3) \end{aligned}$$

Note that in view of Lemma 1, the limit of the first term of the inequality (3), after adding and subtracting the term $P[S_n - S_{[\log n]} \leqslant (x-\epsilon)\, \psi(n)]$, will be bounded below by $G(x-2\epsilon) - G(x+2\epsilon)$, whereas the limit of the third term of the inequality (3), after adding and subtracting the term $P[S_n - S_{[\log n]} \leqslant (x+\epsilon)\, \psi(n)]$, will be bounded above by $G(x+2\epsilon) - G(x-2\epsilon)$, and hence for $n \to \infty$, we get

$$\begin{aligned} -[G(x+2\epsilon) - G(x-2\epsilon)] & \leqslant \lim_{n\to\infty} \{P[S_n \leqslant x\, \psi(n)|A] - P[S_n \leqslant x\, \psi(n)]\} \\ & \leqslant [G(x+2\epsilon) - G(x-2\epsilon)] \end{aligned}$$

Now, allowing $\epsilon \to 0$, we have the result that

$$P[S_n \leqslant x\, \psi(n)|A] - P[S_n \leqslant x\, \psi(n)] \to 0 \text{ as } n \to \infty$$

which proves the lemma.

Lemma 3. *Let*

$$H(x) = P[\sup_{0 \leqslant t \leqslant 1} |\xi(t^a)| \leqslant x],$$

Then

$$\lim_{n\to\infty} P[\max_{1 \leqslant j \leqslant n} |S_j| \leqslant x\, B_n] = H(x)$$

for an appropriately chosen normalizing sequence of positive constants B_n.

Proof: This lemma is due to Sreehari [3], (Theorem : 5. 2) and hence the proof is omitted.

3. Proof of the theorem

We shall prove the theorem for $r=2$; for $r>2$, the proof is analogous.

Let k be a positive integer to be chosen later appropriately.

For convenience we shall denote N_1 by N and N_2 by $1-N$.

Let us denote for a positive integer k,

$$B_i = \{(i/k) < N \leqslant (i+1)/k\} \quad i = 0, 1, 2, \cdots, k-1$$
$$D_{n,k} = \{|(\tau_1(n)/n) - N| < (1/k)\}$$
$$\nu^a(n) = C_1^a(\tau_1(n)) + C_2^a(\tau_2(n))$$
$$J_i(n) = P[\{S_n \leqslant x\,\nu(n)\} \cap D_{n.k} \cap B_i] \qquad i = 1, 2, \ldots, k-2$$
$$\eta_n = P[\{S_n \leqslant x\,\nu(n)\} \cap D_{n.k} \cap \{N > (k-1)/k\}]$$
$$\gamma_n = P[\{S_n \leqslant x\,\nu(n)\} \cap D_{n.k} \cap \{N \leqslant (1/k)\}]$$
$$\xi_n = P[\{S_n \leqslant x\,\nu(n)\} \cap D'_{n.k}],$$

where $D'_{n.k}$ is the complement of $D_{n.k}$ in Ω.

We have

$$P[S_n \leqslant x\,\nu(n)] = \xi_n + \gamma_n + \eta_n + \sum_{i=1}^{k-1} J_i(n). \quad \cdots \quad (3.1)$$

For fixed i, define

$$\alpha_{ni} = [n(i-1)/k], \qquad \beta_{ni} = [n(i+2)/k]$$
$$\theta_{ni} = n - \alpha_{ni}, \qquad \delta_{ni} = n - \beta_{ni}$$
$$\nu_1^a(n, i) = C_1^a(\alpha_{ni}) + C_2^a(\delta_{ni})$$
$$\nu_2^a(n, i) = C_1^a(\beta_{ni}) + C_2^a(\theta_{ni}).$$

On the event $\{B_i \cap D_{n.k}\}$, $\nu(n) \in [\nu_1(n, i), \nu_2(n, i)]$

We first prove the theorem in the case $x \geqslant 0$: when $x < 0$ the steps will be exactly similar.

Let

$$C_{1.i}(n) = P[\{S_n \leqslant x\,\nu_1(n, i)\} \cap D_{n.k} \cap B_i]$$
$$C_{2.i}(n) = P[\{S_n \leqslant x\,\nu_2(n, i)\} \cap D_{n.k} \cap B_i].$$

Then note that

$$C_{1.i}(n) \leqslant P[\{S_n \leqslant x\,\nu(n)\} \cap D_{n.k} \cap B_i] \leqslant C_{2.i}(n). \quad \ldots \quad (3.2)$$

Further

$$C_{2.i}(n)$$
$$= P[\{S_n \leqslant x\,\nu_2(n, i)\} \cap D_{n.k} \cap B_i]$$
$$\leqslant P[\{S_{1,\beta_{ni}} + S_{2.\theta_{ni}} - \max_{\alpha_{ni} \leqslant j \leqslant \beta_{ni}} |S_{1.j} - S_{1.\beta_{ni}}|$$
$$- \max_{\delta_{ni} \leqslant j \leqslant \theta_{ni}} |S_{2.j} - S_{2.\theta_{ni}}| \leqslant x\,\nu_2(n, i)\} \cap D_{n,k} \cap B_i]. \cdots \quad (3.3)$$

Let

$$V_{2,n,i}=\{\max_{\alpha_{ni}\leqslant j\leqslant\theta_{ni}}|S_{1,j}-S_{1,\beta_{ni}}|>\epsilon\,\nu_2(n,i)\}$$

$$W_{2,n,i}=\{\max_{\delta_{ni}\leqslant j\leqslant\theta_{ni}}|S_{2,j}-S_{2,\theta_{ni}}|>\epsilon\,\nu_2(n,i)\}$$

Then using elementary results in probability we get from (3.3)

$$C_{2,i}(n)\leqslant P[\{S_{1,\beta_{ni}}+S_{2,\theta_{ni}}\leqslant(x+2\epsilon)\,\nu_2(n,i)\}\cap B_i]$$
$$+P(V_{2,n,i}\cap B_i)+P(W_{2,n,i}\cap B_i)$$

Now using Lemmas 1, 2 and 3 we get from (3.1)

$$\limsup P[S_n\leqslant x\,\nu(n)]$$
$$\leqslant\sum_{i=1}^{k-2}P(B_i)\,[\limsup P\,[S_{1,\beta_{ni}}+S_{2,\theta_{ni}}\leqslant(x+2\epsilon)\,\nu_2(n,i)]$$
$$+\limsup P\,(V_{2,n,i})+\limsup P\,(W_{2,n,i})]$$
$$+P\{N\leqslant(1/k)\}+P\{N>((k-1)/k)\}$$

$$=\sum_{i=1}^{k-2}P(B_i)\,[G(x+2\epsilon)+2\{1-H((\epsilon(k+3)/3)^{1/\alpha})\}]$$
$$+P\{N\leqslant(1/k)\}+P\{N>(k-1)/k)\}$$
$$\leqslant G(x+2\epsilon) \qquad \text{... (3.4}$$

by letting $k\to\infty$ because H is proper d.f.

Now consider,

$$\sum_{i=1}^{k-2}C_{1,i}(n)=\sum_{i=1}^{k-2}P[\{S_n\leqslant x\,\nu_1(n,i)\}\cap D_{n,k}\cap B_i]$$
$$\geqslant\sum_{i=1}^{k-2}P[\{S_{1,\alpha_{ni}}+S_{2,\delta_{ni}}+\max_{\alpha_{ni}\leqslant j\leqslant\beta_{ni}}|S_{1,j}-S_{1,\alpha_{ni}}|$$
$$+\max_{\delta_{ni}\leqslant j\leqslant\theta_{ni}}|S_{2,j}-S_{2,\delta_{ni}}|\leqslant x\,\nu_1(n,i)\}\cap B_i]$$
$$-P(D'_{n,k})$$
$$\geqslant\sum_{i=1}^{k-2}P(B_i)\,[P[\{S_{1,\alpha_{ni}}+S_{2,\delta_{ni}}\leqslant(x-2\epsilon)\nu_1(n,i)\}|B_i]$$
$$-P(V_{1,n,i}|B_i)$$
$$-P(W_{1,n,i}|B_i)-P(D'_{n,k})$$

where $V_{1,n,i}=\{\max_{\alpha_{ni}\leqslant j\leqslant\beta_{ni}}|S_{1,j}-S_{1,\alpha_{ni}}>\epsilon\,\nu_1(n,i)\}$

$$W_{1,n,i}=\{\max_{\delta_{ni}\leqslant j\leqslant\theta_{ni}}|S_{1,j}-S_{1,\delta_{ni}}|>\epsilon\,\nu_1(n,i)\}$$

By using Lemmas 1, 2 and 3 we get from (3.1)

$$
\begin{aligned}
&\liminf P[S_n \leqslant x\, \nu(n)] \\
&\quad \geqslant \sum_{i=1}^{k-2} P(B_i) \liminf P[\{S_{1\, a_{ni}} + S_{2\, b_{ni}} \leqslant (x-2\epsilon)\, \nu_i(n, i)\}] \\
&\qquad - \limsup P(V_{1\, n,\, i}) - \limsup P(W_{1\, n\, i}) - \limsup P(D'_{n,\, k}) \\
&\qquad - P\{N \leqslant (1/k)\} - P\{N > ((k-1)/k)\} \\
&\quad = \sum_{i=1}^{k-2} P(B_i)\, [G(x-2\epsilon) - 2\{1 - H((\epsilon(k-3)/3)^{1/\alpha})\}] \\
&\qquad - P\{N \leqslant (1/k)\} - P\{N > ((k-1)/k)\} \\
&\geqslant G(x-2\epsilon) \qquad \ldots \quad (3.5)
\end{aligned}
$$

by letting $k \to \infty$.

The required result now follows from (3.4) and (3.5) on allowing $\epsilon \to 0$.

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A NON-UNIFORM RATE OF CONVERGENCE IN THE LOCAL LIMIT THEOREM FOR INDEPENDENT RANDOM VARIABLES*

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SUMMARY Let $\{X_n\}$, $n = 1,2,3,$ be a sequence of independent random variables with corresponding sequence of distribution functions $\{G_n\}$, $n = 1,2,3,$ Suppose that, for every $n, G_n \in \{F_1, F_2, \ , F_m\}$ Let $F_1, F_2, \ , F_m$ belong to the domain of normal attraction of a stable law with index $\alpha, 0 < \alpha < 2$ Define $T_n = X_1 + \ + X_n$ Under fairly mild assumptions, a non-uniform rate of convergence is obtained for the density version of central limit thoerem for normalized sums T_n

1 Introduction

Let $\{X_n\}$ be a sequence of mutually independent random variables (r.v.s) with corresponding sequence of absolutely continuous distribution functions (d.f.s) $\{G_n\}$. Suppose, for each $n, G_n \in \{F_1, F_2, \ , F_m\}$; m being a fixed positive integer. For each n, let $\tau_j = \tau_j(n)$ be the number of r.v s among $X_1, \ , X_n$ which have F_j as their d f., $j = 1, 2, \ , m$. Note that $\sum_{j=1}^{m} \tau_j = n$. Set $T_n = \sum_{i=1}^{n} X_i$, for $n = 1, 2, .$

Suppose that each F_j belongs to the domain of normal attraction of the stable law F_0 with characteristic exponent $\alpha, 0 < \alpha < 2$ By Theorem 3.1 of Sreehari (1970), T_n, properly normalized, converges in distribution to a stable r.v. with d.f. F_0. Kruglov (1968) proved that if the d f $F_j, j = 1, 2, .., m$ are absolutely

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continuous with probability density function (p.d.f.) $v_j, j = 1, 2, \ldots, m$, then

$$\sup_{-\infty<x<\infty} |f_n(x) - v_0(x)| = O(1) \text{ as } n \to \infty,$$

$f_n(x)$ being the p.d.f. of T_n, properly normalized, and v_0 being the p.d.f. of F_0. Basu *et al.* (1979), under certain regularity conditions, obtained a non-uniform rate of convergence in a local limit theorem concerning i.i.d r.v.s in the domain of normal attraction of a stable law. The rate was found to be of the order $n^{1-([\alpha]+1)r}, 0 < \alpha < 2, r = 1/\alpha$; $[x]$ denotes the largest integer less than or equal to x. In this study, we obtain uniform as well as non-uniform rates of convergence of the density f_n to v_0 for the above set up. This improves Kruglov's result and generalizes work of Basu *et al.* (1979) from i.i.d. set up to independent non-identical r.v.s. set up. We state our theorems first:

Theorem 1. *Under the assumptions,* $[A_1] - [A_5]$, *stated below,*

$$\sup_{-\infty<x<\infty} |f_n(x) - v_0(x)| = O\ (n^{1-([\alpha]+1)r}) \text{ as } n \to \infty.$$

Theorem 2. *Under the assumptions,* $[A_1]$ - $[A_5]$, *stated below,*

$$\sup_{-\infty<x<\infty} (1 + |x|^{\alpha})|f_n(x) - v_0(x)| = O\ (n^{1-([\alpha]+1)r}) \quad \text{as} \quad n \to \infty.$$

We prove the theorems in section 4. We introduce the notations and assumptions in section 2. In section 3, we prove some lemmas which will be helpful in section 4. Some of these lemmas are also of independent interest.

2. Notations and assumptions

Let Y_0 denote a stable or a strictly stable r.v. with exponent α, according as $1 < \alpha < 2$ or $0 < \alpha \leq 1$, having the d.f. F_0 with $EY_0 = 0$, whenever it exists, and let w_0 denote its characteristic function (c.f.). We assume $EX_n = 0$, whenever it exits. It is known that $Z_n = T_n/B_n$ converges in distribution to r.v. Y_0, with $B_n^{\alpha} = \sum_{i=1}^{m} d_i^{\alpha} r_i, d_i$ depends only on d.f. F_i. For the sake of simplicity in the proof we shall take $B_n = n^r$ without any loss of generality.

We write

$$\phi_n(t) = E[\exp\ itZ_n] = \prod_{i=1}^{m} \{w_i(tn^{-r})\}^{r_i},$$

where w_i is the c f. corresponding to the d.f. F_i Note that we have from the canonical representation of c f $w_0(t)$ that for all t

$$w_0(t) = \prod_{i=1}^{m} \{w_o(tn^{-r})\}^{\tau_i}. \quad (2\ 1)$$

Then as pointed out in section 1,

$$\lim_{n\to\infty} \phi_n(t) = w_0(t) \text{ for all } t \quad (2\ 2)$$

For each positive integer n and real number x, we define for $k = 0, 1, \ldots, m$; and $j = 1, 2, \ldots, m$

$$\alpha_{k,\tau_j}(t,x) = \int_{|u|\le|x|\tau_j^r} \exp(itu)dF_k(u) \quad (2\ 3)$$

$$\beta_{k,\tau_j}(t,x) = w_k(t) - \alpha_{k,\tau_j}(t,x) \quad (2\ 4)$$

$$A_{k,\tau_j}(t,x) = \{\alpha_{k,\tau_j}(tn^{-r},x)\}^{\tau_j} \quad (2.5)$$

$$\begin{aligned} B_{k,\tau_j}(t,x) &= \{w_k(tn^{-r})\}^{\tau_j} - \{\alpha_{k,\tau_j}(tn^{-r},x)\}^{\tau_j} \\ &= \sum_{h=1}^{\tau_j} \binom{\tau_j}{h} \{\alpha_{k,\tau_j}(tn^{-r},x)\}^{\tau_j-h}\{\beta_{k,\tau_j}(tn^{-r},x)\}^h \end{aligned} \quad (2\ 6)$$

Note that for

$$f_n(u) = (2\pi)^{-1}\int_{-\infty}^{\infty} \phi_n(t)e^{-itu}dt \quad (2\ 7)$$

the inversion integral on right hand side is absolutely convergent. The absolutely convergent integral provides the continuous p d f. that we shall use in our theorems.

In what follows $c, c_0, c_1,$, etc. will denote some positive constants, independent of n and x and their values are not of much importance and may change from one step to another. Also $P_1(\ ), P_2(\ ),$ will denote polynomials in positive variables having non-negative coefficient independent of n and x. They may vary from one step to another. Finally, for any function $g(t)$ and any positive integer k, let $g^{(k)}(t)$ denote the k^{th} derivative of $g(t)$ whenever such a derivative exists. We now make the following assumptions;

[A$_1$] All the d.f s F_j, $j = 1, 2, \ldots, m$ are absolutely continuous.

[A$_2$] If $w_j(t)$ represents the c f. corresponding to the d.f. $F_j, j = 1, 2, \ldots, m$

then for some integer $p \geq 1$,

$$\int_{-\infty}^{\infty} |w_j(t)|^p dt < \infty$$

$[A_3]$ $\int_{-\infty}^{\infty} |u|^{[\alpha]+1} |v_j(u) - v_0(u)| du < \infty,\ j = 1, 2, ..., m.$

$[A_4]$ $\lim_{n\to\infty} \frac{\tau_j}{n} = t_j > 0,\ j = 1, 2, \ \ , m.$

$[A_5]$ F_j belongs to the domain of normal attraction of the stable law F_0.

Further let

$$\{w_j(tn^{-r})\}^n \to w_0(t), \text{ the c.f. of } F_0, \text{ as } n \to \infty$$

From the proof it will be clear that it is sufficient if

$$0 < \underline{lim} \frac{\tau_j}{n} \text{ for all } j \text{ instead of (A4).}$$

3. Preliminary results

Now we mention some preliminary lemmas required to prove the theorems of section 1.

Lemma 1. *For $k = 0, 1, ..., m$ and a .r.v. Y_k with d.f. F_k as $z \to \infty$ we have*

(1)

$$z^\alpha R_k(z) \equiv z^\alpha P(|Y_k| > z) \to C_k > 0; \quad ..(3.1)$$

(2) *whenever* $0 < \alpha < 1$,

$$\int_{|u|\leq z} |u| dF_k(u) = O\ (z^{1-\alpha}); \quad ...(3\ 2)$$

(3) *whenever* $\alpha = 1$,

$$\int_{|u|>z} |u|^{1/2} dF_k(u) = O\ (z^{-1/2}) \quad ..(3\ 3)$$

and

$$\int_{|u|\leq z} u^2 dF_k(u) = O\ (z); \quad ..(3.4)$$

Proof Using the definition of $\phi_n(t)$, the equation (3 1) and by adding and subtracting the terms

$$\prod_{i=1}^{j}\{w_i(tn^{-r})\}^{\tau_i}\prod_{k=i+1}^{m}\{w_0(tn^{-r})\}^{\tau_k} \quad \text{for} \quad j=1,2, \ldots, m-1,$$

we get on simplification

$$\begin{aligned}|\varphi_n(t)-w_0(t)| &\leq \sum_{i=1}^{m}|\{w_i(tn^{-r})\}^{\tau_i}-\{w_0(tn^{-r})\}^{\tau_i}| \\ &\leq n^{1-(|\alpha|+1)r}P_1(|t|)\ \exp\{-c|t|^{\alpha}\} \text{ using Banys' Lemma.}\end{aligned}$$

Lemma 5. *Under the assumption [A_3], there exists a polynomial $P_1(\,)$ such that for large n, the relation*

$$|\alpha_{k,\tau_k}^{\tau_k-j}(tn^{-r},x)-\alpha_{0,\tau_k}^{\tau_k-j}(tn^{-r},x)|$$

$$\leq \tau_k^{1-(|\alpha|+1)r}P_1(|t|)\ exp\{-c|t|^{\alpha}(1-\frac{j}{\tau_k})\}, \ 1\leq j\leq \tau_k,\ k=1,2, \ldots, m; \qquad (3\ 10)$$

holds for all t in the range $|t|\leq\in n^r$ and all x with $|x|\geq 1$

Proof The proof is similar to that of Lemma 2.3 of Basu *et al.* (1980). In fact, the adjustments necessary are rather easy in view of the assumption (A3)

Now we define two functions which will be useful in the proofs of the theorems and some of the lemmas For $j=1,2, \ldots, m$ let

$$d_{\tau_j}(t,x)=\tau_j\{\alpha_{j,\tau_j}(tn^{-r},x)-\alpha_{0,\tau_j}(tn^{-r},x)\} \qquad (3\ 11)$$

$$S_{\tau_j}(t,x)=\tau_j^{-1}\sum_{h=0}^{\tau_j-h}\{\alpha_{j\,\tau_j}(tn^{-r},x)\}^{j}\{\alpha_{0,\tau_j}(tn^{-r},x)\}^{\tau_j-h-1} \qquad (3\ 12)$$

Lemmas 6-8 give bounds on the functions $\alpha_{k,\tau_i}(t,x), d_{\tau_j}(t,x), S_{\tau_j}(t,x)$ and their first and second derivatives with respect to t The proofs are based on the techniques of the proof of Lemma 5 presented abvoe and hence omitted The results of Lemma 8 follow from Lemmas 2 and 7; whereas equations (3 22), (3 23), (3 24) of Lemma 7 follow from Basu *et al* (1980, (3 3) - (3 5))

Lemma 6. *Properties of the function $d_{\tau_1}(t,x)$ 'For all values of t and x with $|x|\geq 1$, we have for $k=1,2, \ldots, m$*

(1) *whenever $0<\alpha<1$,*

$$|d_{\tau_k}(t,x)|\leq \tau_k^{1-r}P_1(|t|), \qquad (3\ 13)$$

$$|d_{\tau_k}^{(1)}(t,x)| \leq c_1 1 \tau_k{}^{1-r}, \quad \ldots (3\ 14)$$

(2) *whenever* $1 \leq \alpha < 2$,

$$|d_{\tau_k}(t,x)| \leq \tau_k^{1-2r} P_2(|t|), \quad \ldots (3.15)$$

$$|d_{\tau_k}^{(1)}(t,x)| \leq \tau_k^{1-2r} P_3(|t|), \quad \ldots (3\ 16)$$

$$|d_{\tau_k}^{(2)}(t,x)| \leq c_1 \tau_k^{1-2r}. \quad \ldots (3.17)$$

Lemma 7. *Properties of the function* $\alpha_{k,\tau_j}(tn^{-r},x)$. *For each fixed n and* $x, \alpha_{k,\tau_j}(tn^{-r},x)$ *is differentiable any number of times under the integral sign,* $k = 0,1,\ldots,m; j = 1,2,\ldots,m$. *For all values of t and x with* $|x| \geq 1$, *we have for* $k = 1,2,\ldots,m; j = 1,2,\ldots,m$:
(1) *whenever* $0 < \alpha < 1$,

$$|\alpha_{k,\tau_j}^{(1)}(tn^{-r},x)| \leq |x|^{1-\alpha} \tau_k^{r-1} \quad \ldots (3\ 18)$$

(2) *whenever* $1 \leq \alpha \leq 2$,

$$|\alpha_{k,\tau_j}^{(1)}(tn^{-r},x)| \leq \tau_k^{r-1} P_k(|t|) \quad \ldots (3.19)$$

$$\leq |x|^{2-\alpha} \tau_k^{r-1} P_k(|t|), \quad \ldots (3.20)$$

$$|\alpha_{k,\tau_j}^{(2)}(tn^{-r},x)| \leq c_1 |x|^{2-\alpha} \tau_k^{r-1} \quad (3\ 21)$$

Also for all $x \neq 0, 0 < \alpha < 2$, *every large integer s, there exists a constant c such that*

$$\int_{-\infty}^{\infty} |\alpha_{k,\tau_j}(t,x)|^{\tau_j} dt = 0(\tau_k^{-r}), \quad (3\ 22)$$

$$\int_{-\infty}^{\infty} |\alpha_{k,\tau_j}(t,x)|^{2s} dt \leq c, \quad \ldots (3.23)$$

$$\int_{-\infty}^{\infty} |\beta_{k,\tau_j}(t,x)|^{2s} dt \leq c. \quad (3\ 24)$$

Lemma 8. *Properties of the function* $S_{\tau_k}(t,x)$. *For all* $|t| \leq \in n^r, |x| \geq 1$ *and all large n and* $\tau_k, k = 1,2,\ldots,m$ *we have*

(1) *for* $0 \leq \alpha < 1$,

$$|S_{\tau_k}(t,x)| \leq C\ exp\{-c|t|^\alpha\}, \quad \ldots (3\ 25)$$

$$|S_{\tau_k}^{(1)}(t,x)| \leq C|x|^{1-\alpha}\ exp\{-c|t|^\alpha\}, \quad \ldots (3.26)$$

(2) *for* $1 \leq \alpha < 2$,

$$|S_{\tau_k}(t,x)| \leq C\, exp\{-c|t|^{\alpha}\}, \tag{3 27}$$

$$|S^{(1)}_{\tau_k}(t,x)| \leq |x|^{2-\alpha}\, exp\{-c|t|^{\alpha}\}P_1(|t|), \tag{3 28}$$

$$|S^{(2)}_{\tau_k}(t,x)| \leq |x|^{2-\alpha}\, exp\{-c|t|^{\alpha}\}P_2(|t|). \tag{3 29}$$

Lemma 9 *There exist polynomials* $P_1(\cdot)$ *and* $P_2(\ )$ *such that for all* t *in the range* $|t| \leq \in n^r, |x| \geq 1$ *and large* n *we have the following*
(1) *for* $0 < \alpha < 1$,

$$|A_{k,\tau_k}(t,x) - A_{0,\tau_k}(t,x)| \leq P_1(|t|)exp\{-c|t|^{\alpha}\}\tau_k^{1-r} \tag{3 30}$$

$$|A^{(1)}_{k,\tau_k}(t,x) - A^{(1)}_{0,\tau_k}(t,x)| \leq |x|^{1-\alpha}P_2(|t|)exp\{-c|t|^{\alpha}\}\tau_k^{1-r} \tag{3 31}$$

(2) *for* $1 \leq \alpha < 2$,

$$|A^{(i)}_{k,\tau_k}(t,x) - A^{(i)}_{0,\tau_k}(t,x)| \leq P_1(|t|)exp\{-c|t|^{\alpha}\}\tau_k^{1-2r} \tag{3 32}$$

$$|A^{(i)}_{k,\tau_k}(t,x) - A^{(i)}_{0,\tau_k}(t,x)| \leq |x|^{2-\alpha}P_{i+1}(|t|)exp\{-c|t|^{\alpha}\}\tau_k^{1-2r} \tag{3.33}$$

$i = 1, 2,\ k = 1, 2, ..., m$

Proof. We will prove (3.30) and (3 31) only. (3 32) - (3.33) can be proved similarly. In view of equations (3.11) and (3.12), observe that

$$A_{k,\tau_k}(t,x) - A_{0,\tau_k}(t,x) = d_{\tau_k}(t,x)S_{\tau_k}(t,x) \qquad ..(3\ 34)$$

Therefore, (3 30) follows from (3 13) and (3.25). Also

$$A^{(1)}_{k,\tau_k}(t,x) - A^{(1)}_{0,\tau_k}(t,x) = d^{(1)}_{\tau_k}(t,x)S_{\tau_k}(t,x) + d_{\tau_k}(t,x)S^{(1)}_{\tau_k}(t,x) \tag{3 35}$$

Using the relations (3.13), (3.14), (3.25) and (3.26), we get (3 31) from (3.35).

Lemma 10 *There exist polynomials* $P_1(\cdot)$ *and* $P_2(\ )$ *such tht for all* t *in the range* $|t| \leq \in n^r, |x| \geq 1$ *and large* n,

(1) *for* $0 < \alpha < 1$,

$$|A^{(1)}_{k,\tau_k}(t,x)| \leq |x|^{1-\alpha}exp\{-c|t|^{\alpha}\}, \qquad .(3.36)$$

(2) *for* $1 \leq \alpha < 2$,

$$|A^{(1)}_{k,\tau_k}(t,x)| \leq P_1(|t|)exp\{-c|t|^{\alpha}\} \qquad ...(3.37)$$

$$\leq |x|^{2-\alpha}P_1(|t|)exp\{-c|t|^{\alpha}\}, \qquad ...(3.38)$$

$$|A^{(2)}_{k,\tau_k}(t,x)| \leq |x|^{2-\alpha}P_2(|t|)exp\{-c|t|^{\alpha}\}. \qquad .(3\ 39)$$

We shall now state a lemma which follows from Lemma 2.4 in Smith et al. (1974).

Lemma 11. *Let* $\in > 0$ *and integer* n_0 *be fixed and let*

$$(H) = \{(t, n, x) | \ |t| > \in, \ n \geq n_0, \ |x| \geq 1\},$$

$$\mu_{0,\tau_k} \equiv \sup_{(H)} |\alpha_{0,\tau_k}(t, x)|, \qquad (3.40)$$

$$\mu_{k,\tau_k} \equiv \sup_{(H)} |\alpha_{k,\tau_k}(t, x)|, \qquad (3.41)$$

for $k = 1, 2, \ldots, m$.

Then it follows that $0 \leq \mu_{0,\tau_k} < 1$ *and* $0 \leq \mu_{j,\tau_k} < 1$. *Let* $\mu = max(\mu_{0,\tau_k}; \mu_{k,\tau_k})$

Lemma 12. *Let* $g(t, x)$ *be a complex-valued function, bounded by some positive constant for* $|x| \geq 1$ *and for all* t. *Then, for* $k = 1, 2, \ldots, m$

$$\left| \int_{-\infty}^{\infty} (B_{k,\tau_k}(t, x) - B_{0,\tau_k}(t, x)) g(t, x) exp(-itx) dt \right|$$

$$= |x|^{-\alpha} \ 0(\tau_k^{1-([\alpha]+1)r}). \qquad (3.42)$$

Proof. The boundedness of the fucntion $g(t, x)$ by some positive constant helps us to obtain (3.42) above on the lines of equations (3.12) to (3.22) of Basu *et al* (1979) with little modificatioans We, therefore, skip the proof of this lemma

Lemma 13. *For all the values of* t *and all* x *with* $|x| \geq 1$,

$$|B_{k,\tau_j}(t, x)| \leq c|x|^{-\alpha}, \qquad (3.43)$$

$k = 0, 1, \ldots, m$; $j = 1, 2, \ldots, m$.

Proof. Using Lemma 1, we get

$$\begin{aligned} |B_{k,\tau_j}(t, x)| &\leq \sum_{h=1}^{\tau_j} \binom{\tau_j}{h} \left| \alpha_{k,\tau_j}^{\tau_j - h}(tn^{-r}, x) \right| \left| \beta_{k,\tau_j}^{h}(tn^{-r}, x) \right| \\ &\leq \sum_{h=1}^{\tau_j} \frac{\tau_j}{h!} \{P[|Y_k| \geq |x|\tau_j^{-r}]\}^h \\ &\leq c|x|^{-\alpha} \end{aligned}$$

4. Proofs of main results

Proof of Theorem 1. We shall prove the theorem for $m = 2$. In case of $m > 2$ but fixed, the proof involves similar steps

We shall prove the relation

$$\sup_{-\infty<x<\infty} |f_n(x) - v_0(x)| = O(n^{1-(|\alpha|+1)r}) \quad (4.1)$$

as $n \to \infty$

The inversion formula for continuous density gives that

$$2\pi |f_n(x) - v_0(x)| \leq I_{1n} + I_{2n} + I_{3n} \quad ..(4.2)$$

where

$$\begin{aligned} I_{1n} &= \int_{|t|\leq \in n^r} |\phi_n(t) - w_0(t)| dt, \\ I_{2n} &= \int_{|t|> \in n^r} |\phi_n(t)| dt \\ &= \int_{|t|> \in n^r} |w_1(tn^{-r})|^{r_1} |w_2(tn^{-r})|^{r_2} dt \end{aligned}$$

and

$$I_{3n} = \int_{|t|> \in n^r} |w_0(tn^{-r})|^n dt,$$

$\in > 0$ being as in Lemma 4.

By Lemma 4 it now follows that

$$I_{1n} = O(n^{1-(|\alpha|+1)r}) \quad as\ n \to \infty \quad (4.3)$$

As the d.f.s F_0, F_1 and F_2 are absolutely continuous and we have from the canonical representation of a stable law the fact that

$$w_0(t) = \{w_0(tn^{-r})\}^{r_1} \{w_0(tn^{-r})\}^{r_2},$$

and that there exists to any $\in > 0$, a $c(\in) > 0$ such that $|w_i(t)| \leq \exp(-c(\in))$, $|t| > \in$, for $i = 0, 1, 2$. Therefore,

$$\begin{aligned} I_{2n} &\leq n^r \int\limits_{|t|>\epsilon} \exp\{-c(\epsilon)(n-2p)\}|w_1(t)|^p|w_2(t)|^p dt \\ &\leq n^r \exp\{-c(\epsilon)(n-2p)\} \int\limits_{|t|>\epsilon} |w_1(t)|^p dt \\ &\leq cn^r \exp\{-c(\epsilon)(n-2p)\} \\ &= O(n^{1-(|\alpha|+1)r}) \end{aligned} \quad (4.4)$$

as $n \to \infty$, and

$$I_{3n} = O(n^{1-(|\alpha|+1)r}) \quad ..(4\ 5)$$

as $n \to \infty$. Thus (4.1) follows from the relation (4.2) through (4.5).

Proof of Thoerem 2 We shall prove the theorem for the case $0 < \alpha < 1$ and $m = 3$. The case $1 \leq \alpha < 2$ can be handled similarly. Also the case $m = 2$ can be worked out exactly on the similar lines. In case $m > 3$ but fixed, the proof will be exactly similar to the case presented here. Modifications necessary for the general case are discussed in the remarks

Note that in view of Thoerem 1, it is sufficient for us to prove

$$\sup_{|x|\geq 1} |x|^\alpha |f_n(x) - v_0(x)| = O(n^{1-r}) \text{ as } n \to \infty, \text{ and} \quad ..\ (4\ 6)$$

Consider, for $|x| \geq 1$,

$$\begin{aligned} &|x|^\alpha |f_n(x) - v_0(x)| \\ &\leq |x|^\alpha \left| \int_{-\infty}^{\infty} \exp(-itx)[\{w_1(tn^{-r})\}^{r_1}\{w_2(tn^{-r})\}^{r_2}\{w_3(tn^{-r})\}^{r_3} \right. \\ &\quad \left. - \{w_0(tn^{-r})\}^{r_1}\{w_0(tn^{-r})\}^{r_2}\{w_0(tn^{-r})\}^{r_3}]dt \right| \\ &\leq \sup_{|x|\geq 1}|x|^\alpha \left| \int_{-\infty}^{\infty} \exp(-itx)\{w_1(tn^{-r})\}^{r_1}\{w_2(tn^{-r})\}^{r_2}[\{w_3(tn^{-r})\}^{r_3} \right. \\ &\quad \left. -\{w_0(tn^{-r})\}^{r_3}]dt \right| \\ &\quad +\sup_{|x|\geq 1}|x|^\alpha \left| \int_{-\infty}^{\infty} \exp(-itx)\{w_1(tn^{-r})\}^{r_1}\{w_0(tn^{-r})\}^{r_3}[\{w_2(tn^{-r})\}^{r_2} \right. \\ &\quad \left. -\{w_0(tn^{-r})\}^{r_2}]dt \right| \\ &\quad +\sup_{|x|\geq 1}|x|^\alpha \left| \int_{-\infty}^{\infty} \exp(-itx)\{w_0(tn^{-r})\}^{r_2}\{w_0(tn^{-r})\}^{r_3}[\{w_1(tn^{-r})\}^{r_1} \right. \\ &\quad \left. -\{w_0(tn^{-r})\}^{r_1}]dt \right| \end{aligned}$$

$$= W_{12} + W_{13} + W_{23} \qquad ..(4.7)$$

In order to prove (4.6) it is sufficient to prove that

$$W_{12} = O(n^{1-r}) \qquad ..(4.8)$$

$$W_{13} = O(n^{1-r}) \qquad (4.9)$$

$$W_{23} = O(n^{1-r}) \qquad .(4.10)$$

as $n \to \infty$.

We shall prove equation (4.8) only Equations (4.9) and (4.10) can be proved similarly Observe that using (2.5) and (2.6), we can write

$$\int_{-\infty}^{\infty} \exp(-itx)\{w_1(tn^{-r})\}^{r_1}\{w_2(tn^{-r})\}^{r_2}[\{w_3(tn^{-r})\}^{r_3} - \{w_0(tn^{-r})\}^{r_3}]dt$$

$$= \int_{-\infty}^{\infty} \exp(-itx)\{A_{1,r_1}(t,x)A_{2,r_2}(t,x) + A_{1,r_1}(t,x)B_{2,r_2}(t,x) + B_{1,r_1}(t,x)A_{2,r_2}(t,x) + B_{1,r_1}(t,x)B_{2,r_2}(t,x)\}[A_{3,r_3}(t,x) - A_{0,r_3}(t,x)]dt$$

$$+ \int_{-\infty}^{\infty} \exp(-itx)\{w_1(tn^{-r})\}^{r_1}\{w_2(tn^{-r})\}^{r_2}[B_{3,r_3}(t,x) - B_{0,r_3}(t,x)]dt$$

$$= I(A_1A_2) + I(A_1B_2) + I(B_1A_2) + I(B_1B_2) + I(B), \text{ say} \qquad (4.11)$$

Estimate of $I(A_1A_2)$. We shall prove that

$$|I(A_1A_2)| = |x|^{-\alpha}O(n^{1-r}) \qquad ..(4.12)$$

as $n \to \infty$.

First of all we consider the integral

$$\int_{-\infty}^{\infty} \exp(-itx)A_{1,r_1}(t,x)A_{2,r_2}(t,x)A_{3,r_3}(t,x)dt$$

Because $A_{k,r_k}(t,x)$, $A^{(1)}_{k,r_k}(t,x)$ and $A^{(2)}_{k,r_k}(t,x)$ are absolutely integrable, simple techniques involving integration by parts give us

$$\begin{aligned}&\int_{-\infty}^{\infty} \exp(-\imath tx) A_{1,\tau_1}(t,x) A_{2,\tau_2}(t,x) A_{3,\tau_3}(t,x) dt \\ = \quad & \imath x^{-1} \sum_{k=1}^{3} \int_{-\infty}^{\infty} \exp(-\imath tx) \prod_{\substack{j=1 \\ j\neq k}}^{3} A_{j,\tau_j}(t,x) A^{(1)}_{k,\tau_k}(t,x) dt\end{aligned} \qquad (4.13)$$

On evaluating

$$\int_{-\infty}^{\infty} \exp(-\imath tx) A_{1,\tau_1}(t,x) A_{2,\tau_2}(t,x) A_{0,\tau_3}(t,x) dt$$

on the lines of (4.13), we have then

$$\begin{aligned}|I(A_1A_2)| &= \Big|\int_{-\infty}^{\infty} \exp(-\imath tx) A_{1,\tau_1}(t,x) A_{2,\tau_2}(t,x) \\ &\qquad [A_{3,\tau_3}(t,x) - A_{0,\tau_3}(t,x)] dt\Big| \\ &\leq |x|^{-1} \left[\int_{|t|\leq\epsilon n^r} + \int_{|t|>\epsilon n^r}\right] |A^{(1)}_{1,\tau_1}(t,x)|\,|A_{2,\tau_2}(t,x)| \\ &\qquad |A_{3,\tau_3}(t,x) - A_{0,\tau_3}(t,x)| dt \\ &+ |x|^{-1} \left[\int_{|t|\leq\epsilon n^r} + \int_{|t|>\epsilon n^r}\right] |A_{1,\tau_1}(t,x)|\,|A^{(1)}_{2,\tau_2}(t,x)| \\ &\qquad |A_{3,\tau_3}(t,x) - A_{0,\tau_3}(t,x)| dt \\ &+ |x|^{-1} \left[\int_{|t|\leq\epsilon n^r} + \int_{|t|>\epsilon n^r}\right] |A_{1,\tau_1}(t,x)|\,|A_{2,\tau_2}(t,x)| \\ &\qquad |A^{(1)}_{3,\tau_3}(t,x) - A^{(1)}_{0,\tau_3}(t,x)| dt \\ &= M_1(x) + \cdots + M_6(x), \text{ say.}\end{aligned} \qquad (4.14)$$

Observe that (3.30) and (3.36) together with Lemma 2 imply that, as $n \to \infty$

$$|M_\imath(x)| = |x|^{-\alpha} O(n^{1-r}), \text{ for } \imath = 1,3, \qquad (4.15)$$

whereas Lemma 2 and (3.31) imply that, as $n \to \infty$,

$$|M_5(x)| = |x|^{-\alpha} O(n^{1-r}). \qquad (4.16)$$

Finally, as a consequence of equation (3.18), the assumption (A_2) and Lemma 11, we get for $\imath = 2,4,6$, as $n \to \infty$,

$$|M_\imath(x)| = |x|^{-\alpha} O(n^{1-r}). \qquad (4.17)$$

Thus from equations (4.14) to (4 17) it follows that, as $n \to \infty$,

$$|I(A_1A_2)| = |x|^{-\alpha}O(n^{1-r}) \qquad (4\ 18)$$

which is same as (4.12)

Estimate of $I(A_1B_2)$ Write $I(A_1B_2)$ as

$$\begin{aligned} I(A_1B_2) &= \left(\int\limits_{|t|\le\epsilon n^r} + \int\limits_{|t|>\epsilon n^r}\right)\exp(-itx)A_{1,\tau_1}(t,x)B_{2,\tau_2}(t,x) \\ &\quad \{A_{3,\tau_3}(,x) - A_{0,\tau_3}(t,x)\}dt \qquad (4\ 19) \\ &= I_1(A_1B_2) + I_2(A_1B_2), \text{ say} \end{aligned}$$

Now,

$$|I_1(A_1B_2)| = |x|^{-\alpha}O(n^{1-r}), \text{ as } n \to \infty, \qquad (4\ 20)$$

is evident from Lemmas 2 and 13 and (3.30); whereas, using Lemmas 11 and 13, we get

$$\begin{aligned} |I_2(A_1B_2)| &\le c|x|^{-\alpha} \int\limits_{|t|>\epsilon n^r} |A_{1,\tau_1}(t,x)|dt \\ &\le c|x|^{-\alpha}n^r\mu^{\tau_1-p}\int\limits_{|t|>\epsilon}|\alpha_{1,\tau_1}(t,x)|^p dt \\ &\le c|x|^{-\alpha}n^r\mu^{\tau_1-p} \end{aligned}$$

Therefore, it follows that, as $n \to \infty$

$$|I_2(A_1B_2)| = |x|^{-\alpha}O(n^{1-r}) \qquad (4.21)$$

Thus,

$$|I(A_1B_2)| = |x|^{-\alpha}O(n^{1-r}), \text{ as } n \to \infty, \qquad (4\ 22)$$

follows from (4 19), (4.20) and (4 21) On similar lines we can prove

$$|I(B_1A_2)| = |x|^{-\alpha}O(n^{1-r}), \text{ as } n \to \infty \qquad (4\ 23)$$

Observing the fact that $|B_{k,\tau_k}(t,x)| \le \max(1, c|x|^{-\alpha})$ for $k = 1,2,3$ and once again using the techniques of estimate of $I(A_1B_2)$ we get

$$|I(B_1B_2)| = |x|^{-\alpha}O(n^{1-r}), \text{ as } n \to \infty \qquad (4\ 24)$$

Estimate of $I(B)$ We have

$$I(B) = \int_{-\infty}^{\infty} \exp(-itx)\{w_1(tn^{-r})\}^{\tau_1}\{w_2(tn^{-r})\}^{\tau_2}$$

$$[B_{3,r_3}(t,x) - B_{0,r_3}(t,x)]dt$$

Note that

$$\{A_{1,r_1}(t,x)A_{2,r_2}(t,x) + A_{1,r_1}(t,x)B_{2,r_2}(t,x)$$
$$+B_{1,r_1}(t,x)A_{2,r_2}(t,x) + B_{1,r_1}(t,x)B_{2,r_2}(t,x)\}$$

is a complex valued function with absolute value of each summand (component) being less than or equal to $\max(1, c|x|^{-\alpha})$. Each component satisfies all the properties of the function $g(t,x)$ introduced in Lemma 12. We therefore take each component $g_j(t,x)$, say, $j = 1,2,3,4$ as $g(t,x)$ of Lemma 12 and apply Lemma 12 Therefore,

$$\begin{aligned}
|I(B)| &= |\int_{-\infty}^{\infty} \exp(-itx)\{A_{1,r_1}(t,x)A_{2,r_2}(t,x) + A_{1,r_1}(t,x)B_{2,r_2}(t,x) \\
&\quad +B_{1,r_1}(t,x)A_{2,r_2}(t,x) + B_{1,r_1}(t,x)B_{2,r_2}(t,x)\} \\
&\quad [B_{3,r_3}(t,x) - B_{0,r_3}(t,x)]dt| \\
&\leq \sum_{j=1}^{4} |\int_{-\infty}^{\infty} \exp(-itx)g_j(t,x)[B_{3,r_3}(t,x) - B_{0,r_3}(t,x)]dt| \\
&\leq \sum_{j=1}^{4} |x|^{-\alpha}O(n^{1-r}) \\
&\leq |x|^{-\alpha}O(n^{1-r}), \text{ as } n \to \infty
\end{aligned} \quad (4.25)$$

using Lemma 12

(4.8) now follows from (4.11), (4.18), (4.22), (4.23), (4.24) and (4.25). In view of the remarks following equations (4.10) the proof of Theorem 2 is complete.

5 GENERAL CASE AND CONCLUDING REMARKS

(i) In the case $m > 3$ (but fixed) in place of (4.7) we will have

$$\begin{aligned}
&\sup_{|x|\geq 1}|x|^{\alpha}\left|\int_{-\infty}^{\infty} \exp(-itx)\left[\prod_{k=1}^{m}|\{w_k(tn^{-r})\}^{r_k} - \prod_{k=1}^{m}\{w_0(tn^{-r})\}^{r_k}\right]dt\right| \\
&\leq \sup_{|x|\geq 1}|x|^{\alpha}|\int_{-\infty}^{\infty} \exp(-itx)\prod_{k=1}^{m}\{w_k(tn^{-r})\}^{r_k}[\{w_m(tn^{-r})\}^{r_k} - \{w_0(tn^{-r})\}^{r_k}]dt| \\
&\quad +\sum_{s=2}^{m-1}\sup_{|x|>1}|x|^{\alpha}|\int_{-\infty}^{\infty} \exp(-itx)\prod_{k=1}^{m-s}\{w_k(tn^{-r})\}^{r_k}\prod_{j=m-s+2}^{m}\{w_0(tn^{-r})\}^{r_j} \\
&\quad [\{w_{m-s+1}(tn^{-r})\}^{r_{m-s+1}} - \{w_0(tn^{-r})\}^{r_{m-s+1}}]dt|
\end{aligned}$$

$$+\sup_{|x|\geq 1}|x|^{\alpha}|\int_{-\infty}^{\infty}\exp(-itx)\prod_{j=2}^{m}\{w_0(tn^{-r})\}^{\tau_j}$$
$$[\{w_1(tn^{-r})\}^{\tau_1}-\{w_0(tn^{-r})\}^{\tau_1}]dt|$$

As in Theorem 2 we shall consider 1st term only. Proceeding as before, this can be expressed as sum of 5 terms say I_1, I_2, I_3, I_4 and I_5, where I_1 has in the integrand the product term involving $(m-1)$ A_i's with $(A_m - A_0), I_2$ is the sum of $(m-1)$ integrals with each integrand containing the product of one $B_i, i \leq m-1$ with $(m-2)$ A_i's and $(A_m - A_0), I_3$ is the sum of $2^{m-1}-(m+1)$ integrals with each integrand being the product of $m-1$ terms with $(A_m - A_0)$ of which atleast two are B_i's and atleast one is A_i : I_4 is an integral whose integrand is the product of $(m-1)$ B_i with $(A_m - A_0)$ and I_5 is an integral whose integrand is the product

$$\prod_{k=1}^{m-1}\{w_k(tn^{-r})\}^{\tau_k}(B_m - B_0)$$

Proceeding as in equations (4.12) to (4.25) we get $O(n^{(1-r)}$.

(ii) It is well known that limit distribution of normalized sums of independent r.v.s. exists irrespective of the sampling scheme under consideration (See : Sreehari (1970)). We are unable to prove the rate in the local theorem of this result in the case $\tau_i/n \to \infty$ as $n \to 0$ for some i, mainly because of the failure of some of our estiamtes to hold in this case.

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