CHAPTER 2

BASIC PROPERTIES OF STABLE DISTRIBUTIONS AND THE DISTRIBUTIONS ATTRACTED TO THEM

2.1 INTRODUCTION:

This chapter contains a summary of basic properties of stable distributions and of the distributions 'attracted' to them which are needed in the following chapters. The proofs of well-known statements are omitted; they can be found in cited literature.

All the r.v.s are assumed to be defined on a probability space $(\Omega, \mathbb{F}, P)$.

We shall be discussing definitions and properties of stable distributions, domains of attractions, in Section 2.2. In Section 2.3, properties of distributions in the normal attraction of non-normal stable laws are discussed in detail.

2.2 STABLE DISTRIBUTIONS:

2.2.1 MAIN DEFINITIONS AND BASIC PROPERTIES:

Let $(\Omega, \mathbb{F}, P)$ be a probability space and let Y be a r.v. defined on it. Let G and g denote the d.f. and the corresponding c.f. of r.v. Y respectively.

Definition 2.2.1: A d.f. G (or a r.v. Y) is said to be **stable** if for any positive a_1 and a_2 there exist real

numbers $a > 0$ and b such that $g(a_1t)g(a_2t) = e^{itb}g(at)$.
If $b = 0$ then the d.f. G (or r.v. Y) is called **strictly stable**.

Theorem 2.2.1: *A r.v.* Y *is stable iff its c.f.* g(t) *can be represented in the form*

$$g(t) = \exp[i\mu t - C|t|^{\alpha}\{1 - i\beta \operatorname{sgn}(t)\, w(t,\alpha)\}], \qquad \ldots(2.2.1)$$

where μ *is a real constant,* $C \geq 0$, $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$ *and*

$$w(t,\alpha) = \begin{cases} \tan(\alpha\pi/2) & \ldots\text{if } \alpha \neq 1 \\ -(2/\pi)\log|t| & \ldots\text{if } \alpha = 1. \end{cases}$$

Remark 2.2.1: For the proof, see Ibragimov and Linnik (1971, Canonical Representation of Stable Laws, Theorem 2.2.2, p.43). Also see P.Hall (1981) for his remark on this representation.

Remark 2.2.2: The sub-class with $\beta = 0 = \mu$ comprises of the **symmetric stable distributions** if $\alpha \leq 2$. The parameter α is called **characteristic exponent** or **index** or simply **exponent**.

It should be noted that a strictly stable distribution with $\alpha = 1$ is symmetric stable.

Theorem 2.2.2: *All proper stable distributions are absolutely continuous.*

Remark 2.2.3: For the proof, see Gnedenko and Kolmogorov (1954, p.183).

Let $p_{\alpha,\beta}(x)$ denote the p.d.f. of the stable d.f. $G_{\alpha,\beta}(x)$ with parameters (α,β). It is well known that the explicit expressions for stable densities in terms of elementary functions are known only in few cases viz. normal distribution ($\alpha = 2$), the Cauchy distribution ($\alpha = 1$, $\beta = 0$) and the Levy distribution ($\alpha = 1/2$, $\beta = \pm 1$).

Finally, if G is a stable d.f. with index $\alpha < 2$, absolute moment of order β exists iff $\beta < \alpha$.

2.2.2 DOMAINS OF ATTRACTIONS OF THE STABLE LAWS:

Let $X, X_1, \ldots, X_n$ be a sequence of independent r.v.s having the common d.f. F. Further, let $\{A_n\}$ and $\{B_n\}$ be sequences of constants such that $B_n > 0$, and the d.f.s of the normalized sums

$$Z_n = (S_n - A_n)/B_n \qquad \ldots(2.2.2)$$

converge weakly to some d.f.G. Then we say that d.f. F is **attracted** to d.f. G or d.f. F is in the domain of attraction of d.f. G.

It is well-known that the class of stable distributions coincides with the set of distributions that are limits of distributions of Z_n at (2.2.2).

Definition 2.2.2: The set of all d.f.s attracted to the stable law $G_{\alpha,\beta}$ is called the **domain of attraction** of $G_{\alpha,\beta}$.

Theorem 2.2.1 shows that these stable distributions form a four parameter family $G(\alpha, \beta, \mu, C)$. In view of the discussions in Section 2.3 (p. 47-48) of Ibragimov and Linnik (1971), we may restrict ourselves either to $\beta \geq 0$ or alternatively to $x \geq 0$.

Theorem 2.2.3: *A d.f.* F *belongs to the domain of attraction*

(i) *of the standard normal law* $\Phi(x)$ *iff*

$$R(x) = F(-x) + (1-F(x))$$

$$= O(x^{-2} \int_{|u|<x} u^2 dF(u)) \text{ as } x \to \infty, \qquad \ldots(2.2.3)$$

(ii) *of a non-normal stable d.f.* $G_{\alpha,\beta}(x)$ *iff*

$$F(x) = (d_1+O(1))|x|^{-\alpha}h(|x|) \text{ as } x \to -\infty \text{ and}$$

$$1-F(x) = (C_1+O(1))x^{-\alpha}h(x) \text{ as } x \to \infty, \qquad \ldots(2.2.4)$$

where $h(x)$ *is a slowly varying function, and* d_1 *and* C_1 *are constants depending upon the parameters* α, β *and* C, *with the condition that* $d_1+C_1 > 0$.

Further, if F belongs to the domain of attraction of a stable law then $B_n = n^{1/\alpha}h(n)$, where $h(n)$ is a slowly varying function. More precisely, B_n can be taken to be the largest solution of the equation

$$n \int_{|u|\le B_n} u^2 dF(u) = B_n^2, \quad 0< \alpha\le 2. \qquad \ldots(2.2.5)$$

B_n can be replaced by other sequences B_n^* with $\lim_{n\to\infty} B_n^*/B_n = 1$.

Definition 2.2.3: A d.f. F (or r.v. X) belongs to the **domain of normal attraction** of a stable law $G_{\alpha,\beta}$ if the d.f. of sum Z_n in (2.2.2) weakly converges to $G_{\alpha,\beta}$ with $B_n = Cn^{1/\alpha}$.

Remark 2.2.4: When a d.f. F belongs to the domain of normal attraction of a stable law G with index α, we denote this by $F \in \mathcal{D}_{NA}(\alpha)$.

Definition 2.2.4: A d.f. F (or r.v. X) belonging to the domain of attraction of the stable law $G_{\alpha,\beta}$ but not belonging to the domain of normal attraction of the stable law $G_{\alpha,\beta}$ is said to belong to the **domain of non-normal attraction** of the stable law $G_{\alpha,\beta}$.

Remark 2.2.5: When a d.f. F belongs to the domain of non-normal attraction of a stable law G with index α, we denote this by $F \in \mathcal{D}_{NNA}(\alpha)$.

Remark 2.2.6: The domain of attraction of a stable law with index α, denoted as $\mathcal{D}_A(\alpha)$, is defined by $\mathcal{D}_A(\alpha) =$

$D_{NA}(\alpha) \cup D_{NNA}(\alpha)$.

Theorem 2.2.4: *A d.f.* F *(or r.v.* X*) belongs to domain of normal attraction*

(i) *of the standard normal law* $\Phi(x)$ *iff* $EX^2 < \infty$, *and*

(ii) *of a stable law* $G_{\alpha,\beta}(x)$ *with* $\alpha < 2$ *iff*

$$F(x) = (d_1+S_1(x))|x|^{-\alpha} \text{ as } x \to -\infty$$

$$1-F(x) = (d_2+S_2(x))x^{-\alpha} \text{ as } x \to \infty, \qquad \ldots(2.2.6)$$

where d_1, d_2 *are constants depending on the parameters of the stable distribution in such a way that* $d_1+d_2 \geq 0$ *and* $S_i(x) \to 0$, $i = 1, 2$.

Following is a consequence of the above Theorem.

Theorem 2.2.5: *If a r.v.* X *belongs to the domain of attraction of* $G_{\alpha,\beta}$ *then* $E|X|^{\delta} < \infty \Leftrightarrow 0 \leq \delta < \alpha \leq 2$.

Remark 2.2.7: For the proof, see Gnedenko and Kolmogorov (1954, p.179).

If a d.f. F belongs to the domain of attraction of a stable law G, then the structure of the c.f. f corresponding to d.f. F can be characterized in the neighbourhood of origin as follows:

Theorem 2.2.6: *In order that the d.f.* F *with c.f.* f(t) *belongs to the domain of attraction of the stable law* G *with c.f.* g(t) *at (2.2.1), it is necessary and sufficient that, in the neighbourhood of the origin,*

$$\log f(t) = i\mu t - C|t|^{\alpha}\tilde{h}(t)(1-i\beta\ \mathrm{sgn}(t)\ w(t,\alpha)), \qquad \ldots(2.2.7)$$

where μ *is a constant, and* $\tilde{h}(t)$ *is slowly varying as* $t \to 0$.

Remark 2.2.8: For the proof of this Theorem, see Ibragimov and Linnik (p.85, 1971).

2.3 D.F.S IN THE DOMAIN OF NORMAL ATTRACTION OF A NON-NORMAL STABLE LAW (FURTHER PROPERTIES):

Throughout this section we suppose that a d.f. F_1 of r.v. X_1 belongs to the domain of normal attraction of a stable law F_0 of r.v. X_0 with index α, $0< \alpha< 2$. Here we study the behaviour of tail function, tail sum function and that of the truncated moments of the d.f.s F_k, $k = 0, 1$. Throughout the thesis we shall let $\gamma = 1/\alpha$.

Further, we assume, without any loss of generality, that $\int_{-\infty}^{\infty} u\,dF_k(u) = 0$, $k = 0, 1$, whenever such an integral exists. Also, $v_k^*(.)$ and $w_k(.)$ will denote the p.d.f. and c.f. corresponding to the d.f. F_k, $k = 0, 1$, in this section.

Lemma 2.3.1: For $k = 0, 1$, and a r.v. X_k with d.f. F_k, as $z \to \infty$, we have

(i) $z^{\alpha} R_k(z) \equiv z^{\alpha} P(|X_k| > z) \to c_k > 0;$ ...(2.3.1)

(ii) whenever $0 < \alpha < 1$,

$$\int_{|u| \le z} |u| dF_k(u) = O(z^{1-\alpha}); \qquad \text{...(2.3.2)}$$

(iii) whenever $\alpha = 1$,

$$\int_{|u| > z} |u|^{1/2} dF_k(u) = O(z^{-1/2}) \qquad \text{...(2.3.3)}$$

and

$$\int_{|u| \le z} u^2 dF_k(u) = O(z); \qquad \text{...(2.3.4)}$$

(iv) whenever $1 < \alpha < 2$,

$$\int_{|u| > z} |u| dF_k(u) = O(z^{1-\alpha}) \qquad \text{...(2.3.5)}$$

and

$$\int_{|u| \le z} u^2 dF_k(u) = O(z^{2-\alpha}). \qquad \text{...(2.3.6)}$$

Proof: Since F_k, $k = 0, 1$, belongs to the domain of normal attraction of F_O, $R_k(z) = O(z^{-\alpha})$ as $z \to \infty$. Simple calculations involving integration by parts and the relation $R_k(z) = O(z^{-\alpha})$ then lead to the results given above.□

For each positive integer n and real number x, we define, for $k = 0, 1$,

$$\alpha_{k,n}(t,x) = \int_{|u| \le |x| n^{\gamma}} e^{itu} dF_k(u), \qquad \text{...(2.3.7)}$$

$$\beta_{k,n}(t,x) = w_k(t) - \alpha_{k,n}(t,x), \qquad \text{...(2.3.8)}$$

$$A_{k,n}(t,x) = \{\alpha_{k,n}(tn^{-\gamma},x)\}^n, \qquad \ldots(2.3.9)$$

$$B_{k,n}(t,x) = \{w_k(tn^{-\gamma})\}^n-\{\alpha_{k,n}(tn^{-\gamma},x)\}^n$$

$$= \sum_{r=1}^{n}\binom{n}{r}\{\alpha_{k,n}(tn^{-\gamma},x)\}^{n-r}\{\beta_{k,n}(tn^{-\gamma},x)\}^{r-1}. \qquad \ldots(2.3.10)$$

The following result is given in Basu and Maejima (1980). This result being crucial for our further study we present it for completeness.

Lemma 2.3.2: Suppose that an absolutely continuous d.f F_1 is in the domain of normal attraction of a stable law F_o with index α, $\alpha< 2$ (and in addition strictly stable for $0< \alpha \le 1$). There exist positive constants ε, c and C_1 such that for k = 0, 1,

$$|A_{k,n}(t,x)| \le C_1 e^{-c|t|^\alpha} \qquad \ldots(2.3.11)$$

for all t with $|t|\le \varepsilon n^\gamma$, all x with $|x|\ge 1$ and all large n.

Proof: We first observe that by the lemma in Gnedenko and Kolmogorov (1968, p.238), there exist positive constants ε and c such that, for $|t|\le \varepsilon$,

$$|w_k(t)| \le e^{-c|t|^\alpha}. \qquad \ldots(2.3.12)$$

Therefore, for sufficiently large n,

$$|w_k(tn^{-\gamma})|^n \le e^{-c|t|^\alpha} \qquad \ldots(2.3.13)$$

for all $|t|\le \varepsilon n^\gamma$ for k = 0, 1.

Therefore, for all t with $|t|\le \varepsilon n^\gamma$, all x with $|x|\ge 1$ and sufficiently large n,

$$|A_{k,n}(t,x)| \le |\{w_k(tn^{-\gamma})\} - \{\beta_{k,n}(tn^{-\gamma},x)\}|^n$$

$$\le \sum_{r=0}^{n} \binom{n}{r} |w_k(tn^{-\gamma})|^{n-r} |\beta_{k,n}(tn^{-\gamma},x)|^r$$

$$\le \sum_{r=0}^{n} \binom{n}{r} e^{-c|t|^{\alpha}(1-r/n)} \{P(|X_k| \ge n^{\gamma})\}^r, \text{ using } (2.3.13)$$

$$\le \sum_{r=0}^{n} (n^r/r!) e^{-c|t|^{\alpha}} e^{c\varepsilon r} C_1 n^{-r}, \text{ using } (2.3.1)$$

$$\le e^{-c|t|^{\alpha}} \sum_{r=0}^{n} (ne^{c\varepsilon}(C_1 n^{-1}))^r / r!$$

$$\le C_1 e^{-c|t|^{\alpha}}. \square$$

In addition to the assumptions made at the beginning of this section, we make some or all of the following assumptions in the following lemmas.

[A1] F_k, $k = 0, 1$, are absolutely continuous and $F_k^{(1)}(u) = v_k^*(u)$.

[A2] $\int_{-\infty}^{\infty} |u|^{[\alpha]+1} |v_1^*(u) - v_o^*(u)| du < \infty$.

[A3] The d.f. $F_1 \in \mathcal{D}_{NA}(\alpha)$ and F_o is stable with index $\alpha < 2$. In case $0 < \alpha \le 1$, F_o is strictly stable.

[A4] $\Xi = \{(t, n, x): |t| \le \varepsilon n^{\gamma}, n \ge n_o, |x| \ge 1\}$ where ε is as determined in Lemma 2.3.2 and n_o is large.

[A5] $\Theta = \{(t, n, x): |t| > \varepsilon, n \geq n_o, |x| \geq 1\}$ where ε is as determined in Lemma 2.3.2 and n_o is large.

The following result is stated in Basu, Maejima and Patra (1979) without proof. We prove the result.

Lemma 2.3.3: Under the assumptions **[A1]**, **[A2]**, **[A3]** and **[A4]**, for all $(t, n, x) \in \Xi$, we have,

$$|\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-r}-\{\alpha_{0,n}(tn^{-\gamma},x)\}^{n-r}|$$

$$\leq n^{1-([\alpha]+1)\gamma}P(|t|)e^{-c|t|^{\alpha}} \qquad \ldots(2.3.14)$$

for $r = 0, 1, 2, \ldots, n$.

Proof: Write $\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-r}-\{\alpha_{0,n}(tn^{-\gamma},x)\}^{n-r}$

$$= [\{\alpha_{1,n}(tn^{-\gamma},x)\}-\{\alpha_{0,n}(tn^{-\gamma},x)\}]$$

$$\sum_{k=1}^{n-r}\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-r-k}\{\alpha_{0,n}(tn^{-\gamma},x)\}^{k-1}$$

$$= \{n[\{\alpha_{1,n}(tn^{-\gamma},x)\}-\{\alpha_{0,n}(tn^{-\gamma},x)\}]\}$$

$$\{n^{-1}\sum_{k=1}^{n-r}\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-r-k}\{\alpha_{0,n}(tn^{-\gamma},x)\}^{k-1}\}$$

$$= I_1 I_2, \text{ say.} \qquad \ldots(2.3.15)$$

Estimation of I_1:

Consider the case $0 < \alpha < 1$.

Note that, in view of (2.3.7),

$$I_1 = n\int_{|u|\leq|x|n^{\gamma}} e^{itun^{-\gamma}}d(F_1(u)-F_0(u))$$

$$= n \int_{|u| \le |x| n^{\gamma}} \{e^{itun^{-\gamma}} - 1\} d(F_1(u) - F_0(u))$$

$$-n \int_{|u| > |x| n^{\gamma}} d(F_1(u) - F_0(u)).$$

Therefore,

$$|I_1|$$

$$\le n \int_{|u| \le |x| n^{\gamma}} |tun^{-\gamma}| |v_1^*(u) - v_0^*(u)| du$$

$$+ n \int_{|u| > |x| n^{\gamma}} |v_1^*(u) - v_0^*(u)| du$$

$$\le n^{1-\gamma} |t| \int_{|u| \le |x| n^{\gamma}} |u| |v_1^*(u) - v_0^*(u)| du$$

$$+ n^{1-\gamma} |x|^{-1} \int_{|u| > |x| n^{\gamma}} |u| |v_1^*(u) - v_0^*(u)| du$$

$$\le n^{1-\gamma} |t| \int_{-\infty}^{\infty} |u| |v_1^*(u) - v_0^*(u)| du$$

$$+ n^{1-\gamma} \int_{-\infty}^{\infty} |u| |v_1^*(u) - v_0^*(u)| du$$

$$\le n^{1-\gamma} P(|t|), \qquad \text{...(2.3.16a)}$$

by the hypothesis of the lemma.

Now, consider the case $\alpha = 1$.

$$I_1 = n \int_{|u| \le |x| n} e^{itun^{-1}} d(F_1(u) - F_0(u))$$

$$= n \int_{|u| \le |x| n} \cos(tun^{-1}) d(F_1(u) - F_0(u))$$

$$= n \int_{|u| \le |x| n} (\cos(tun^{-1}) - 1) d(F_1(u) - F_0(u))$$

$$-n \int_{|u| > |x| n} d(F_1(u) - F_0(u)).$$

Hence,

$$|I_1| \le n \int_{|u|\le|x|n} ((tun^{-1})^2/2)\,|v_1^*(u)-v_0^*(u)|\,du$$

$$+|x|^{-2}n^{-1} \int_{|u|>|x|n} u^2|v_1^*(u)-v_0^*(u)|\,du$$

$$\le n^{-1}(t^2/2) \int_{-\infty}^{\infty} (u^2)\,|v_1^*(u)-v_0^*(u)|\,du$$

$$+n^{-1} \int_{-\infty}^{\infty} u^2|v_1^*(u)-v_0^*(u)|\,du$$

$$\le n^{-1}P(|t|), \qquad \ldots(2.3.16b)$$

by the hypothesis of the lemma.

Finally, consider the case $1< \alpha< 2$.

Again note that in view of (2.3.7)

$$I_1 = n \int_{|u|\le|x|n^{\gamma}} e^{itun^{-\gamma}} d(F_1(u)-F_0(u))$$

$$= n \int_{|u|\le|x|n^{\gamma}} (e^{itun^{-\gamma}}-1-itun^{-\gamma})\,d(F_1(u)-F_0(u))$$

$$+n \int_{|u|\le|x|n^{\gamma}} d(F_1(u)-F_0(u))+itn^{1-\gamma} \int_{|u|\le|x|n^{\gamma}} u\,d(F_1(u)-F_0(u)).$$

Therefore, in view of the assumption of finiteness of $\int_{-\infty}^{\infty}|u|^{[\alpha]+1}|v_1^*(u)-v_0^*(u)|\,du$, we have

$$|I_1| \le n \int_{|u|\le|x|n^{\gamma}} ((tun^{-\gamma})^2/2)\,|v_1^*(u)-v_0^*(u)|\,du$$

$$+n \int_{|u|>|x|n^{\gamma}} |v_1^*(u)-v_0^*(u)|\,du$$

$$+|t|n^{1-\gamma} \int_{|u|>|x|n^{\gamma}} (u^2/|u|)\,|v_1^*(u)-v_0^*(u)|\,du,$$

because $EX_1 = EX_0 = 0$.

$$\leq n^{1-2\gamma}(t^2/2) \int_{-\infty}^{\infty} u^2 |v_1^*(u) - v_o^*(u)| du$$

$$+|x|^{-2} n^{1-2\gamma} \int_{-\infty}^{\infty} u^2 |v_1^*(u) - v_o^*(u)| du$$

$$+|x|^{-1} |t| n^{1-2\gamma} \int_{-\infty}^{\infty} u^2 |v_1^*(u) - v_o^*(u)| du$$

$$\leq n^{1-2\gamma} P(|t|). \qquad \ldots(2.3.16c)$$

Thus, in general, we get from (2.3.16a), (2.3.16b) and (2.3.16c),

$$|I_1| \leq n^{1-([\alpha]+1)\gamma} P(|t|), \quad \text{for all } t. \qquad \ldots(2.3.16)$$

Estimation of I_2: Note that, in view of Lemma 2.3.2, we have

$$|I_2| \leq |n^{-1} \sum_{k=1}^{n-r} \{\alpha_{1,n}(tn^{-\gamma}, x)\}^{n-r-k} \{\alpha_{0,n}(tn^{-\gamma}, x)\}^{k-1}|$$

$$\leq n^{-1} \sum_{k=1}^{n-r} \{C_1 e^{-c|t|^\alpha}\}^{(1-(r+k)/n)} \{C_1 e^{-c|t|^\alpha}\}^{((k-1)/n)}$$

$$= n^{-1} C_1 e^{-c|t|^\alpha} \sum_{k=1}^{n-r} C_1^{-(r+1)/n} e^{c|t|^{\alpha}((r+1)/n)}$$

$$\leq n^{-1} C_1 e^{-c|t|^\alpha} \sum_{k=1}^{n-r} C^* e^{c\varepsilon(r+1)}$$

$$= C_1 e^{-c|t|^\alpha}, \text{ for } |t| \leq \varepsilon n^{\gamma}. \qquad \ldots(2.3.17)$$

Combining the estimates of (2.3.16) and (2.3.17), we get

$$|I_1 I_2| \leq n^{1-([\alpha]+1)\gamma} P(|t|) e^{-c|t|^\alpha}. \square$$

Next we define two functions.

$$d_n(t,x) = n[\{\alpha_{1,n}(tn^{-\gamma}, x)\} - \{\alpha_{0,n}(tn^{-\gamma}, x)\}] \qquad \ldots(2.3.18)$$

$$S_n(t,x) = n^{-1} \sum_{k=1}^{n} \{\alpha_{1,n}(tn^{-\gamma}, x)\}^{n-k} \{\alpha_{0,n}(tn^{-\gamma}, x)\}^{k-1}.$$

$$\ldots(2.3.19)$$

In what follows, Lemmas 2.3.4 - 2.3.6 give bounds on the functions $\alpha_n(t,x)$, $d_n(t,x)$, $S_n(t,x)$ and their first and second derivatives with respect to t.

Properties of the function $d_n(t,x)$

Lemma 2.3.4: Under the assumption **[A1]**, **[A2]** and **[A3]**, for all values of t, x with $|x| \geq 1$ and large n, we have

(i) whenever $0 < \alpha < 1$,

$$|d_n(t,x)| \leq n^{1-\gamma} P_1(|t|), \quad \ldots(2.3.20)$$

$$|d_n^{(1)}(t,x)| \leq C_1 n^{1-\gamma}; \quad \ldots(2.3.21)$$

(ii) whenever $1 \leq \alpha < 2$,

$$|d_n(t,x)| \leq n^{1-2\gamma} P_2(|t|), \quad \ldots(2.3.22)$$

$$|d_n^{(1)}(t,x)| \leq n^{1-2\gamma} P_3(|t|), \quad \ldots(2.3.23)$$

$$|d_n^{(2)}(t,x)| \leq C_2 n^{1-2\gamma}. \quad \ldots(2.3.24)$$

Remark 2.3.1: Although in the assumptions $\alpha = 1$ is clubbed with interval (0, 1), while discussing most properties, we notice that the case of $\alpha = 1$ can be clubbed with interval (1, 2). Further we need second derivatives of certain functions in the case of $\alpha = 1$ as well.

Proof: Note that I_1 of Lemma 2.3.3 at (2.3.15) is $d_n(t,x)$. Therefore, from (2.3.16), we have $|d_n(t,x)| = |I_1| \leq n^{1-([\alpha]+1)\gamma} P(|t|)$, $0 < \alpha < 2$ and (2.3.20) and (2.3.22) are proved. Now it remains to prove (2.3.21), (2.3.23) and (2.3.24).

Consider the case $0< \alpha< 1$. Here

$$d_n^{(1)}(t,x) = (d/dt)\{n \int_{|u|\le|x|n^\gamma} e^{itun^{-\gamma}} d(F_1(u)-F_0(u))\}$$

$$= n^{1-\gamma} i \int_{|u|\le|x|n^\gamma} u e^{itun^{-\gamma}} d(F_1(u)-F_0(u)),$$ using DCT.

Hence,

$$|d_n^{(1)}(t,x)| = |(d/dt)\{n \int_{|u|\le|x|n^\gamma} e^{itun^{-\gamma}} d(F_1(u)-F_0(u))\}|$$

$$\le n^{1-\gamma} \int_{|u|\le|x|n^\gamma} |u||v_1^*(u)-v_0^*(u)|du$$

$$\le n^{1-\gamma} \int_{-\infty}^{\infty} |u||v_1^*(u)-v_0^*(u)|du$$

$$= C_1 n^{1-\gamma},$$ using the hypothesis of the lemma.

This proves (2.3.21).

Now consider the case $\alpha = 1$.

$$d_n^{(1)}(t,x)$$

$$= (d/dt)\{n \int_{|u|\le|x|n} \cos\ (tun^{-1}) d(F_1(u)-F_0(u))\}$$

$$= \int_{|u|\le|x|n} u \sin\ (tun^{-1}) d(F_1(u)-F_0(u)),$$ using DCT.

Hence,

$$|d_n^{(1)}(t,x)|$$

$$\le \int_{|u|\le|x|n} |u||tun^{-1}||v_1^*(u)-v_0^*(u)|du,$$ since $|\sin(x)/x|\le 1$.

$$\le |t|n^{-1} \int_{-\infty}^{\infty} u^2|v_1^*(u)-v_0^*(u)|du$$

$$= n^{-1}P(|t|),$$ using the hypothesis of the lemma.

Thus, (2.3.23) is proved for $\alpha = 1$.

$|d_n^{(2)}(t,x)|$

$= |(d/dt)\{-\int_{|u|\le|x|n} u \sin(tun^{-1})d(F_1(u)-F_0(u))\}|$

$\le |\int_{|u|\le|x|n} u.un^{-1}\cos(tun^{-1})d(F_1(u)-F_0(u))|$, using DCT.

$\le n^{-1} \int_{-\infty}^{\infty} u^2|v_1^*(u)-v_0^*(u)|du$

$=C\, n^{-1}$,

using the hypothesis of the lemma, which proves (2.3.24) for $\alpha = 1$.

Finally we consider the case $1< \alpha< 2$.

Observe that

$d_n^{(1)}(t,x)$

$=(d/dt)\{n\int_{|u|\le|x|n^{\gamma}} e^{itun^{-\gamma}}d(F_1(u)-F_0(u))\}$

$=n^{1-\gamma}\int_{|u|\le|x|n^{\gamma}} ue^{itun^{-\gamma}}d(F_1(u)-F_0(u))$, using DCT.

Therefore,

$|d_n^{(1)}(t,x)|$

$\le n^{1-\gamma}|\int_{|u|\le|x|n^{\gamma}} u(e^{itun^{-\gamma}}-1)d(F_1(u)-F_0(u))|$

$+n^{1-\gamma}|\int_{|u|>|x|n^{\gamma}} ud(F_1(u)-F_0(u))|$, because $EX_1 = EX_0 = 0$.

$\le n^{1-2\gamma}|t| \int_{-\infty}^{\infty} u^2|v_1^*(u)-v_0^*(u)|du$

$+ n^{1-2\gamma}|x|^{-1} \int_{-\infty}^{\infty} u^2|v_1^*(u)-v_0^*(u)|du$

$\le n^{1-2\gamma}P(|t|)$, which establishes (2.3.23).

The inequality at (2.3.24) can be proved similarly.□

Properties of the function $\alpha_n(t,x)$

Lemma 2.3.5: Under the assumptions **[A1]** and **[A3]**, for each fixed n and x, $\alpha_n(tn^{-\gamma},x)$ is differentiable any number of times under the integral sign. For all values of t and x with $|x|\geq 1$, we have, for k = 0, 1,

(i) whenever $0<\alpha<1$,

$$|\alpha_{k,n}^{(1)}(tn^{-\gamma},x)|\leq C_1|x|^{1-\alpha}n^{\gamma-1}; \qquad \ldots(2.3.25)$$

(ii) whenever $1\leq\alpha<2$,

$$|\alpha_{k,n}^{(1)}(tn^{-\gamma},x)|\leq n^{\gamma-1}P_1(|t|) \qquad \ldots(2.3.26)$$

$$\leq |x|^{2-\alpha}n^{\gamma-1}P_1(|t|), \qquad \ldots(2.3.27)$$

$$|\alpha_{k,n}^{(2)}(tn^{-\gamma},x)|\leq C_1|x|^{2-\alpha}n^{2\gamma-1}. \qquad \ldots(2.3.28)$$

(iii) If, in addition to assumptions **[A1]** and **[A3]**, $\int_{-\infty}^{\infty}|w_1(t)|^p dt<\infty$, for an integer $p\geq 1$, then, for all $x\neq 0$, $0<\alpha<2$ and every sufficiently large but fixed integer s, there exists a constant C such that for k = 0, 1,

$$\int_{-\infty}^{\infty}|\alpha_{k,n}(t,x)|^n dt = O(n^{-\gamma}) \qquad \ldots(2.3.29)$$

$$\int_{-\infty}^{\infty}|\alpha_{k,n}(t,x)|^{2s}dt\leq C \qquad \ldots(2.3.30)$$

$$\int_{-\infty}^{\infty}|\beta_{k,n}(t,x)|^{2s}dt\leq C. \qquad \ldots(2.3.31)$$

Proof: Consider the case $0<\alpha<1$.

Note that $|\alpha_{k,n}^{(1)}(tn^{-\gamma},x)|$

$$=|(d/dt)\int_{|u|\leq|x|n^{\gamma}}e^{itu}dF_k(u)|_{t\to tn^{-\gamma}}|$$

$$=|\int_{|u|\leq|x|n^{\gamma}}ue^{itun^{-\gamma}}dF_k(u)|,$$ using DCT

$= \int_{|u|\leq|x|n^{\gamma}} |u| dF_k(u)$

$= O(|x|^{1-\alpha} n^{\gamma-1})$

$\leq C_1 |x|^{1-\alpha} n^{\gamma-1}$,

using (2.3.2) of Lemma 2.3.1, which proves (2.3.25).

The proof in case of $\alpha = 1$ is as follows.

Note that $|\alpha_{k,n}^{(1)}(tn^{-1},x)|$

$= |(d/dt) \int_{|u|\leq|x|n} \cos(tu) dF_k(u)|_{t \to tn^{-1}}$,

$= |-\int_{|u|\leq|x|n} u\sin(tun^{-1}) dF_k(u)|$

$\leq \int_{|u|\leq|x|n} ||u||\sin(tun^{-1})/(tun^{-1})||tun^{-1}| dF_k(u)$

$\leq n^{-1}|t| \int_{|u|\leq|x|n} |u^2 dF_k(u)$, since $|\sin(x)/x| \leq 1$

$= n^{-1}|t|\ O(|x|n)$, using (2.3.4)

$= |x| P_1(|t|)$.

For the case of $1 < \alpha < 2$, we split the term

$|\int_{|u|\leq|x|n^{\gamma}} u e^{itun^{-\gamma}} dF_k(u)|$ as follows:

$|\int_{|u|\leq|x|n^{\gamma}} u e^{itun^{-\gamma}} dF_k(u)|$

$= |\int_{|u|\leq|x|n^{\gamma}} u(e^{itun^{-\gamma}}-1) dF_k(u) + \int_{|u|\leq|x|n^{\gamma}} u dF_k(u)|$

$\leq |t|n^{-\gamma} \int_{|u|\leq n^{\gamma}} u^2 dF_k(u) + \int_{n^{\gamma}<|u|\leq|x|n^{\gamma}} |u||e^{itun^{-\gamma}}-1| dF_k(u)$

$+ \int_{|u|>|x|n^{\gamma}} |u| dF_k(u)$

$\leq |t|n^{-\gamma} \int_{|u|\leq n^{\gamma}} u^2 dF_k(u)\ +\ 3 \int_{|u|>|x|n^{\gamma}} |u| dF_k(u)$,

as $|e^{itun^{-\gamma}}-1| \leq 2$.

Now using (2.3.5) and (2.3.6) we obtain the desired results. (2.3.27) follows from (2.3.26) since $|x| \geq 1$. The proof of (2.3.28) is also similar and needs no modifications. The proofs of (2.3.29), (2.3.30) and (2.3.31) are on the lines of Smith and Basu (1974, p.370).□

Remark 2.3.2: Although (2.3.26) is sharper than (2.3.27) we mention it because of its utility later.

Properties of the function $S_n(t,x)$

Lemma 2.3.6: Under the assumptions **[A1]**, **[A3]** and **[A4]**, for all $(t, n, x) \in \Xi$, we have

(i) whenever $0 < \alpha < 1$

$$|S_n(t,x)| \leq C_1 e^{-c|t|^{\alpha}}, \qquad \ldots(2.3.32)$$

$$|S_n^{(1)}(t,x)| \leq C_1 |x|^{1-\alpha} e^{-c|t|^{\alpha}}; \qquad \ldots(2.3.33)$$

(ii) whenever $1 \leq \alpha < 2$,

$$|S_n(t,x)| \leq C e^{-c|t|^{\alpha}}, \qquad \ldots(2.3.34)$$

$$|S_n^{(1)}(t,x)| \leq |x|^{2-\alpha} e^{-c|t|^{\alpha}} P_1(|t|), \qquad \ldots(2.3.35)$$

$$|S_n^{(2)}(t,x)| \leq |x|^{2-\alpha} e^{-c|t|^{\alpha}} P_2(|t|). \qquad \ldots(2.3.36)$$

Proof: Observe that for $r = 0$, the quantity I_2 of Lemma 2.3.3 is same as $S_n(t,x)$. And, therefore, using (2.3.17), we obtain (2.3.32) and (2.3.34). Next, in order to obtain an upper bound on $|S_n^{(1)}(t,x)|$, we find $S_n^{(1)}(t,x)$.

$$S_n^{(1)}(t,x)$$

$$= n^{-1}\sum_{k=2}^{n}\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-k}(k-1)\{\alpha_{0,n}(tn^{-\gamma},x)\}^{k-2}$$

$$\{\alpha_{0,n}^{(1)}(tn^{-\gamma},x)\}n^{-\gamma}$$

$$+n^{-1}\sum_{k=1}^{n-1}(n-k)\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-k-1}\{\alpha_{1,n}^{(1)}(tn^{-\gamma},x)\}$$

$$\{\alpha_{0,n}(tn^{-\gamma},x)\}^{k-1}n^{-\gamma}$$

$$= S_n^{(1)}(t,x,1) + S_n^{(1)}(t,x,2), \text{ say.} \qquad \ldots(2.3.37)$$

Observe that, in view of the techniques used at (2.3.17) and (2.3.37), we have

$$|S_n^{(1)}(t,x,1)|$$

$$\le n^{-1}\sum_{k=2}^{n}|\alpha_{1,n}(tn^{-\gamma},x)|^{n-k}(k-1)|\alpha_{0,n}(tn^{-\gamma},x)|^{k-2}$$

$$|\alpha_{0,n}^{(1)}(tn^{-\gamma},x)|n^{-\gamma}$$

$$\le n^{1-\gamma}\sum_{k=2}^{n}|\alpha_{1,n}(tn^{-\gamma},x)|^{n(1-k/n)}|\alpha_{0,n}(tn^{-\gamma},x)|^{n((k-2)/n)}$$

$$|\alpha_{0,n}^{(1)}(tn^{-\gamma},x)|$$

$$\le n^{1-\gamma}C_1e^{-c|t|^{\alpha}}C_2|x|^{1-\alpha}n^{\gamma-1}, \text{ using (2.3.11) and (2.3.25).}$$

$$=C_1|x|^{1-\alpha}e^{-c|t|^{\alpha}}. \qquad \ldots(2.3.38)$$

Similarly it can be shown that

$$|S_n^{(1)}(t,x,2)|\le C_1|x|^{1-\alpha}e^{-c|t|^{\alpha}}. \qquad \ldots(2.3.39)$$

Now, (2.3.33) follows from (2.3.37), (2.3.38) and (2.3.39). Inequalities (2.3.35) and (2.3.36) in case of $1\le \alpha< 2$ can similarly be proved.□

Properties of the function $A_{k,n}(t,x)$

Lemma 2.3.7: Under the assumptions **[A1]**, **[A2]**, **[A3]** and **[A4]**, for all $(t, n, x) \in \Xi$, there exist polynomials $P_1(.)$ and $P_2(.)$ in $|t|$ such that, we have the following:

(i) whenever $o< \alpha< 1$,

$$|A_{1n}(t,x)-A_{on}(t,x)| \le n^{1-\gamma}e^{-c|t|^{\alpha}}P_1(|t|), \qquad ..(2.3.40)$$

$$|A_{1n}^{(1)}(t,x)-A_{on}^{(1)}(t,x)|$$

$$\le |x|^{1-\alpha}n^{1-\gamma}e^{-c|t|^{\alpha}}P_2(|t|); \qquad ...(2.3.41)$$

(ii) whenever $1\le \alpha< 2$,

$$|A_{1n}(t,x)-A_{on}(t,x)| \le n^{1-2\gamma}e^{-c|t|^{\alpha}}P_1(|t|), \qquad ...(2.3.42)$$

$$|A_{1n}^{(i)}(t,x)-A_{on}^{(i)}(t,x)|$$

$$\le |x|^{2-\alpha}n^{1-2\gamma}e^{-c|t|^{\alpha}}P_i(|t|), \qquad ...(2.3.43)$$

$i = 1, 2.$

Proof: In view of (2.3.18) and (2.3.19), we observe that

$$A_{1n}(t,x)-A_{on}(t,x) = d_n(t,x)S_n(t,x). \qquad ...(2.3.44)$$

On differentiating the above identity on both the sides with respect to t once or twice according as $0< \alpha< 1$ or $1\le \alpha< 2$, and then using the estimates of $d_n(t,x)$, $S_n(t,x)$ and their first and second order derivatives from lemmas 2.3.4 and 2.3.6, we obtain the desired results.□

Lemma 2.3.8: Under the assumptions **[A1]**, **[A3]** and **[A4]**, there exist polynomials $P_1(.)$ and $P_2(.)$ such that, for all $(t, n, x) \in \Xi$ and $k = 0, 1$, we have

(i) whenever $0 < \alpha < 1$,

$$|A_{k,n}^{(1)}(t,x)| \le C_1 |x|^{1-\alpha} e^{-c|t|^\alpha}; \qquad \ldots(2.3.45)$$

(ii) whenever $1 \le \alpha < 2$,

$$|A_{k,n}^{(1)}(t,x)| \le e^{-c|t|^\alpha} P_1(|t|) \qquad \ldots(2.3.46)$$

$$\le |x|^{2-\alpha} e^{-c|t|^\alpha} P_1(|t|), \qquad \ldots(2.3.47)$$

$$|A_{k,n}^{(2)}(t,x)| \le |x|^{2-\alpha} e^{-c|t|^\alpha} P_2(|t|). \qquad \ldots(2.3.48)$$

Proof: Note that $|A_{k,n}^{(1)}(t,x)| = |(d/dt)\{\alpha_{k,n}(tn^{-\gamma},x)\}^n|$

$$= |n^{1-\gamma}\{\alpha_{k,n}(tn^{-\gamma},x)\}^{n-1}\{\alpha_{k,n}^{(1)}(tn^{-\gamma},x)\}|$$

$$\le n^{1-\gamma}|\alpha_{k,n}(tn^{-\gamma},x)|^{n(1-(1/n))}|\alpha_{k,n}^{(1)}(tn^{-\gamma},x)|$$

$$\le n^{1-\gamma}\{Ce^{-c|t|^\alpha}\}^{(1-(1/n))}C_1|x|^{1-\alpha}n^{\gamma-1},$$

using (2.3.12) and (2.3.25)

$$\le Ce^{-c|t|^\alpha}C^{-1/n}n^{1-\gamma}e^{c|t|^\alpha/n}C_1|x|^{1-\alpha}$$

$\le C_1|x|^{1-\alpha}e^{-c|t|^\alpha}$, which proves (2.3.45).

Results for the case $1 \le \alpha < 2$ can be proved exactly on the same lines.□

The following result is due to Smith and Basu (Lemma 2.4, p.370, 1974).

Lemma 2.3.9: Under the assumptions **[A1]**, **[A3]** and **[A5]**, let $\varepsilon > 0$, and let n_o be a fixed integer. Let $\mu_k \equiv \sup_{\Theta} |\alpha_{k,n}(t,x)|$. Then $0 \le \mu_k < 1$.

Proof: First of all note that μ_k can not be greater than unity. If possible let $\mu_k = 1$. Then, there must be sequences of reals $\{t_n\}$ such that $t_n > t_o$, and $\{y_n\}$ such that $y_n \to \infty$, with the property that $\int_{-y_n}^{y_n} e^{it_n u} v_k^*(u)\,du \to 1$ as $n \to \infty$; here v_k^* is the p.d.f. corresponding to d.f.F_k, $k = 0, 1$. But, since $\int_{|u|>y_n} v_k^*(u)\,du \to 0$, this implies $\varphi_k(t_n) \to 1$, $k = 0, 1$, as $n \to \infty$; here $\varphi_k(u)$ represents the c.f. corresponding to d.f.F_k, $k = 0, 1$. By the Riemann-Lebesgue Lemma it follows that $\{t_n\}$ is a bounded sequence for otherwise $\phi_k(t_n) \to 0$ as $t_n \to \infty$. Thus $\{t_n\}$ must have a finite limit point t^*, say, and continuity of $\varphi_k(t)$ then requires $\varphi_k(t^*) = 1$. But we have $t^* \ge \varepsilon$ and are forced to the contradiction that $F_k(x)$ is lattice.□

Properties of the function $B_{k,n}(t,x)$

Lemma 2.3.10: Assume that **[A1]**, **[A2]** and **[A3]** hold. Let $g(t,x)$ be a complex-valued continuous function such that $|g(t,x)| \le \max(1, c|x|^{-\alpha})$ for all x with $|x| \ge 1$ and for all t. If we have $\int_{-\infty}^{\infty} |w_1(t)|^p dt < \infty$ for an integer $p \ge 1$; then

$$\left|\int_{-\infty}^{\infty}\{B_{1,n}(t,x) - B_{0,n}(t,x)\}g(t,x)\exp(-itx)dt\right|$$
$$=|x|^{-\alpha}O(n^{1-([\alpha]+1)\gamma}). \qquad ...(2.3.49)$$

Proof: We shall prove the result for the case $0< \alpha< 1$. The other case $1\leq \alpha< 2$ follows analogously. In view of (2.3.11), we have

$$\int_{-\infty}^{\infty}\{B_{1,n}(t,x) - B_{0,n}(t,x)\}g(t,x)\exp(-itx)dt$$
$$= \sum_{j=1}^{n}\binom{n}{j}\int_{-\infty}^{\infty}\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-j}\{\beta_{1,n}(tn^{-\gamma},x)\}^{j}$$
$$-\{\alpha_{o,n}(tn^{-\gamma},x)\}^{n-j}\{\beta_{o,n}(tn^{-\gamma},x)\}^{j}g(t,x)\exp\{-itx\}dt.$$
$$...(2.3.50)$$

Split the summation in (2.3.50) into three parts as $\sum_1 + \sum_2 + \sum_3$ where $\sum_1$ is over the range $1\leq j\leq [n/2]$, $\sum_2$ is over the range $[n/2]+1\leq j\leq n-2s$ and $\sum_3$ is over the range $n-2s+1\leq j\leq n$; s being some fixed positive integer with $n > 6s$ for n large. Then denoting

$$\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-j}\{\beta_{1,n}(tn^{-\gamma},x)\}^{j}$$
$$-\{\alpha_{o,n}(tn^{-\gamma},x)\}^{n-j}\{\beta_{o,n}(tn^{-\gamma},x)\}^{j}$$

$= W(t,x,n)$, we have the right hand side of (2.3.50) given by

$$\left|\sum_{j=1}^{n}\binom{n}{j}\int_{-\infty}^{\infty}W(t,x,n)g(t,x)\exp(-itx)dt\right|$$
$$= \left|\sum\nolimits_1\binom{n}{j}\int_{-\infty}^{\infty}W(t,x,n)g(t,x)\exp(-itx)dt\right.$$
$$+ \sum\nolimits_2\binom{n}{j}\int_{-\infty}^{\infty}W(t,x,n)g(t,x)\exp(-itx)dt$$
$$\left.+ \sum\nolimits_3\binom{n}{j}\int_{-\infty}^{\infty}W(t,x,n)g(t,x)\exp(-itx)dt\right|$$

$= |J_1(x)+J_2(x)+J_3(x)|$, say. ...(2.3.51)

Write

$J_1(x)$

$= \sum_1 \binom{n}{j} \int_{-\infty}^{\infty} \{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-j}$

$[\{\beta_{1,n}(tn^{-\gamma},x)\}^{j}-\{\beta_{o,n}(tn^{-\gamma},x)\}^{j}]g(t,x)\exp(-itx)dt$

$+\sum_1 \binom{n}{j} \int_{-\infty}^{\infty} \{\beta_{o,n}(tn^{-\gamma},x)\}^{j}$

$[\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-j}-\{\alpha_{o,n}(tn^{-\gamma},x)\}^{n-j}]g(t,x)\ \exp(-itx)dt$

$= J_{11}(x)+J_{12}(x)$, say. ...(2.3.52)

Let us first consider $J_{11}(x)$. In view of assumption **[A2]**, we notice that, for $1\le j\le [n/2]$ and $|x|\ge 1$,

$|\{\beta_{1,n}(tn^{-\gamma},x)\}^{j}-\{\beta_{o,n}(tn^{-\gamma},x)\}^{j}|$

$= |\beta_{1,n}(tn^{-\gamma},x)-\beta_{o,n}(tn^{-\gamma},x)|$

$$|\sum_{i=1}^{j}\{\beta_{1,n}(tn^{-\gamma},x)\}^{j-i}\{\beta_{o,n}(tn^{-\gamma},x)\}^{i-1}|$$

$$\le |\int_{|u|\ge|x|n^{\gamma}} e^{itun^{-\gamma}}d(F_1(u)-F_o(u))|$$

$$\sum_{i=1}^{j}\{R_1(|x|n^{\gamma})\}^{j-i}\{R_o(|x|n^{\gamma})\}^{i-1}$$

$$\le \{|x|^{-1}n^{-\gamma}\int_{|u|\ge|x|n^{\gamma}}|u||v_1(u)-v_o(u)|du\}$$

$[j(C_2|x|^{-\alpha}n^{-1})^{j-1}]$, using (2.3.1)

$\le C_1|x|^{-1}n^{-\gamma}j(C_2/n)^{j-1}$. ...(2.3.53)

Using Lemma 2.3.6 and inequality at (2.3.53) and the fact that $|g(t,x)|\le \max(1,\ c|x|^{-\alpha})$, we have for large n,

$|J_{11}(x)|$

$$\le \Sigma_1 \binom{n}{j} C|x|^{-1} n^{-\gamma} j (C_2/n)^{j-1} {}_{-\infty}\!\int^{\infty} |\alpha_{1,n}(tn^{-\gamma},x)|^{n-j} dt$$

$$(1+C|x|^{-\alpha})$$

$$\le C_1 \; \Sigma_1 \binom{n}{j} C|x|^{-1} n^{-\gamma} j (C_2/n)^{j-1} {}_{-\infty}\!\int^{\infty} |\alpha_{1,n}(tn^{-\gamma},x)|^{n/2} dt$$

$$\le C_1 \; \Sigma_1 \binom{n}{j} C|x|^{-1} n^{-\gamma} j \; (C_2/n)^{j-1} n^{\gamma} {}_{-\infty}\!\int^{\infty} |\alpha_{1,n}(t,x)|^{n/2} dt$$

$$\le C_1 \; \Sigma_1 \binom{n}{j} C|x|^{-1} n^{-\gamma} j (C_2/n)^{j-1} n^{\gamma} {}_{-\infty}\!\int^{\infty} |\alpha_{1,\frac{n}{2}}(t,2^{\gamma}x)|^{n/2} dt,$$

using (2.3.7)

$$\le \Sigma_1 \binom{n}{j} C|x|^{-1} n^{-\gamma} j (C_2/n)^{j-1}$$

$$\le C|x|^{-1} n^{1-\gamma}, \qquad \ldots(2.3.54)$$

the last but one inequality follows as a consequence of (2.3.29).

Now let us consider $J_{12}(x)$:

$J_{12}(x)$

$$= \Sigma_1 \binom{n}{j} {}_{-\infty}\!\int^{\infty} \{\beta_{o,n}(tn^{-\gamma},x)\}^{j}$$

$$[\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-j} - \{\alpha_{o,n}(tn^{-\gamma},x)\}^{n-j}] g(t,x) \exp(-itx) dt$$

$$= \Sigma_1 \binom{n}{j} \int_{|t| \le \varepsilon n^{\gamma}} \{\beta_{o,n}(tn^{-\gamma},x)\}^{j}$$

$$[\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-j} - \{\alpha_{o,n}(tn^{-\gamma},x)\}^{n-j}] g(t,x) \exp(-itx) dt$$

$$+ \Sigma_1 \binom{n}{j} \int_{|t| > \varepsilon n^{\gamma}} \{\beta_{o,n}(tn^{-\gamma},x)\}^{j}$$

$$[\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-j} - \{\alpha_{o,n}(tn^{-\gamma},x)\}^{n-j}] g(t,x) \exp(-itx) dt.$$

$$\ldots(2.3.55)$$

By Lemma 2.3.3,

$$\int_{|t|\leq \varepsilon n^{\gamma}} [\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-j} - \{\alpha_{o,n}(tn^{-\gamma},x)\}^{n-j}] g(t,x) \exp\{-itx\}dt$$

$$= O(n^{1-\gamma}) \qquad \ldots(2.3.56)$$

because $1\leq j\leq [n/2]$. Now we shall prove that the second integral in (2.3.55) is of the order $O(n^{1-\gamma})$.

Note that for each j, $1\leq j\leq [n/2]$, $o\leq m\leq n-j-1$

$$|\alpha_{1,n}(tn^{-\gamma},x)|^{n-j-m-1}|\alpha_{o,n}(tn^{-\gamma},x)|^{m}$$

$$\leq |\alpha_{q,n}(tn^{-\gamma},x)|^{[n/4]} \qquad \ldots(2.3.57)$$

where $q = 0$ ... if $[n/4]+1\leq m\leq n-j-1$

$= 1$ ... if $0\leq m\leq [n/4]$.

And from the integrability of $|w_q(t)|^p$, $p\geq 1$, $q = 0, 1$ and (2.3.31), it follows that for all large n and some fixed $h> 0$, the right hand side of (2.3.57) is integrable in the range $|t|> \varepsilon n^{\gamma}$ and

$$\int_{|t|>\varepsilon n^{\gamma}} |\alpha_{q,n}(tn^{-\gamma},x)|^{[n/4]}dt\leq C\mu^{[n/8]-h}\, n^{\gamma} \qquad ..(2.3.58)$$

where $\mu = \max\{\mu_o,\mu_1\}$; and μ_o and μ_1 being defined as in Lemma 2.3.9.

In view of Lemma 2.3.9, then (2.3.57) and (2.3.58) imply that

$$\left| \int_{|t|>\varepsilon n^{\gamma}} [\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-j}-\{\alpha_{o,n}(tn^{-\gamma},x)\}^{n-j}] g(t,x)\exp\{-itx\}dt\right|$$

$$\leq 2\sum_{m=0}^{n-j-1}\int_{|t|>\varepsilon n^{\gamma}} |\alpha_{1,n}(tn^{-\gamma},x)|^{n-j-m-1}|\alpha_{o,n}(tn^{-\gamma},x)|^{m}dt$$

$$\leq 2\sum_{m=0}^{n-j-1} \int_{|t|>\varepsilon n^{\gamma}} |\alpha_{q,n}(tn^{-\gamma},x)|^{[n/4]}dt$$

$$= O(n^{1-\gamma}). \qquad \ldots(2.3.59)$$

Therefore, using (2.3.56) and (2.3.59) and Lemma 2.3.1, we now have for all $|x| \geq 1$,

$$|J_{12}(x)| \leq Cn^{1-\gamma}\Sigma_1 \binom{n}{j} R_o^{\,j}(|x|n^{\gamma})$$

$$\leq C|x|^{-\alpha}n^{1-\gamma} \qquad \ldots(2.3.60)$$

for all large n.

Using (2.3.52), (2.3.54) and (2.3.60) one gets

$$|J_1(x)| \leq C|x|^{-\alpha}n^{1-\gamma}, \text{ for all large } n. \qquad \ldots(2.3.61)$$

Again because of (2.3.30) and Lemma 2.3.1, we have

$$|J_2(x)| \leq \Sigma_2 \binom{n}{j} [\{R_1(|x|n^{\gamma})\}^j \int_{-\infty}^{\infty} |\alpha_{1,n}(tn^{-\gamma},x)|^{n-j}dt$$

$$+ \{R_o(|x|n^{\gamma})\}^j \int_{-\infty}^{\infty} |\alpha_{o,n}(tn^{-\gamma},x)|^{n-j}dt\}]$$

$$\leq \Sigma_2 \binom{n}{j} [\{R_1(|x|n^{\gamma})\}^j \int_{-\infty}^{\infty} |\alpha_{1,n}(tn^{-\gamma},x)|^{2s}dt$$

$$+ \{R_o(|x|n^{\gamma})\}^j \int_{-\infty}^{\infty} |\alpha_{o,n}(tn^{-\gamma},x)|^{2s}dt\}]n^{\gamma}$$

$$\leq C\Sigma_2 \binom{n}{j} [\{R_1(|x|n^{\gamma})\}^j + \{R_o(|x|n^{\gamma})\}^j]n^{\gamma}$$

$$\leq C|x|^{-\alpha}o(n^{1-\gamma}), \qquad \ldots(2.3.62)$$

for all x with $|x| \geq 1$.

Finally, we consider the estimate for $J_3(x)$. Observe that for $n > 6s$ and all x with $|x| \geq 1$, we have

$$|J_3(x)| \le \Sigma_3 \binom{n}{j} [\{R_1(|x|n^{\gamma})\}^{j-2s} {}_{-\infty}\int^{\infty} |\beta_{1,n}(tn^{-\gamma},x)|^{2s} dt$$

$$+ \{R_o(|x|n^{\gamma})\}^{j-2s} {}_{-\infty}\int^{\infty} |\beta_{o,n}(tn^{-\gamma},x)|^{2s} dt]$$

$$\le \Sigma_3 \binom{n}{j} [\{R_1(|x|n^{\gamma})\}^{j-2s} + \{R_o(|x|n^{\gamma})\}^{j-2s}] n^{\gamma},$$

using (2.3.31)

$$\le C|x|^{-\alpha} n^{6s-n-1}$$

$$\le |x|^{-\alpha} o(n^{1-\gamma}). \qquad \ldots(2.3.63)$$

The lemma now follows immediately from (2.3.61), (2.3.62) and (2.3.63).□

Lemma 2.3.11: Under the assumptions **[A1]** and **[A3]**, we have, for all the values of t, all x with $|x| \ge 1$, and n large,

$$|B_{k,n}(t,x)| \le C|x|^{-\alpha}, \text{ for } k = 0, 1. \qquad \ldots(2.3.64)$$

Proof: Using Lemma 2.3.1, we get

$$|B_{k,n}(t,x)| \le \sum_{h=1}^{n} \binom{n}{h} |\alpha_{k,n}(tn^{-\gamma},x)|^{n-h} |\beta_{k,n}(tn^{-\gamma},x)|^{h}$$

$$\le \sum_{h=1}^{n} n^h \{1-F_k(|x|n^{\gamma})+F_k(-|x|n^{\gamma})\}^h / h!$$

$$\le C|x|^{-\alpha}. \square$$

Lemma 2.3.12: Let $\varepsilon > 0$ and $c > 0$ be as same as in Lemma 2.3.2. Under the assumptions of Lemma 2.3.3, we have

$$|\{w_1(tn^{-\gamma})\}^n - w_o(t)| \le n^{1-([\alpha]+1)\gamma} P(|t|) e^{-c|t|^{\alpha}} \qquad \ldots(2.3.65)$$

for all $(t, n, x) \in \Xi$.

Proof: We prove the result for the case $0< \alpha< 1$; the case $1\le \alpha< 2$ can be similarly proved.

In view of equations (2.3.10) and (2.3.40), we have

$$|\{w_1(tn^{-\gamma})\}^n-w_o(t)|$$

$$=|\{w_1(tn^{-\gamma})\}^n-\{w_o(tn^{-\gamma})\}^n|$$

$$\le |A_{1n}(t,x)-A_{on}(t,x)|+|B_{1n}(t,x)-B_{on}(t,x)|$$

$$\le n^{1-\gamma}P_1(|t|)e^{-c|t|^{\alpha}}+|B_{1n}(t,x)-B_{on}(t,x)|. \qquad \text{...(2.3.66)}$$

Hence, in order to establish (2.3.65), it is enough to prove that

$$|B_{1n}(t,x) - B_{on}(t,x)|\le n^{1-\gamma}e^{-c|t|^{\alpha}}P_2(|t|). \qquad \text{...(2.3.67)}$$

We, therefore, consider

$$|B_{1n}(t,x) - B_{on}(t,x)|$$

$$\le \sum_{m=1}^{n}\binom{n}{m}|\alpha_{1,n}(tn^{-\gamma},x)|^{n-m}|\beta_{1,n}(tn^{-\gamma},x)\}^m-\{\beta_{o,n}(tn^{-\gamma},x)\}^m|$$

$$+\sum_{m=1}^{n}\binom{n}{j}|\beta_{o,n}(tn^{-\gamma},x)|^m$$

$$|\{\alpha_{1,n}(tn^{-\gamma},x)\}^{n-m}-\{\alpha_{o,n}(tn^{-\gamma},x)\}^{n-m}|$$

$$=T_{1n}(x)+T_{2n}(x),\ \text{say.} \qquad \text{...(2.3.68)}$$

We first estimate T_{1n}. Consider the quantity

$$|\{\beta_{1,n}(tn^{-\gamma},x)\}^m - \{\beta_{o,n}(tn^{-\gamma},x)\}^m|$$

$$= |\int_{|u|>|x|n^{\gamma}} e^{itun^{-\gamma}}d(F_1(u)-F_o(u))|$$

$$|\sum_{h=0}^{m-1}\{\beta_{1,n}(tn^{-\gamma},x)\}^{m-h-1}\{\beta_{o,n}(tn^{-\gamma},x)\}^h|$$

$$\leq \{ | \int_{|u|>|x|n^{\gamma}} (e^{itun^{-\gamma}}-1)d(F_1(u)-F_o(u))|$$

$$+| \int_{|u|>|x|n^{\gamma}} d(F_1(u)-F_o(u))| \ \{m(c|x|^{-\alpha}n^{-1})^{m-1}\},$$

using (2.3.1)

$$\leq \{ |t|n^{-\gamma} \int_{|u|>|x|n^{\gamma}} |u||v_1^*(u)-v_o^*(u)|du$$

$$+ \int_{|u|>|x|n^{\gamma}} |v_1^*(u)-v_o^*(u)|du\} \ \{m(c|x|^{-\alpha}n^{-1})^{m-1}\}$$

$$\leq \{ |t|n^{-\gamma} \int_{|u|>|x|n^{\gamma}} |u||v_1^*(u)-v_o^*(u)|du$$

$$+|x|^{-1}n^{-\gamma} \int_{|u|>|x|n^{\gamma}} |u||v_1^*(u)-v_o^*(u)|du\} \ \{m(c|x|^{-\alpha}n^{-1})^{m-1}\}$$

$$\leq P_2(|t|)n^{-\gamma} \ \{m(c|x|^{-\alpha}n^{-1})^{m-1}\}. \qquad \ldots(2.3.69)$$

Last inequality follows from the assumption **[A2]**.

Therefore, using (2.3.11) we get form (2.3.69),

$$T_{1n} \leq \sum_{m=1}^{n} \binom{n}{m} \ \{Ce^{-c|t|^{\alpha}}\}^{(1-m/n)} \ P(|t|)n^{-\gamma}\{m(c|x|^{-\alpha}n^{-1})^{m-1}\}$$

$$\leq Ce^{-c|t|^{\alpha}}P_2(|t|)n^{-\gamma}\sum_{m=1}^{n}(n^m/m!)$$

$$C^{-m/n}e^{Cm(|t|/n^{\gamma})^{\alpha}}c^{m-1}|x|^{-\alpha}n^{1-m} \ m$$

$$\leq Ce^{-c|t|^{\alpha}}P_2(|t|)n^{1-\gamma}\sum_{m=1}^{\infty}(Ce^{c\varepsilon^{\alpha}})^{m-1}/((m-1)!)$$

$$\leq e^{-c|t|^{\alpha}}P_2(|t|)n^{1-\gamma}. \qquad \ldots(2.3.70)$$

Now direct use of (2.3.14) of Lemma 2.3.3 and the fact that $|\beta_{on}(tn^{-\gamma},x)| \leq \{1 - F_o(|x|n^{\gamma}) + F_o(|x|n^{\gamma})\}$, give us

$$T_{2n} \leq e^{-c|t|^{\alpha}}P_3(|t|)n^{1-\gamma}. \qquad \ldots(2.3.71)$$

Combining results (2.3.68), (2.3.70) and (2.3.71), we obtain (2.3.67).□

CONCLUDING REMARKS:

In this chapter we have discussed several results under the assumption that the limit law is non-normal stable (and strictly stable in case of $0< \alpha< 1$). These will be used with slight modifications in Chapters 5 and 6. It was assumed that the d.f. F_1 of the summands X_j is in the domain of normal attraction of stable law here. We shall be relaxing the condition of normal attraction in Chapter 4 (but consider some special cases). In Chapter 3, we shall discuss the case where the limit law is normal.