CHAPTER-4

CONTROLLABILITY OF NONLINEAR INTEGRO-DIFFERENTIAL THIRD ORDER DISPERSION SYSTEM

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4.1 Introduction

The problem of controllability of nonlinear systems described by ordinary differential equation in finite and infinite dimensional spaces has been studied by several authors. Naito and Seidman [106] studied the invarience of approximately reachable set under nonlinear perturbation whereas Yamamoto and Park [136] discussed the controllability of parabolic systems with uniformly bounded nonlinear terms. Recently, Balachandran and Sakthivel ([26],[25]) studied the controllability of integro-differential systems, also with delay term. Here our aim is to study the controllability of the nonlinear integro-differential third-order dispersion system by using Schaefer fixed point theorem (refer [123]). The semilinear partial integro-differential system considered here in the abstract formulation arises in some of the applicable fields of engineering. This work extends the work done in Chapter 3.

4.2 Hypothesis and Formulation of problem

Consider the nonlinear third-order dispersion equation

$$\frac{\partial w(x,t)}{\partial t}+\frac{\partial^3 w(x,t)}{\partial x^3}=(Gu)(x,t)+f\Big(t,w(x,t),\int_0^t g(t,s,w(x,s))ds\Big) \qquad (4.2.1)$$

On the domain $t\geq 0$, $t\in[0,b]=J$, $0\leq x\leq 2\pi$, with periodic boundary conditions.

$$\frac{\partial^k w(0,t)}{\partial x^k}=\frac{\partial^k w(2\pi,t)}{\partial x^k}, k=0,1,2. \qquad (4.2.2)$$

and initial condition

$$w(x,0)=0 \qquad (4.2.3)$$

where

$$(Gu)(x,t)=g_1(x)\Big\{u(x,t)-\int_0^{2\pi} g_1(s)u(s,t)ds\Big\} \qquad (4.2.4)$$

where $g_1(s)$ is a piece-wise continuous non-negative function on $[0,2\pi]$ such that

$$[g_1]\stackrel{\text{def}}{=}\int_0^{2\pi} g_1(x)dx=1$$

and
$f:J\times R\times R\to R$ is a continuous nonlinear function. Here the state $w(.,t)$ takes values in a Banach space X with the norm $\|.\|$ and the control function $u(.)$ is given in $L^2(J,U)$, a Banach space of admissible control functions, with $U=L^2(0,2\pi)$ as a Banach space, $g:\Delta\times X\to X$, $f:J\times X\times X\to X$ are nonlinear functions, where $\Delta=\Big\{(t,s);0\leq s\leq t\leq b\Big\}$ and $t\in J=[0,b]$.
Define an operator A with domain $D(A)$ given by

$$D(A)=\Big\{w\in H^3(0,2\pi):\frac{\partial^k w(0)}{\partial x^k}=\frac{\partial^k w(2\pi)}{\partial x^k};k=0,1,2.\Big\}$$

such that

$$Aw=-\frac{\partial^3 w}{\partial x^3} \qquad (4.2.5)$$

It follows from Lemma 8.5.2 and Korteweg-de Vries equation of Pazy[114] that A is the infinitesimal generator of a Co-group of isometry on U. Let $\{\Phi(t)\},t\geq 0$, be the Co-group generated by A. Obviously, one can show for all $w\in D(A)$,

$$\langle Aw,w\rangle_{L^2(0,2\pi)}=0 \qquad (4.2.6)$$

Also, there exists a constant $M>0$ such that

$$\text{Sup}\Big\{\|\Phi(t)\|:t\in[0,b]\Big\}\leq M \qquad (4.2.7)$$

We know that system (4.2.1) is controllable on the interval $J = [0, b]$ if for every $w_1 \in X = L^2(0, 2\pi)$ there exists a control $u \in L^2(0, b; L^2(0, 2\pi)) = L^2(J, U)$ such that the corresponding solution $w(., t)$ of (4.2.1) satisfies $w(., b) = w_1$.

We use the fixed-point theorem due to Schaefer [123] to obtain our main result. We assume the following hypotheses.

1. A is the infinitesimal generator of a Co-group of bounded linear operators $\Phi(t), t \geq 0$ in X. So there exist constants $M_1 \geq 1$ and $\alpha \in R^+$ such that $\|\Phi(t)\| \leq M_1 e^{\alpha t}; t \geq 0$,

2. The linear operator $W : L^2(J, U) \to X$, defined by

$$Wu = \int_0^b \Phi(b - s)(Gu)(x, s)ds$$

   has an inverse operator W^{-1} which takes the values in $L^2(J, U)/\ker W$ and there exists positive constants M_2, M_3 such that $\|G\| \leq M_2$ and $\|W^{-1}\| \leq M_3$.

3. g satisfies caratheodory condition.
   i.e. for each $(t, s) \in \Delta$, $g(t, s, .) : X \to X$ is continuous and for each $x \in X$, $g(., ., x) : \Delta \to X$ is strongly measurable.

4. f satisfies caratheodory condition.
   i.e. for each $t \in J$, $f(t, ., .) : R \times R \to R$ is continuous and for each $x, y \in X$, $f(., x, y) : J \to X$ is strongly measurable.

5. For every positive integer k, there exists $h_k \in L^1(X, J)$ such that for *a.e.* $t \in J$

$$sup_{\|w\| \leq k} \left\| f\left(t, w(x, t), \int_0^t g(t, s, w(x, s))ds\right) \right\| \leq h_k(t)$$

6. There exists a continuous function $q : J \to R$ such that

$$\left\| \int_0^t g(t, s, w(x, s))ds \right\| \leq q(t)\Omega(\|w(x, t)\|); \ t \in J, w \in X$$

   where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

7. There exists a continuous function $p : J \to R$ such that

$$\|f(t, w, v)\| \leq p(t)\Omega(\|w(x, t)\|) + \|v\|, \ t \in J, w, v \in X$$

8. $\Phi(t), t > 0$ is compact.

The system (4.2.1) has a mild solution of the following form

$$w(.,t) = \int_0^t \Phi(t-s)(Gu)(.,s)ds + \int_0^t \Phi(t-s)f\Big(s, w(.,s), \int_0^s g(s,\tau,w(.,\tau))\Big)d\tau ds \tag{4.2.8}$$

In order to study the controllability of system (4.2.1), we introduce a parameter $\lambda \in (0,1)$ and consider the following system:

$$\frac{\partial w(x,t)}{\partial t} + \frac{\partial^3 w(x,t)}{\partial x^3} = \lambda(Gu)(x,t) + \lambda f\Big(t, w(x,t), \int_0^t g(t,s,w(x,s))\Big)ds \tag{4.2.9}$$

$$w(x,0) = 0; \qquad \lambda \in (0,1), \quad t \in J$$

For the system (4.2.9), there exists a mild solution of the following form:

$$w(x,t) = \lambda \int_0^t \Phi(t-s)\Big[(Gu)(s) + f\Big(s, w(x,s), \int_0^s g(s,\tau,w(x,\tau))d\tau\Big)\Big]ds$$

4.3 Controllability Result

THEOREM 4.3.1 *If Hypotheses 1-8 are satisfied and also*

$$\int_0^b \widehat{m}(s)ds < \int_0^\infty \frac{ds}{1+s+\Omega(s)}$$

where

$$\widehat{m}(t) = max\Big\{\alpha, M_1 N, M_1[p(t)+q(t)]\Big\}$$

and

$$N = M_2 M_3\left[\|w_1\| + M_1\int_0^b e^{\alpha(b-s)}[p(s)+q(s)]\Omega(s)ds\right],$$

then the system (4.2.1) is controllable on J.

Proof: Consider the space $C = C(J,X)$, the Banach space of all continuous functions from J into X with sup norm.
Using the hypothesis (2) for an arbitrary function $w(.,t)$, define the control

$$u(x,t) = W^{-1}\left[w_1 - \int_0^b \Phi(b-s)f\Big(s, w(x,s), \int_0^s g(s,\tau,w(x,\tau))d\tau\Big)ds\right](t)$$

We shall now show that when using this control, operator $F : C \to C$ defined by

$$(Fw)(x,t) = \int_0^t \Phi(t-s)\left[(Gu)(x,s) + f\Big(s, w(x,s), \int_0^s g(s,\tau,w(x,\tau))d\tau\Big)\right]ds$$

has a fixed point. This fixed point is then a solution of equation (4.2.8). Clearly $(Fw)(x, b) = w_1$, which means that the control u steers the integro-differential system from the initial state to final state w_1 in time b, provided we can obtain a fixed point of the nonlinear operator F.

We shall prove that the operator $F : C \to C$ defined by

$$(Fw)(x,t) = \int_0^t \Phi(t-\eta)Q(\eta)d\eta,$$

where

$$Q(\eta) = GW^{-1}\left[w_1 - \int_0^b \Phi(b-s)f\Big(s, w(x,s), \int_0^s g(s,\tau,w(x,\tau))d\tau\Big)ds\right](\eta)$$
$$+f\left(\eta, w(x,\eta), \int_0^\eta g(\eta,\tau,w(x,\tau))d\tau\right)$$

is completely continuous operator.
Obviously,

$$\|Q(\eta)\| \le M_2 M_3\left[\|w_1\| + M_1\int_0^b e^{\alpha(b-s)}h_k(s)ds\right] + h_k(\eta)$$

Step-1 F maps B_k into equicontinuous family.

Let $B_k = \Big\{w \in C : \|w(x,t)\| \le k\Big\}$ for some $k \ge 1$.

We first show that F maps B_k into an equicontinuous family. Let $w(.,t) \in B_k$ and $t_1, t_2 \in J$ with $\epsilon > 0$. Then if, $0 < \epsilon < t_1 < t_2 \le b$,

$$\begin{aligned}\|(Fw)(x,t_1) - (Fw)(x,t_2)\| \le\; & \int_0^{t_1-\epsilon} \|\Phi(t_1-\eta) - \Phi(t_2-\eta)\|\|Q(\eta)\|d\eta \\ & + \int_{t_1-\epsilon}^{t_1} \|\Phi(t_1-\eta) - \Phi(t_2-\eta)\|\|Q(\eta)\|d\eta \\ & + \int_{t_1}^{t_2} \|\Phi(t_2-\eta)\|\|Q(\eta)\|d\eta\end{aligned}$$

The right hand side is independent of $w(.,t)$ and tends to zero as $(t_2 - t_1) \to 0$ for sufficiently small ϵ, since the compactness of $\Phi(t), t > 0$, implies the equi-continuity in the uniform operator topology. Thus F maps B_k in to an equicontinuous family of functions. It is easy to see that the family FB_k is uniformly bounded.

Step-2 $\overline{FB_k}$ is compact.

Since we have shown FB_k is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that F maps B_k into a precompact set in X.

Let $0 < t \le b$ be fixed and ϵ be a real number satisfying $0 < \epsilon < t$. For $w(.,t) \in B_k$, we define

$$\begin{aligned}(F_\epsilon w)(x,t) &= \int_0^{t-\epsilon} \Phi(t-\eta)Q(\eta)d\eta \\ &= \Phi(\epsilon)\int_0^{t-\epsilon} \Phi(t-\eta-\epsilon)Q(\eta)d\eta\end{aligned}$$

Since $\Phi(t)$ is a compact operator, the set $Y_\epsilon(t) = \Big\{(F_\epsilon w)(x,t) : w(.,t) \in B_k\Big\}$ is precompact in X for every $\epsilon, 0 < \epsilon < t$. Moreover, for every $w(.,t) \in B_k$ we have

$$\begin{aligned}\|(Fw)(x,t) - (F_\epsilon w)(x,t)\| &\le \int_{t-\epsilon}^{t} \|\Phi(t-\eta)\|\Big\{M_2M_3\Big[\|x_1\| \\ &\quad + M_1\int_0^b e^{\alpha(b-s)}h_k(s)ds\Big] + h_k(\eta)\Big\}d\eta\end{aligned}$$

Therefore there are precompact sets arbitrary close to the set $\Big\{(Fw)(x,t) : w(.,t) \in B_k\Big\}$. Hence, the set $\Big\{(Fw)(x,t) : w(.,t) \in B_k\Big\}$ is precompact in X.

Step-3 $F : C \to C$ is continuous.

Let $\{w_n\}_{n=0}^{\infty} \subseteq C$ with $w_n \to w$ in C. Then, $\exists$ an integer r such that $\|w_n(x,t)\| \le r$ for all n and $t \in J$, so $w_n \in B_r$ and $w \in B_r$.
For each $t \in J$ and by using assumptions (4) and (5),

$$\|f\Big(t, w_n(x,t), \int_0^t g(t,s,w_n(x,s))ds\Big) - f\Big(t, w(x,t), \int_0^t g(t,s,w(x,s))ds\Big)\| \le 2h_r(t)$$

This implies that

$$f\Big(t, w_n(x,t), \int_0^t g(t,s,w_n(x,s))ds\Big) \to f\Big(t, w(x,t), \int_0^t g(x,s,w(x,s))ds\Big)$$

Therefore, we have by dominated convergence theorem,

$$\begin{aligned}\|Fw_n - Fw\| &\le \int_0^b \|\Phi(t-\eta)\|M_2M_3 \\ &\quad \times\Big[M_1\int_0^b e^{w_1(b-s)}\|f\Big(s, w_{\tilde{n}}(x,s), \int_0^s g(s,\tau,w_n(x,\tau))d\tau\Big)\end{aligned}$$

$$-f\Big(s, w(x,s), \int_0^s g(s,\tau,w(x,\tau))d\tau\Big)\|\Big]d\eta$$
$$+\int_0^b \|\Phi(t-s)\|\|f\big(s, w_n(x,s), \int_0^s g(s,\tau,w_n(x,\tau))d\tau\big)$$
$$-f\big(s, w(x,s), \int_0^s g(s,\tau,w(x,\tau))d\tau\big)\|ds \to 0 \text{ as } n \to \infty$$

Thus F is continuous.

Step-4 Bound for the solution $w(x,t)$.

Finally we obtain a priori bound for the solution

$$\begin{aligned} w(x,t) &= \lambda \int_0^t \Phi(t-\eta)GW^{-1} \\ &\Big[w_1 - \int_0^b \Phi(b-s)f\big(s, w(x,s), \int_0^s g(s,\tau,w(x,\tau))d\tau\big)ds\Big](\eta)d\eta \\ &+\lambda \int_0^t \Phi(t-s)f\big(s, w(x,s), \int_0^s g(s,\tau,w(x,\tau))d\tau\big)ds. \end{aligned}$$

We have,

$$\begin{aligned} \|w(x,t)\| &\leq \lambda \int_0^t \|\Phi(t-\eta)\|\|G\|\|W^{-1}\|\Big[\|w_1\| + \int_0^b \|\Phi(b-s)\| \\ &\|f\big(s, w(x,s), \int_0^s g(s,\tau,w(x,\tau))d\tau\big)\|ds\Big]d\eta + \lambda \int_0^t \|\Phi(t-s)\| \\ &\|f\big(s, w(x,s), \int_0^s g(s,\tau,w(x,\tau))d\tau\big)\|ds \\ &\leq M_1 \int_0^t e^{\alpha(t-s)} M_2 M_3 \Big[\|w_1\| + M_1 \int_0^b e^{\alpha(b-s)}[p(s)+q(s)] \\ &\Omega(\|w(x,s)\|)\Big]ds + M_1 \int_0^t e^{\alpha(t-s)}[p(s)+q(s)]\ \Omega(\|w(x,s)\|)ds \\ &\leq M_1 N e^{\alpha t} \int_0^t e^{-\alpha s}ds + M_1 e^{\alpha t} \int_0^t e^{-\alpha s}[p(s)+q(s)]\ \Omega(\|w(x,s)\|)ds \end{aligned}$$

Thus we have

$$e^{-\alpha t}\|w(x,t)\| \leq M_1 N \int_0^t e^{-\alpha s}ds + M_1 \int_0^t e^{-\alpha s}[p(s)+q(s)]\ \Omega(\|w(x,s)\|)ds$$

Denoting the right hand side of above inequality by $v(t)$, we have

$$\begin{aligned}
v(0) &= 0 \text{ and} \|w(x,t)\| \leq e^{\alpha t} v(t) \\
v'(t) &\leq M_1 N e^{-\alpha t} + M_1 e^{-\alpha t}[p(t)+q(t)]\Omega(\|w(x,t)\|) \\
\Rightarrow e^{\alpha t} v'(t) &\leq M_1 N + M_1[p(t)+q(t)]\Omega(\|w(x,t)\|) \\
\Rightarrow e^{\alpha t} v'(t) &\leq M_1 N + M_1[p(t)+q(t)]\Omega(e^{\alpha t} v(t))
\end{aligned}$$

Also,

$$\begin{aligned}
v(t) &= M_1 N \int_0^t e^{-\alpha s} ds + M_1 \int_0^t e^{-\alpha s}[p(s)+q(s)]\ \Omega(\|w(x,s)\|) ds \\
\Rightarrow e^{\alpha t} v(t) &= M_1 N e^{\alpha t} \int_0^t e^{-\alpha s} ds + M_1 e^{\alpha t} \int_0^t e^{-\alpha s}[p(s)+q(s)]\ \Omega(\|w(x,s)\|) ds
\end{aligned}$$

and

$$\begin{aligned}
(e^{\alpha t} v(t))' &\leq M_1 N\Big[1 + \alpha e^{\alpha t} \int_0^t e^{-\alpha s} ds\Big] + M_1\Big[\alpha e^{\alpha t} \int_0^t e^{-\alpha s}[p(s)+q(s)] \\
&\quad \Omega(\|w(x,s)\|) ds\Big] M_1[p(t)+q(t)]\ \Omega(\|w(x,t)\|) \\
&\leq M_1 N + \alpha e^{\alpha t}\Bigg[M_1 N \int_0^t e^{-\alpha s} ds + M_1 \int_0^t e^{-\alpha s}[p(s)+q(s)] \\
&\quad \Omega(\|w(x,s)\|) ds\Bigg] + M_1[p(t)+q(t)]\ \Omega(\|w(x,t)\|) \\
&\leq M_1 N + \alpha e^{\alpha t} v(t) + M_1[p(t)+q(t)]\ \Omega(\|w(x,t)\|) \\
&\leq \hat{m}(t) + \hat{m}(t) e^{\alpha t} v(t) + \hat{m}(t)\Omega(\|w(x,t)\|); \\
&\quad \text{where } \hat{m}(t) = \max\{M_1 N, \alpha, M_1[p(t)+q(t)]\} \\
&= \hat{m}(t)\big[1 + e^{\alpha t} v(t) + \Omega(\|w(x,t)\|)\big] \\
&\leq \hat{m}(t)\big[1 + e^{\alpha t} v(t) + \Omega(e^{\alpha t} v(t))\big]
\end{aligned}$$

This implies that

$$\int_0^b \frac{ds}{1+s+\Omega(s)} \leq \int_0^b \hat{m}(s) ds < \int_0^\infty \frac{ds}{1+s+\Omega(s)}; \quad t \in J$$

This inequality implies that $e^{\alpha t}v(t) < \infty$. So there *exists* a constant K such that $v(t) \leq K$ and hence $\|w(x,t)\| \leq K, t \in J$, where K depends only on b and on the functions $\hat{m}$ and Ω.

This completes the proof that F is completely continuous.

Finally, the set $\xi(F) = \Big\{w(.,t) \in C : w(x,.) = \lambda F w(.,t), \lambda \in (0,1)\Big\}$ is bounded, as proved above. Therefore by applying Schaefer's theorem, the operator F has a fixed point in C. This means that any fixed point of F is a mild solution of the system (4.2.1) on J satisfying $(Fw)(x,t) = w(x,t)$. Thus, the system (4.2.1) is controllable on J. This completes the proof. ∎