CHAPTER-6

EXACT CONTROLLABILITY OF GENERALIZED HAMMERSTEIN TYPE INTEGRAL EQUATION WITH APPLICATIONS

Chapter 6

EXACT CONTROLLABILITY OF GENERALIZED HAMMERSTEIN TYPE INTEGRAL EQUATION WITH APPLICATIONS

6.1 Introduction

Let X and V be Hilbert spaces and $I = [0, T]$, where $0 < T < \infty$. Let $Y = L^2(0, T; X)$ be the solution space and $U = L^2(0, T; V)$ be the control function space. We consider the following nonlinear control problem:

$$x(t) = \int_0^t h(t,s)u(s)ds + \int_0^t k(t,s,x)\, f(s, x(s))ds; \quad 0 \leq t \leq T < \infty. \tag{6.1.1}$$

Here, the state of the system $x(t) \in X$ and $u(t) \in V$ is the control at time t. The nonlinear function $f : I \times X \mapsto X$ and for each $t, s \in I, x \in Y$, the kernel $k(t, s, x) : X \mapsto X$ and $h(t, s) : V \mapsto X$ are bounded linear operators.

REMARK 6.1.1 *Note that in equation (6.1.1), the kernel k depends on the whole function x, but not depends on pointwise. That is, the system has to be treated separately if we consider the kernel $k(t, s, x(s))$.*

REMARK 6.1.2 *The equation (6.1.1) satisfies the initial condition $x(0) = 0 \in X$, but one can incorporate any initial state $x(0) = x_0$ which will not alter the results.*

We know that the system (6.1.1)is **exactly controllable** over the interval $[0, T]$, if for any given $x_1 \in X$, there exists a control $u \in U$ such that the corresponding solution x of (6.1.1) satisfies $x(T) = x_1$.

A large amount of literature is available regarding the existence and uniqueness of the above type of equation as well as related equations. See, Petry [115], Stuart [130], Leggett [94], Backwinkel-Schilling [8], Srikanth-Joshi [128] to name a few and the references therein.

The corresponding linear control system

$$x(t) = \int_0^t h(t, s)u(s)ds; \quad 0 \leq t \leq T < \infty, \tag{6.1.2}$$

is quite standard and one can give various conditions to ensure the exact controllability of the linear system (6.1.2). Throughout the chapter, we assume that the linear system is exactly controllable.

The exact controllability of related nonlinear systems are also available. See, for example, [9], [25], [26] and for approximate controllability of non-autonomous semilinear system [60]. In [79], Joshi- George established the exact controllability for nonlinear systems in finite dimensional settings, using the monotone operator theory and fixed point theorems. Our aim in this chapter is to generalize these results to infinite dimensional systems. Here, we will present some abstract results along with some useful corollaries. Numbers of well-known models of dynamical control systems can be represented in a above frame work. The application of abstract results to specific examples both from ordinary and partial differential equations are discussed.

The layout of the chapter is as follows. In Section 6.2, we give main assumptions on system components and some preliminary estimates of system operators. We transform the controllability problem to that of a solvability problem. An operator W corresponding to the linear system will be introduced and controllability depends on the compactness of this operator. We prove the compactness under various sufficient conditions in Section 6.3. In Section 6.4, we establish the exact controllability result. Finally in Section 6.5, we demonstrate some applications to illustrate our theory.

6.2 Assumptions and Estimates

Here we provide some sets of sufficient conditions which give guarantee the existence of the solution operator W, and study its behaviour under various assumptions.

Define the following operators

(2.1) for $x \in Y$, $K(x) : Y \mapsto Y$ by

$$(K(x)y)(t) = \int_0^t k(t,s,x)y(s)ds,$$

(2.2) $H : U \mapsto Y$ by $(Hu)(t) = \int_0^t h(t,s)u(s)ds$,

(2.3) $N : Y \mapsto Y$ by $(Nx)(t) = f(t, x(t))$ and

(2.4) $W : U \mapsto Y$ by $Wu = f(.,x(.))$, where $x(.)$ is the solution of (6.1.1) corresponding to $u \in U$.

First, we reduce the controllability problem to a solvability problem. The results on solvability crucially depend on the compactness of W. We make the following assumptions.

Assumptions [A]

$[A_1]$ $\{\int_0^T \int_0^t \|k(t,s,x)\|^2 ds\, dt\}^{\frac{1}{2}} \equiv k(x) < k_0 < \infty.$

$[A_2]$ $\{\int_0^T \int_0^t \|h(t,s)\|^2 ds\, dt\}^{\frac{1}{2}} \equiv h_0 < \infty.$

$[A_3]$ The function f satisfies caratheodory conditions. i.e., $t \rightarrow f(t,.)$ is measurable and $x \rightarrow f(.,x)$ is continuous.

$[A_4]$ The function f satisfies the following growth condition:

$$\|f(t,x)\| \leq a_0 \|x\| + b(t),$$

where $a_0 > 0$ is a constant and $b_0(t) \geq 0$ and $b_0 \in L^2(I)$.

LEMMA 6.2.1 [Estimates:] *The operators K, H and N satisfy the following estimates.*

$$\|K(x)y\|_Y \leq k\|y\|_Y \quad \forall x, y \in Y. \tag{6.2.1}$$

$$\|Hu\|_Y \leq h\|u\|_U \quad u \in U. \tag{6.2.2}$$

$$\|Nx\|_Y \leq \sqrt{2}\,(a_0\|x\|_Y + b_0) \quad \forall x \in Y, \tag{6.2.3}$$

where $b_0 = \|b_0\|_{L2(I)}$.

Proof: The estimate (6.2.1) follows from Cauchy-Schwartz inequality as:

$$\begin{aligned}
\|K(x)y\|_Y^2 &= \int_0^T \|((Kx)(y)(t))\|_X^2\,dt \\
&\leq \int_0^T\Big(\int_0^t \|\,k(t,s,x)\|\ \ \|y(s)\|_X\,ds\Big)^2 dt \\
&\leq \int_0^T\Big(\int_0^t \|k(t,s,x)\|^2\,ds\Big)\Big(\int_0^t \|y(s)\|^2\,ds\Big)dt \\
&\leq k_0^2\,\|y\|_Y^2.
\end{aligned}$$

The inequality (6.2.2) follows in a similar fashion. Now

$$\begin{aligned}
\|Nx\|_Y^2 &= \int_0^T \|Nx(t)\|_X^2\,dt = \int_0^T \|f(t,x(t))\|_X^2\,dt \\
&\leq 2\int_0^T [a_0^2\,\|x(t)\|^2 + b_0(t)^2]dt \\
&\leq 2\,[a_0^2\,\|x\|_Y^2 + b_0^2]\ \leq\ 2\,[a_0\,\|x\|_Y + b_0]^2.
\end{aligned}$$

Hence (6.2.3). ■

Operator form of the equation: With the notations as earlier, we may write the equation (6.1.1) as

$$x(t) = (Hu)(t) + (K(x)(Nx))(t) \tag{6.2.4}$$

or, equivalently

$$x = Hu + K(x)(Nx). \tag{6.2.5}$$

The following theorem gives the existence of solution x of (6.2.5) for a given u which can be proved along the lines as in [79].

THEOREM 6.2.2 *[Existence and Uniqueness:] Assume the following:*

[AK1] There exists a constant $\mu > 0$ such that

$$\int_0^T \langle \int_0^t k(t,s,x)x(t)ds, x(t)\rangle ds \geq \mu \int_0^T \left\| \int_0^t k(t,s,x)x(t)ds \right\|^2 dt \ \ \forall x \in Y. \quad (6.2.6)$$

[AF1] The function f is monotone in the sense that

$$< f(t,x) - f(t,y), x - y > \leq 0 \ \ \forall x, y \in X, t \in I. \quad (6.2.7)$$

i.e. f is monotone.

Then, given $u \in U$, there exists a unique solution $x \in Y$ of (6.2.5) and x satisfies a growth condition

$$\|x\|_Y \leq \frac{b_0}{\mu} + \left(\frac{b_0}{\mu} + 1\right) h_0 \|u\|_U. \quad (6.2.8)$$

■

LEMMA 6.2.3 *Under the assumptions [AK1], [AF1] and the assumptions [A], the Nemytskii operator W is well-defined and continuous. Moreover it satisfies the following growth condition:*

$$\|Wu\|_Y \leq \sqrt{2}\left(\frac{b_0}{\mu} + 1\right) a_0 h_0 \ \|u\|_U + \sqrt{2}\left(\frac{1}{\mu} + 1\right) b_0. \quad (6.2.9)$$

The proof follows from the assumptions and estimate (6.2.8). ■

6.3 Compactness of the operator W

We make the following further assumptions in this section to guarantee the compactness of W .

Assumptions [B]

$[B_1]$ There exists $\tilde{k} > 0$ such that

$$\left\| \int_s^t k(t,\tau,x)x(\tau)d\tau \right\|_X \leq \tilde{k}(t-s)\|x\|_Y, \ \ 0 \leq s < t \leq T.$$

$[B_2]$ There exists $\tilde{h} > 0$ such that

$$\left\| \int_s^t h(t,\tau)u(\tau)d\tau \right\|_X \leq \tilde{h}(t-s)\, \|u\|_U\,, \quad 0 \leq s < t \leq T.$$

$[B_3]$ The operators k and h satisfy the uniform continuity in the following sense: Given $\varepsilon > 0$ there exists $h > 0$ small such that

$$\|k(r+h,s,x) - k(r,s,x)\|_{BL(X)} \leq \varepsilon$$

and

$$\|h(r+h,s) - h(r,s)\|_{BL(X)} \leq \varepsilon, \quad 0 \leq r < r+h \leq T.$$

$[B_4]$ There exists a space $\hat{X}$ such that $X \mapsto \hat{X}$ is a compact embedding.

$[B_5]$ Assume that f can be extended to $I \times \hat{X} \mapsto X$ such that f is caratheodory and $x \mapsto f(.,x(.))$ is continuous from $L^2(I;\hat{X}) \mapsto L^2(I;X)$.

THEOREM 6.3.1 *Under the Assumptions [B], the operator W is compact.*

Proof: Let $\{u_n\}$ be a bounded sequence in U. We have to show that $\{Wu_n\} = \{f(.,x_n(.))\}$ has a convergent subsequence. First of all $\{f(.,x_n(.))\}$ is bounded in Y by Lemma 6.2.3. Therefore there exists a constant $M > 0$ such that

$$\int_0^T \|f(t,x_n(t))\|_X^2\, dt \leq M^2,$$

where, x_n is the corresponding solution of u_n. We show that the family $\{x_n(.)\}$ is equicontinuous in $C(I;X)$.

$$x_n(t) = \int_0^t k(t,\tau,x_n) f(\tau, x_n(\tau)) d(\tau) + \int_0^t h(t,\tau)u_n(\tau)d(\tau)$$

Let $t = r + h_0$. We have

$$\begin{aligned} \|x_n(t) - x_n(r)\| \leq &\left\| \int_0^r \{k(t,\tau,x_n) - k(r,\tau,x_n)\}\, f(\tau,x_n(\tau))d\tau \right\| \\ &+ \left\| \int_r^t k(t,\tau,x_n)\, f((\tau,x_n(\tau))d\tau \right\| \\ &+ \left\| \int_0^r \{h(t,\tau) - h(r,\tau)\}u_n(\tau)d\tau \right\| + \left\| \int_r^t h(t,\tau)u_n(\tau)d\tau \right\| \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Now by (B_3) and (B_1) respectively, we get

$$I_1 \le \varepsilon \int_0^r \|f(\tau, x_n(\tau))\|_X d\tau \le \quad \varepsilon\, r^{\frac{1}{2}} M \le \quad \varepsilon\, M\, T^{\frac{1}{2}}$$

and

$$I_2 \le \tilde{k}\, h_0 \,\|f(.,x_n(.))\|_Y .$$

Similarly, I_3 and I_4 can be estimated as

$$I_3 \le \varepsilon\, T^{\frac{1}{2}}\, \|u_n\|_U \text{ and } I_4 \le \tilde{h}\, h_0 \quad \|u_n\|_U .$$

The above estimates shows that $\{x_n(.)\}$ is equicontinuous in $C(I;X)$ as $\|u_n\|$ is bounded. Further, $\{x_n(.)\}$ is also uniformly bounded in $C(I;X)$. Now, using the compact inclusion $X \hookrightarrow \hat{X}$ and applying general form of Arzela-Ascoli theorem [1], we deduce that $\{x_n(.)\}$ is relatively compact in $C(I;\hat{X})$. Thus along a subsequence $\{x_{n_k}\}$ converges in $C(I;\hat{X})$ and so converges in $L^2(0,T;\hat{X})$.

Then from the assumption (B_5), it follows that $f(.,x_{n_k}(.))$ converges in $Y = L^2(0,T;X)$. Thus the operator is compact and the proof is complete. ∎

REMARK 6.3.2 *If $h(t,s)$ is a compact operator, then it is easy to show that W is compact. In such situations, the exact controllability in the whole space may be impossible ([135], [124]) for different conditions to ensure the compactness of W with non-compact $h(t,s)$.*

Also, it is possible to give various more specific conditions under which the operator W is compact.

When W is assumed to be compact, the assumption [AK1] can be weakened by imposing strong monotonicity on f.i.e. by making [AK1] stronger which is shown in the following lemma.

LEMMA 6.3.3 *Assume that*

[AK2] $\int_0^T \langle \int_0^t k(t,s,x)x(s)ds, x(t)\rangle_X dt \ge 0 \qquad \forall x \in Y$

[AF2] There exists a constant $\beta > 0$, such that

$$\langle f(t,x) - f(t,y), x-y\rangle \;\ge \beta \|x-y\|^2$$

[AF3] Assumptions of [B] are satisfied.

Then the operator W is well defined and continuous. Further it satisfies the growth condition

$$\|Wu\| \leq C_0 + C\|u\|_U,$$

where, $C_0 = b_0 + a_0mbTe^{ma_0T}$ and $C = a_0h_0T\ e^{ma_0T}$, with m is a positive constant satisfying

$$\|k(t,s,x)\| \leq m(x) < m \quad \forall t,s \in I$$

Proof : By hypotheses, the operator $K(x)$ and N satisfies the following:

$$\langle K(x)x, x\rangle_Y \geq 0, \qquad \langle Nx - Ny, x - y\rangle \ \geq \beta\|x-y\|^2 \qquad \forall x, y \in Y$$

Also $[AF3]$ implies that $K(x_n)Nx_n$ has a convergent subsequence for every bounded sequence u_n, where x_n is the corresponding solution of u_n. Now the proof follows from and Grownwall's inequality and Lemma 2.2 of ([60]) and then use the similar argument given in the Theorem (6.2.2) and Lemma (6.2.3). ∎

When f is Lipschitz continuous, we have the following lemma giving different conditions to guarantee that W is well defined and Lipschitz continuous. The proof of it follows from [60] and [79].

Let us make the following assumptions on f.

[AF4] $\exists\ \alpha > 0$ such that

$$\|f(t,x) - f(t,y)\| \leq \alpha\|x-y\| \qquad \forall x, y, \in X, t \in I$$

[AF5] $\exists\ \beta > 0$ such that

$$\langle f(t,x) - f(t,y), x - y\rangle \ \leq -\beta\|x-y\|^2 \qquad \forall x, y \in X, t \in I$$

LEMMA 6.3.4 *In each of the following cases, the solution operator W is well defined and Lipschitz continuous.*

[Case(i)]: Assumption $[AF4]$ holds with $k_0\alpha < 1$

[Case(ii)]: Assumption $[AF4]$ and $[AF5]$ hold with $\beta > k_0\alpha^2$

[Case(iii)]: Assumption [AF4] hold with

$$\|k(t,s,x)\| \leq m(x) < m \ \forall t,s \in I, m > 0$$

[Case(iv)]: Assumption [AF4] holds.

Further the Lipschitz constants for W in above cases are, respectively

$$\frac{\alpha k_0 h_0}{1-k_0\alpha}, \qquad \frac{k_0\alpha^3 h_0}{\beta(\beta - k_0\alpha^2)}, \qquad \alpha T h_0 e^{ma_0T}, \qquad \frac{k_0 h_0 \alpha}{1-\varepsilon}; \qquad \varepsilon > 0$$

being an arbitrary small constant. ∎

REMARK 6.3.5 *Here [AF4] is sufficient to prove the existence of W and Lipschitz continuity of W. The additional assumptions only give better estimation on the Lipschitz constant of the solution operator W.*

When f is locally Lipschitz continuous, then also we can show that W is well-defined, shown in the following lemma. The proof follows along the same line as in the proof of the Lemma 2.4 of [60].

LEMMA 6.3.6 *Under the following assumptions, the operator W is well-defined and continuous.*

(i) There exists a constant $\alpha(r)$ such that

$$\|f(t,x) - f(t,y)\| \leq \alpha(r)\|x-y\| \qquad \forall x,y \in X \quad \text{such that} \quad \|x\| \leq r, \|y\| \leq r$$

(ii) There exists $m > 0$ such that $\|k(t,s,x)\| \leq m \qquad \forall t,s \in I$

(iii) f satisfies the growth condition $[A_4]$.

Moreover, W satisfies a growth condition $[A_4]$

$$\|Wu\|_Y \leq (b_0 + a_0 mbTe^{ma_0T}) + a_0h_0Te^{ma_0T}\|u\|_U.$$

Proof : Since, by the local Lipschitz condition, $\exists$ a unique solution to the equation (6.1.1) in a maximal interval $[0, t_{max}]$, $t_{max} \leq t$. If $t_{max} < t$ then $\lim_{t \mapsto t_{max}} \|x(t,s)\|_X = \infty$(refer [131]). In other words, if $\lim_{t \mapsto t_{max}} \|x(t,s)\|_X = \infty$, then $\exists$ a unique solution in the interval $[0, t]$. We have already shown in the proof of Lemma 6.3.3 that $\|x(t,s)\|_X < \infty$ for each u. Thus W is well-defined and the growth condition follows from the proof of Lemma 6.3.3. ∎

We now move on to the exact controllability under the assumption that the operator W is compact.

6.4 Exact Controllability

We first reduce the controllability problem to a solvability problem which in turn imply the conditions for controllability of system (6.1.1). Define the control operator $C : U \mapsto X$ by

$$Cu = \int_0^T h(T,s)\, u(s)\, ds. \tag{6.4.1}$$

The operator C is bounded linear and in fact, is a control operator for the linear system

$$x(t) = \int_0^t h(t,s)\, u(s)\, ds, \quad x(0) = 0 \quad . \tag{6.4.2}$$

Let $N(C) = \{u \in U : \ Cu = 0\}$ be the null space and $Z = [N(C)]^{\perp} = \{u \in U : \langle u, v\rangle = 0 \ \forall\, v \in N(C)\}$.

A bounded linear operator $S : X \mapsto Z$ is a **Steering Operator** if S steers the linear system (6.4.2) from 0 to x_1. In other words, if $u = Sx_1$, $(x_1 \in X)$, then

$$x(T) = \int_0^T h(T,s)(Sx_1)(s)ds = x_1$$

Clearly $CS = I$, identity operator on X. Thus, if there exists a steering operator S, then $u = Sx_1$ acts as a control and the linear system (6.4.2) is controllable. Conversely, if the linear system is controllable, then for any $x_1 \in X$ there exists $u \in U$ such that $Cu = x_1$, i.e., C is onto. Thus, we can define a generalized inverse $C^{\#} = (C|_Z)^{-1} : X \mapsto Z$ and $S = C^{\#}$ will be a steering operator. Thus, one gets the following result.

THEOREM 6.4.1 *The linear system (6.4.2) is exactly controllable if and only if there exists a steering operator.*

∎

Here we note that $C^{\#}Cu = u$ for $\forall u \in z$ and $C^{\#}Cu = v$ for $u \in U$, where v is the projection of u on z.

We now assume the controllability of the linear system and proceed to prove the exact controllability of the nonlinear system. Define an operator $F : Z \mapsto X$ by

$$Fu = \int_0^T k(T, s, x)(Wu)(s)ds,$$

where x is the solution of the system (6.1.1) corresponding to the control u. Let S be the steering operator of the linear system. Let $x_1 \in X$ and $u_0 = Sx_1$ be the control which steers the linear system from 0 to x_1. The exact controllability of (6.1.1) is equivalent to the existence of $u \in Z$ (let x be the corresponding solution (6.1.1)) such that

$$x_1 = x(T) = \int_0^T k(T, s, x)(Wu)(s)ds + \int_0^T h(T, s)u(s)ds.$$

That is

$$x_1 = Fu + Cu.$$

Applying S on both sides, we get

$$u_0 = SFu + u$$

in z, where u_0 is the control, steering the linear system from 0 to x_1.

Thus, the problem of controllability reduces to solvability problem of the operator equation :

$$\begin{cases} \text{Solve } u \in Z \\ (I + SF)u = u_0. \end{cases} \tag{6.4.3}$$

We now state our controllability result. For the sake of generality, we state the theorem by imposing indirect conditions on W and F. The explicit conditions on k, h, f can be given to verify the conditions on W and F. The corollaries follow are direct verfication of the conditions of the main theorem.

THEOREM 6.4.2 *Assume the linear system (6.4.2) is exactly controllable with the steering operator S. Further assume that the operator W is well defined and compact and satisfies*

$$\|SFu\| \leq a_0 \|u\| + b_0, \text{ with } a_0 < 1, \; b_0 \geq 0$$

Then the system (6.4.3) is solvable in Z.

Proof: We look for the solvability of the operator $R : Z \mapsto U$, where

$$Ru = [I + SF]u.$$

Then

$$\begin{aligned}\langle Ru, u\rangle &= \langle u, u\rangle + \langle SFu, u\rangle \\ &\geq \|u\|^2 - a_0 \|u\| - b_0 \|u\|,\end{aligned}$$

which implies

$$\lim_{\|u\| \to \infty} \frac{\langle Ru, u\rangle}{\|u\|} = \infty.$$

Thus, R is coercive operator. Again compactness of W implies that SF is compact.

Now, R is compact perturbation of the identity operator and hence R is of **type (M)**. See [78] for a definition of type(M). Since any coercive operator of type(M) is onto [78], the proof of the theorem is complete. ∎.

COROLLARY 6.4.3 *Assume the linear system is exactly controllable with a steering operator S. Assume the conditions $[AK1]$ and $[AF1]$ and the assumptions $[B]$. Then the nonlinear system (6.1.1) is controllable if*

$$\|S\| k_0(b_0 + \mu)a_0 h_0 < \mu.$$

∎

THEOREM 6.4.4 *Suppose that the system (6.1.1) satisfies followings:*

(1) The linear part is exactly controllable.

(ii) W is well defined and compact.

(iii) *SF is uniformly bounded i.e.$|SFu| \leq C$, for some $C > 0$. Then the system (6.1.1) is exactly controllable.*

Proof: Let R be the operator as defined in the proof of the Theorem (6.4.2). We have

$$\langle Ru, u\rangle > \|u\|^2 - C\|u\|$$
$$\Rightarrow lim_{\|u\|\mapsto\infty}\langle Ru, u\rangle = \infty$$

By following the same argument in the proof of Theorem (6.4.2), we have that R is a coercive operator of type (M) and hence it is onto.
This completes the proof. ∎

In the above result we do not require the Lipschitz continuity of W but we need F to be uniformly bounded. If f is uniformly bounded then it is not hard to show that SF is also uniformly bounded. When f is uniformly bounded, we have the following result which follows as particular case of Theorem (6.4.4).

COROLLARY 6.4.5 *Suppose that the linear system (6.4.2) is exactly controllable (i.e linear part of (6.1.1) is exactly controllable) and the nonlinear term f is uniformly bounded. Further suppose that the assumptions in Theorem (6.2.2), Lemma (6.2.3) and assumption [B] hold true. Then the system (6.1.1) is exactly controllable.*

∎

When f is Lipschitz continuous, we have the following result.

THEOREM 6.4.6 *Suppose that the system (6.1.1) satisfies the following:*

(i) *The linear part is exactly controllable.*

(ii) *There exists $\alpha \in (0,1)$ such that $\|SFu - SFv\| \leq \alpha\|u - v\| \qquad \forall u, v \in Z$.*

Then the system (6.1.1) is exactly controllable. Further, if u_0 is the steering control for the linear system (6.4.2), to steer the system from 0 to x_1; then the control u, *approximated from the following iterative scheme, steers the state of the nonlinear system (6.1.1) from 0 to x_1 in the same time interval $[0,T]$*

$$u^{(n+1)} = u_0 - SFu^{(n)}$$
$$u^{(0)} = u_0.$$

Proof : Since SF is the contraction, the solvability of (6.4.3) and the approximating scheme follow from Banach Contraction Principle (refer [78]). ∎

The following corollary follows from Theorem 6.4.6 using Lemma 6.3.4.

COROLLARY 6.4.7 *Suppose that the linear system (6.4.2) is exactly controllable with steering operator S. Then under each of the following cases the nonlinear system (6.1.1) is exactly controllable.*

[Case (i)]: Assumption $[AF4]$ holds with $k(x)\alpha < 1$ and

$$\alpha k_0 h_0 k_0 \|S\| < (1 - k_0\alpha)$$

[Case(ii)]: Assumption $[AF4]$ and $[AF5]$ holds with $\beta > k_0\alpha^2$ and

$$\|S\|.k_0 k_0 h_0 \alpha^3 < \beta(\beta - k_0\alpha^2)$$

[Case (iii)]: Assumption $[AF4]$ hold with $\|k(t,s,x)\| \leq m \;\; \forall t,s \in I, m > 0$ and

$$\|S\| k_0 k_0 h_0 \alpha e^{m\alpha_0 T} < 1$$

[Case (iv)]: Assumption $[AF4]$ folds with $\|S\|.k_0 k_0 h_0 \alpha < (1-\epsilon)$ where $\epsilon > 0$ being an arbitrary small constant.

Proof: The proof of all the cases follow by using proof of respective cases of the Lemma 6.3.3 and by using [60]. ∎

6.5 Applications

One can put nonlinear evolution systems with internal control in above frame work to study the exact controllability. It is also possible to use the above results to study the exact controllability problems associated with the partial differential equations with boundary controls.

(a) Nonlinear evolution system with internal control

$$\frac{dx}{dt} = A(t)x + B(t)u + f(t,x); 0 < t, \leq T < \infty \tag{6.5.1}$$
$$x(0) = 0$$

where, $A(t)$ is a linear operator for each $t \in [0,T]$, but not necessarily bounded, $B(t)$ is a bounded linear operator and f is a nonlinear operator in a suitable Hilbert space. Let X and U be the state space and space of control functions, respectively. Assume that, for each $t \in [0,T], A(t)$ generates a strongly continuous evolution system $\phi(t,s)$ on X. By using the variation of constant formula, a mild solution of (6.5.1) can be written as (refer [114], pp106)

$$x(t) = \int_0^t \phi(t,s)f(s,x(s))ds + \int_0^t \phi(t,s)B(s)u(s)ds \tag{6.5.2}$$

This equation is in the form of (6.1.1) and can be written in the form

$$u + K(x)Nx = 0, \tag{6.5.3}$$

with $k(t,s,x) = \phi(t,s)$ and $h(t,s) = \phi(t,s)b(s)$. We apply our main result to deduce controllability.

In this case it is easy to show that the linear part of (6.5.1) is exactly controllable if and only if there exists $\lambda > 0$ such that

$$\langle \int_0^T \phi(T,S)B(s)B^*(s)\phi^*(T,S)vds, v\rangle \geq \lambda \|v\|^2 \quad \forall V \in X$$

where $\phi^*(t,s), B^*(s)$ are the adjoint operators of $\phi(t,s)$ and $B(s)$, respectively.

LEMMA 6.5.1 *Under the condition $\langle -A(t)x, x\rangle_X \geq \mu\|x\|^2 \quad \forall x \in D(A(t))$, the reduced form of the assumption [AK1], that is*

$$[AK3] : \int_0^T \langle \int_0^t k(t,s)x(s)ds, x(t)\rangle_X dt \geq \mu \int_0^T \| \int_0^t k(t,s)x(s)ds\|^2 dt, \qquad \forall x \in Y$$

holds good for (6.5.5)

Proof: Let
$$f(t) = \int_0^t \phi(t,s)x(s), \qquad x \in Y \tag{6.5.4}$$
$$\Rightarrow f'(t) = x(t) + A(t)\int_0^t \phi(t,s)x(s)ds.$$
Therefore,
$$\begin{aligned}\int_0^T \langle \int_0^t \phi(t,s)x(s)ds, x(t)\rangle_X dt &= \int_0^T \langle f(t), f'(t) - A(t)\int_0^t \phi(t,s)x(s)ds\rangle dt \\ &= \int_0^T \langle f(t), f'(t)\rangle dt + \int_0^T \langle f(t), -A(t)f(t)\rangle dt \quad (6.5.5)\end{aligned}$$
But,
$$\begin{aligned}\int_0^T \langle f(t), f'(t)\rangle dt &= \langle f(t), f(t)\rangle|_0^T - \int_0^T \langle f'(t), f(t)\rangle dt \\ \Rightarrow \int_0^T \langle f(t), f'(t)\rangle dt &= \frac{1}{2}\|f(t)\|^2 \geq 0\end{aligned}$$
Therefore, R.H.S. of (6.5.5)

$$\begin{aligned}&\geq \int_0^T \langle f(t), -A(t)f(t)\rangle dt \\ &\geq \mu\int_0^T \|f(t)\|^2 \quad \text{(by hypothesis)} \\ &\geq \mu\int_0^T \langle \int_0^t \phi(t,s)x(s)ds, \int_0^t \phi(t,s)x(s)ds\rangle dt\end{aligned}$$

Hence,
$$\int_0^T \langle \int_0^t \phi(t,s)x(s)ds, x(t)\rangle dt \geq \mu\int_0^T \| \int_0^t \phi(t,s)x(s)ds\|^2 dt; \quad \forall x \in Y$$
This completes the proof. ∎

Similarly, one can impose other conditions on $A(t), B(t)$and$f(t,x)$ to verify that the assumptions made on system (6.5.1) are not redundant. Thus by using the main theorem, one can obtain different sets of verifiable conditions for exact controllability of the nonlinear system (6.5.1).

(b) The autonomous parabolic system with boundary control

$$\frac{dx}{dt} = Ax + f(t,x) \qquad \textit{on } [0,T] \times \Omega \tag{6.5.6}$$

$$\beta x = u$$
$$x(0) = 0$$

where, A is an elliptic differential operator (eg. second order or fourth order), f is a nonlinear operator and β is a boundary operator(eg. Dirichlet or Neumann) in some appropriate space. Here u is the boundary control. Ω is a bounded open domain in R^n with boundary $\partial\Omega$. Assume that $D(A)$ includes homogeneous boundary conditions $\beta x = 0$. Let $L^2(\Omega)$ be the state space X and $L^2(\Gamma)$ be the control space V for some choice of $\Gamma \subset \partial\Omega$. Assume that 0 is not an eigen value of A

Define a Green's operator $D : V \mapsto X$ with $Ax = 0, \beta x = u$. Now the standard trace and regularity theory for these elliptic operators implies that $A^\theta D : V \mapsto X$ is bounded for $\theta < 3/4$. Using the variation of parameter formula, solution of (6.5.6) can be written as

$$x(t) = \int_0^t \phi(t-s) f(s, x(s)) ds + \int_0^t \phi(t-s) ADu(s) ds$$

where $\phi(t-s)$ is the strongly continuous semigroup generated by the elliptic operator A. Thus the system (6.5.6) can be represented in the form (6.5.1) with $k(t,s,x) = \phi(t-s)$ and $h(t,s) = \phi(t-s)AD$. Hence we can make the use of the main results of Section 6.4 to obtain controllability criterion for (6.5.6).

(c) Nonlinear Euler-Bernoulli equation with boundary control

$$\frac{\partial^2}{\partial t^2} w(t,y) = \Delta^2 w(t,y) + g(t, w(t,y), w_t(t,y)) \quad \text{in } [0,T] \times \Omega \tag{6.5.7}$$

$$\begin{aligned} w(0,y) &= w_t(0,y) = 0 \quad \text{in } \Omega \\ w|_{\Sigma} &= u_1 \quad \text{in} \sum \equiv [0,T] \times \Gamma \\ \Delta w|_\Sigma &= u_2 \quad \text{in} \Sigma \end{aligned}$$

where Ω is an open and bounded domain of R^n with sufficiently smooth boundary Γ. Here u_1 and u_2 are the boundary controls.

Let $A : L^2(\Omega) \mapsto L^2(\Omega)$ be the positive self-adjoint operator defined by

$$Ah = \Delta^2 h, \text{with } D(A) = \{h \in H^4(\Omega) : h|_\Gamma = \Delta h|_\Gamma = 0\}$$

So that $A^{1/2} h = -\Delta h$ and $Ah = \Delta^2 h$.

Let $X = D(A) \times L^2(\Omega)$, where $D(A) = H^2(\Omega) \cap H_0(\Omega)$. Define Green's operators G_1 and G_2 as follows: $G_1 : H^s(\Gamma) \mapsto H^{s+1/2}(\Omega)$ is continuous such that

$$\begin{aligned} G_1 u_1 &= h \\ \Delta^2 h &= 0 \text{ in } \Omega \\ h &= u_1 \text{ on } \Gamma \\ \Delta h &= 0 \text{ on } \Gamma. \end{aligned}$$

$G_2 : H^s(\Gamma) \mapsto H^{s+5/2}(\Omega)$ is continuous such that

$$\begin{aligned} G_2 u_2 &= y \\ \Delta^2 y &= 0 \text{ in } \Omega \\ y &= 0 \text{ in } \Gamma \\ \Delta y &= u_2 \text{ in } \Gamma. \end{aligned}$$

Define on operator B as follows:

$$B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ A(G_1 u_1 + G_2 u_2) \end{bmatrix}$$

A generates a strongly continuous cosine operator $C(t)$ on $L^2(\Omega)$ with $S(t) = \int_0^t C(\tau)d\tau$. Define on operator A as follows:

$$A = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$$

where $D(A) = D(A) \times D(A^{1/2})$.

A generates a unitary strongly continuous semigroup e^{At} given by

$$e^{At} = \begin{bmatrix} C(t) & S(t) \\ -AS(t) & C(t) \end{bmatrix}$$

Using variation of constant formula, the solution of (6.5.7), can be written in the form (6.5.1), where

$$x(t) = \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$

$$h(t,s)u = e^{A(t-s)}Bu = \begin{bmatrix} S(t-s)A(G_1u_1 + G_2u_2) \\ C(t-s)A(G_1u_1 + G_2u_2) \end{bmatrix}, \quad f(t, x(t)) = \begin{bmatrix} 0 \\ g(t, w, w_t) \end{bmatrix}$$

It is well-known that the linear part is exactly controllable (refer ([91])). Thus by using the main results of Section 6.4, one can obtain verifiable assumptions on g to achieve exact controllability for (6.5 7).

REMARK 6.5.2 *As a particular case of the above example, one can also consider the following nonlinear Euler-Bernoulli equations with boundary control only in* $\Delta w|_{\sum}$

$$\frac{\partial^2}{\partial t^2}w(t,y) = \Delta^2 w(t,y) + g(t, w(t,y), w_t(t,y)) \quad in\ (0,T)\times\Omega \tag{6.5.8}$$

$$\begin{aligned} w(0,y) &= w_t(0,y) = 0 \qquad in\ \Omega \\ w|_{\sum} &= 0 \qquad in\ (0,T)\times\Gamma = \textstyle\sum \\ \Delta w|_{\sum} &= u \qquad in \textstyle\sum \end{aligned}$$

where Ω is an open bounded domain in R^n with sufficiently smooth boundary $\partial\Omega = \Gamma$. Here u is the only boundary control. As in the case of above example, controllability of the linear part is established in Lasiecka and Triggiani [92].

By using the main result in Section 6.4, we can get the verifiable assumptions on g to achieve exact controllability for the system (6.5.8).

REMARK 6.5.3 *We consider the system governed by parabolic initial boundary value problem*

$$\begin{aligned} \frac{\partial}{\partial t}y(t,x) + Ay(t,x) &= u(t,x) + g(t, y(t,x), y_t(t,x)) \ in\ Q = (0,t)\times\Omega, \quad (6.5.9) \\ y(\cdot, x) &= 0 \qquad on \textstyle\sum = (0,T)\times\partial\Omega \\ y(0) &= y_0 \qquad on\ \ \Omega, \end{aligned}$$

where Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$, $y_0 \in H_o^1(\Omega)$ and $u \in L^2(Q)$. Let A be the second order elliptic differential operator given by

$$Ay = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{i,j}(x)\frac{\partial y}{\partial x_j}\Big) + c(x)y$$

with the assumptions that $c \geq 0$ on $\overline{\Omega}$ and the matrix $(a_{ij}(x))$ is symmetric and positive definite.

As an exact controllability problem of linear part of system (6.5.9), Cao and Gunzburger [141] proved that for given function $y_0, \hat{y} \in L^2(\Omega)$, a function $y = y(t,x)$ and a control $u(t,x)$ both defined for $(t,x) \in Q$ such that y, u satisfy (6.5.9) together with $y(T,x) = \hat{y}(x)$ for $x \in \Omega$.

For the nonlinear portion, we can follow the method given in example (c).

(d) Consider the partial functional integro-differential system of the form

$$\begin{aligned} x_t(y,t) &= x_{yy}(y,t) + e^{(t-s)}u(y,t) + \int_0^t (t-s)\{e^{-\int_0^1 \|x(u)\| du}\}p(s,x(y,s))ds; \quad (6.5.10) \\ & \quad 0 < y < 1, t \in I = [0,1] \\ x(0,t) &= x(1,t) = 0, t > 0 \end{aligned}$$

where, $u \in L^2(I,V)$ and $X = L^1[(0,1);R]$.

Let $f(t,w(t))(y) = p(t,w(t,y))$; $0 < y < 1$ and Let $A : X \to X$ be defined by $Aw = w$" with domain $D(A)$ defined as
$D(A) = \{w \in X; w, w'$ are absolutely continuous, $w" \in X, w(0) = w(1) = 0\}$.
Then

$$Aw = \Sigma_{n=1}^{\infty} - n^2(w,w_n)w_n \qquad w \in D(A).$$

where $w_n(s) = \sqrt{2}\ \ sinns, n = 1,2,3,\cdots$ is the orthogonal sets of eigen vectors of A. (w,w_n) is the Fourier expansion of w". Here A is an infinitesimal generator of an analytic semigroup $T(t); t \geq 0$ in X and is given by

$$T(t)w = \Sigma_{n=1}^{\infty} exp(-n^2 t)(w,w_n)w_n; w \in X$$

where $T(t)$ satisfies $|T(t)| \leq M_1 e^{\omega t}; t \geq 0$ for some $M_1 \geq 1, \omega \in R$. Here $h(t,s) = e^{(t-s)}$ and $k(t,s,x) = (t-s)\{e^{-\int_0^1 \|x(u)\| du}.\}$ Further function $p : J \times R \mapsto R$ is continuous, bounded and strongly measurable such that

$$\|p(t,w(t,y))\| \leq a(t)\|w(t,y)\| + b(t); \ a > 0, b(\cdot) = \|b(\cdot)\|_{L^2(I)}.$$

Thus all the conditions of our main theorem are satisfied. Hence system (6.5.10) is exactly controllable on I.