CHAPTER-8

CONTROLLABILITY OF SECOND ORDER SEMI-LINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS IN BANACH SPACES

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8.1 Introduction

Let X be a Banach space with norm $|.|$ and U be another Banach space taking the control values. In this chapter, we would like to consider the controlled neutral functional second order inclusion system with nonlocal conditions

$$\left.\begin{array}{l}\frac{d}{dt}[x'(t) - f(t, x_t)] \in Ax(t) + Bu(t) + F(t, x_t, x'(t)),\ t \in J \\ x_0 = \phi,\ x'(0) = y_0.\end{array}\right\} \qquad (8.1.1)$$

Here the state $x(t)$ takes values in X and the control $u \in L^2(J, U)$, the space of admissible controls, where $J = (0, \infty)$. Further, we assume A is the infinite generator of strongly continuous Cosine family $\{C(t) : t \in R\}$ defined on X and $B : U \to X$ is a bounded linear operator. The map $F : J \times C_r \times X \to 2^X$ is a bounded, closed, convex multi-valued map. Let $r > 0$ be the delay time and $C_r = C([-r, 0], X)$ be the Banach space of all continuous functions with the norm $\|\phi\| = sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}$. Let $J_0 = [-r, 0]$ and non-local condition $\phi \in C_r$ and $y_0 \in X$ be the given initial values. Also for any continuous function x defined on the interval $J_1 = [-r, \infty)$ with values in X and for any $t \in J$, we denote by x_t an element of $C(J_0, X)$ defined by $x_t(\theta) = x(t + \theta), \theta \in J_0$.

Our aim is to study the exact controllability of the above abstract system which will have applications to many interesting systems including PDE systems. We reduce the controllability problem (8.1.1) to the search for fixed points of a suitable multi-valued map on a subspace of the Frechet space $C(J, X)$. In order to prove the existence of fixed points, we shall rely on a theorem due to Ma [98], which is an extension of Schaefer's theorem [123] to multi-valued maps between locally convex topological spaces.

Much attention has been received in recent years regarding the existence of mild, strong and classical solutions for differential and integro-differential equations in abstract spaces with nonlocal conditions. We refer to the paper of Byszewski [41] who studied the existence and uniqueness of solution of semi-linear evolution nonlocal Cauchy problem. Ntouyas and Tsamatos [110] discussed global existence for semi-linear evolution equations with nonlocal conditions.

The controllability of second-order system with local and nonlocal conditions are also very interesting and researchers are engaged in it. Many times, it is advantageous to treat the second-order abstract differential equations directly rather than to convert them to first-order system. For example, refer Fitzgibbon [59] and Ball [27]. In [59], Fitzgibbon used the second-order abstract system for establishing the boundedness of solutions of the equation governing the transverse motion of an extensible beam. A useful tool in the study of abstract second-order equations is the theory of strongly continuous cosine families ([132]- [133]). Balachandran and Marshal Anthoni ([20]-[23]) discussed the controllability of second-order ordinary and delay, differential and integro-differential systems with the proper illustrations, without converting them to first-order by using the cosine operators and Leray Schauder alternative. Quinn and Carmichael [117] have first shown that the controllability problem in Banach spaces can be converted in to a fixed point problem for a single valued map. Benchohra and Ntouyas [30] proved the existence and controllability results for nonlinear differential inclusions with nonlocal conditions. Also they considered controllability of functional differential and integro-differential inclusions in Banach spaces [32]. In both the papers they used a fixed point theorem for the condensing maps due to Martelli. Then they demonstrated the controllability results for multi-valued semi-linear neutral functional equation [33]. Benchohra, Gorniewicz and Ntouyas [29] paid there attention to show the controllability on infinite time horizon for first and second-order functional differential inclusions in Banach spaces. The existence of the system considered in [29] was also proved by them. They used here the fixed point theorem due to Ma [98]. Our intention in this chapter is to study the controllability on infinite time horizon for second-order semi-linear neutral functional differential inclusion in Banach spaces. We consider the multi-valued map which is function of both the delay term as well the derivative of the unknown function. We will take the help of fixed point theorem due to Ma, which is an extension of Schaefer's theorem to locally convex topological spaces, semigroup method [114] and set-valued analysis [53], [74].

The layout of the chapter is as follows. In the following section, we give the necessary preliminaries so that the system can be put in the integral form which gives the existence of a mild solution. In Section 8.3, we represent the state of the system in terms of the Cosine and Sine family and reduce the controllability to that of finding a fixed point of a multi-valued map. We, then establish the existence of a fixed point by applying a fixed point theorem due to Ma [98]. Finally, in Section 8.4, we present an example to illustrate our theory.

8.2 Definitions and Hypotheses

In this section, we introduce notations and preliminary facts from multi-valued analysis which are used throughout this chapter. Let $J_m = [0, m]$, $m \in \mathbb{N}$. The space $C(J, X)$ is the Frechet space of continuous functions from J into X with the metric (see [56])

$$d(x, z) = \sum_{m=0}^{\infty} \frac{2^{-m}\|x - z\|_m}{1 + \|x - z\|_m}, \text{ for each } x, z \in C(J, X)$$

where

$$\|x\|_m := \sup\Big\{|x(t)| : t \in J_m\Big\}$$

Let $B(X)$ be the Banach space of bounded linear operators from X to X with the standard norm. A measurable function $x : J \to X$ is Bochner integrable if and only if $|x|$ is Lebesgue integrable. For properties of the Bochner integral, we refer to [38]. Let $L^1(J, X)$ denotes the Banach space of Bochner integrable functions and Up denotes a neighbourhood of 0 in $C(J,X)$ defined by

$$Up := \left\{x \in C(J,X) : \|x\|_m < p\right\}$$

The convergence in $C(J,X)$ is the uniform convergence in the compact intervals, i.e. $x_j \to x$ in $C(J,X)$ if and only if $\|x_j - x\|_m \to 0$ in $C(J_m,X)$ as $j \to \infty$ for each $m \in \mathrm{N}$. A set $M \subseteq C(J,X)$ is a bounded set if and only if there exists a positive function $\xi \in C(J,R_+)$ such that

$$|x(t)| \leq \xi(t) \text{ for all } t \in J \text{ and } x \in M.$$

The Arzela-Ascoli theorem says that a set $M \subseteq C(J,X)$ is compact if and only if for each $m \in \mathrm{N}$, M is a compact set in the Banach space $(C(J_m,X), \|.\|_m)$.

For the definition and some useful lemmas of strongly continuous cosine operators and for some of the terminology of set valued analysis, we refer Chapter 2.

We assume the following hypotheses:

Let **(H1)**holds, (refer Chapter 2).

(H2) $C(t)$, $t > 0$ is compact.

(H3) $Bu(t)$ is continuous in t and M_2 be constant such that $|B| \leq M_2$.

(H4) Let $m \in \mathrm{N}$ be fixed. Let $W : L^2(J,U) \to X$ be the linear operator defined by

$$Wu = \int_0^m S(m-s)Bu(s)ds$$

Then $W : L^2(J,U)/KerW \to X$ induces a bounded invertible operator W^{-1} and there exists positive constant M_3 such that and $|W^{-1}| \leq M_3$. For construction of W^{-1}, refer [11].

(H5) The function $f : J \times C_r \to X$ is completely continuous and for any bounded set $B \subseteq C(J_1,X)$, the family $\{t \mapsto f(t,x_t) : x \in B\}$ is equi-continuous in $C(J,X)$. Further assume, there exist constants $0 \leq c_1 < 1$ and $c_2 \geq 0$ such that for all $t \in J$, $\phi \in C_r$, we have

$$|f(t,\phi)| \leq c_1\|\phi\| + c_2.$$

(H6) The multi-valued map $(t,\psi,x) \mapsto F(t,\psi,x)$ is measurable with respect to t for each $\psi \in C_r$ and $x \in X$ and F is u.s.c. with respect to second and third variable for each $t \in J$. Moreover for each fixed $z \in C(J_1,X)$ and $x \in C(J,X)$, the set

$$S_{F,z,x} = \{v \in L^1(J,X) : v(t) \in F(t,z_t,x(t)) \text{ for a.e. } t \in J\}$$

is nonempty.

(H7) We assume F satisfies the following estimate. Given $\psi \in C_r$ and $x \in X$, there exist $p \in L^1(J, R_+)$

$$\|F(t,\psi,x)\| := \sup\{|v| : v \in F(t,\psi,x)\} \leq p(t)\Psi(\|\psi\| + |x|),$$

where $\Psi : R_+ \to (0,\infty)$ is continuous and increasing and there is a $c > 0$ such that the integral $\int_c^\infty \frac{ds}{s+\Psi(s)}$ is sufficiently large (an explicit lower bound and expression for c can be given). For example one can take Ψ such that

$$\int_c^\infty \frac{ds}{s+\Psi(s)} = \infty.$$

(H8) For $z \in C(J_1, X)$ and $x \in C(J, X)$ varies in a neighborhood of 0 and $t \in J$, the set

$$\Big\{C(t)\phi(0) + S(t)[y_0 - f(0,\phi)] \int_0^t C(t-s)f(s,x_s)ds$$
$$+ \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)v(s)ds;\ v \in S_{F,z,x}\Big\}$$

is relatively compact.

Then the integral equation formulation of the system (8.1.1) can be written as [110]

$$\left.\begin{array}{rcl} x(t) &=& \phi(t),\ t \in J_0 \\ x(t) &=& C(t)\phi(0) + S(t)[y_0 - f(0,\phi)] + \int_0^t C(t-s)f(s,x_s)ds \\ && + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)v(s)ds,\ t \in J, \end{array}\right\} \tag{8.2.1}$$

where $v \in S_{F,x,x'} = \{v \in L^1(J,X) : v(t) \in F(t, x_t, x'(t))$ for a.e. $t \in J\}$ is called the mild solution on J of the inclusion (8.1.1).

REMARK 8.2.1 *If dim $X < \infty$ and J is a compact real interval, then $S_{F,x,x'} \neq \phi$ (see [93]).*

We note that the system (8.1.1) is said to be controllable on J if for every $\phi \in C_r$ with $\phi(0) \in D(A)$, $y_0 \in X_1$, $x_1 \in X$ and for each m, there exists a control $u \in L^2(J_m, U)$ such that the solution $x(.)$ of (8.1.1) satisfies $x(m) = x_1$.
where,
$D(A) = \{x \in X : C(t)x$ is twice continuously differentiable in $t\}$.

i.e.

$$D(A) = \{x \in X : C(.)x \in C^2(R, X)\}$$

and $X_1 = \{x \in X : C(t)x$ is once continuously differentiable in $t\}$.

The following lemmas are crucial in the proof of our main theorem.

LEMMA 8.2.2 *([93]) Let $I = J_m$ be the compact real interval and X be a Banach space. Let F be a multi-valued map satisfying (H6) and let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$, then the operator*

$$\Gamma o S_F : C(I,X) \to BCC(C(I,X)) \text{ defined by } x \to (\Gamma o S_F)(x) := \Gamma(S_{F,x_t,x'})$$

is a closed graph operator.

8.3 Controllability Result

We now state and prove the main controllability result.

THEOREM 8.3.1 *Assume that the hypotheses $(H1)-(H8)$ are satisfied and system (8.1.1) is controllable for all y_0 and x_1 satisfying $(H1)-(H8)$. Then the system (8.1.1) is controllable on J.*

Proof: Fix $m \in N$. Consider the space

$$Z = \Big\{x \in C([-r, m], X) : x|_{[0,m]} \in C^1([0, m], X)\Big\}$$

with the norm

$$\|x\|_Z = \max\{\|x\|_{C([-r,m],X)}, \|x'\|_{C^1([0,m],X)}\}.$$

Using the hypothesis (H4) for $x \in Z$, we define the control formally as

$$u(t) = W^{-1}\Bigg[x_1 - C(m)\phi(0) - S(m)[y(0) - f(0,\phi)] - \int_0^m C(m-s)f(s,x_s)ds \\ - \int_0^m S(m-s)v(s)ds\Bigg](t) \tag{8.3.1}$$

Using the above control, define a multi-valued map $N_1 : Z \to 2^Z$ by

$$(N_1 x)(t) = \phi(t) \text{ for } -r \le t \le 0$$

and for $m \ge t \ge 0$

$$N_1 x := \{h \in C(J, X) : h \text{ satisfies } (8.3.2)\},$$

where h is given by

$$\begin{aligned} h(t) = C(t)\phi(0) + S(t)[y_0 - f(0,\phi)] + \textstyle\int_0^t C(t-s) f(s, x_s) ds \\ + \textstyle\int_0^t S(t-s) v(s) ds + \int_0^t S(t-\eta) Bu(\eta) d\eta. \end{aligned} \tag{8.3.2}$$

Here u is defined as in (8.3.1) and $v \in S_{F,x_t,x'}$. Our aim is to prove the existence of a fixed point for N_1. This fixed point will then be a solution of equation (8.2.1). Clearly $(N_1 x)(m) = x_1$ which means that the control u steers the system from initial state x_0 to x_1 in time m, provided we obtain a fixed point of the nonlinear operator N_1.

In order to obtain the fixed point of N_1, we need to verify the various conditions in Lemma 2.2.12.

Step 1: The set $\Omega := \{x \in Z : \lambda x \in N_1(x), \lambda > 1\}$ is bounded. To see this, let $x \in \Omega$. Then x has the representation for $t \ge 0$

$$\begin{aligned} x(t) = \lambda^{-1} h(t) = \lambda^{-1} C(t)\phi(0) + \lambda^{-1} S(t)[y_0 - f(0,\phi)] + \lambda^{-1} \textstyle\int_0^t C(t-s) f(s, x_s) ds \\ + \lambda^{-1} \textstyle\int_0^t S(t-s) v(s) ds + \lambda^{-1} \int_0^t S(t-\eta) Bu(\eta) d\eta. \end{aligned} \tag{8.3.3}$$

where u is defined as in (8.3.1). It is, then easy to observe that x is a mild solution of the system

$$\frac{d}{dt}[x'(t) - \lambda^{-1} f(t, x_t)] \in \lambda^{-1} A x(t) + \lambda^{-1} Bu(t) + \lambda^{-1} F(t, x_t, x'(t)), \quad t \in J. \tag{8.3.4}$$

Thus we have to obtain bounds on x and x' independent of $\lambda > 1$ which will prove the boundedness of Ω.

Using the assumptions, it is easy to obtain positive constants C_1, C_2, C_3 depends on the initial values, m and bounds on the Cosine and Sine operators such that

$$|x(t)| \le C_1 + C_2 \int_0^t \|x_s\| ds + C_3 \int_0^t p(s) \Psi(\|x_s\| + |x'(s)|) ds, \text{ for all } -r \le t \le m.$$

Denoting by $v(t)$, the right-hand side of the above inequality, we get

$$\mu(t) \le v(t).$$

Here the function μ is defined by

$$\mu(t) = \sup\{|x(s)| : -r \leq s \leq t\} : -r \leq t \leq m.$$

Further $v(0) = C_1$ and

$$\begin{aligned} v'(t) &\leq C_2\mu(t) + C_3p(t)\psi(\mu(t) + x'(t)) \\ &\leq C_2v(t) + C_3p(t)\Psi(v(t) + |x'(t)|), \ t \in J. \end{aligned}$$

Now

$$\begin{aligned} x'(t) &= \lambda^{-1}AS\phi(0) + \lambda^{-1}C(t)[y_0 - f(0,\phi)] + \lambda^{-1}f(t,x_t) \\ &\quad + \lambda^{-1}\int_0^t AS(t-s)f(s,x_s)ds + \lambda^{-1}\int_0^t C(t-\eta)BW^{-1} \\ &\quad \Big[x_1 - C(m)\phi(0) - S(m)[y_0 - f(0,\phi)] - \int_0^m C(b-s)f(s,x_s)ds \\ &\quad - \int_0^m S(m-s)v(s)ds\Big](\eta)d\eta + \lambda^{-1}\int_0^t C(t-s)v(s)ds, t \in J. \end{aligned}$$

We can estimate x' in a similar fashion. There exist positive constants C_4, C_5, C_6, C_7 such that

$$\begin{aligned} |x'(t)| &\leq C_4 + C_5\|x_t\| + C_6\textstyle\int_0^t \|x_s\|ds + C_7\int_0^t p(s)\Psi(\|x_s\| + |x'(s)|)ds \\ &\leq C_4 + C_5\mu(t) + C_6\textstyle\int_0^t \|x_s\|ds + C_7\int_0^t p(s)\Psi(\|x_s\| + |x'(s)|)ds \\ &\leq C_4 + C_5v(t) + C_6\textstyle\int_0^t \|x_s\|ds + C_7\int_0^t p(s)\Psi(\|x_s\| + |x'(s)|)ds. \end{aligned}$$

Denoting by $r(t)$ the right-hand side of the above inequality , we have

$$\begin{aligned} |x'(t)| &\leq r(t), t \in J \\ r(0) &= C_4 + C_5C_1 \end{aligned}$$

and

$$\begin{aligned} r'(t) &\leq C_5v'(t) + C_6\mu(t) + C_7p(t)\Psi(\mu(t)) + |x'(t)|) \\ &\leq C_5v'(t) + C_6v(t) + C_7p(t)\Psi(v(t) + r(t)) \\ &\leq (C_2C_5 + C_6)v(t) + (C_3C_5 + C_7)p(t)\Psi(v(t) + r(t)), \end{aligned}$$

where the last inequality is obtained from the estimate of $v'(t)$. Let

$$w(t) = v(t) + r(t), \ t \in J.$$

Then

$$c := w(0) = v(0) + r(0) = C_1 + C_4 + C_1C_5$$

and

$$\begin{aligned} w'(t) &= v'(t) + r'(t) \\ &\leq (C_2 + C_2C_5 + C_6)v(t) + (C_3 + C_3C_5 + C_7)p(t)\Psi(v(t) + r(t)) \\ &= (C_2 + C_2C_5 + C_6)w(t) + (C_3 + C_3C_5 + C_7)p(t)\Psi(w(t)) \\ &\leq m(t)[w(t) + \Psi(w(t))], \end{aligned}$$

where $m(t) := \max\{C_2 + C_2C_5 + C_6, C_3 + C_3C_5 + C_7)\}$. This implies that

$$\int_c^{w(t)} \frac{ds}{s + \Psi(s)} = \int_{w(0)}^{w(t)} \frac{ds}{s + \Psi(s)} \leq \int_0^m m(s)ds < \int_c^{\infty} \frac{ds}{s + \Psi(s)},$$

where the last inequality follows from assumption (H7). This implies that there exists a constant L such that

$$w(t) = v(t) + r(t) \leq L,\ t \in J_m.$$

Thus

$$\begin{aligned} \|x(t)\| &\leq v(t) \leq L,\ t \in J_m \\ \|x'(t)\| &\leq r(t) \leq L,\ t \in J_m \end{aligned}$$

and hence Ω is bounded.

Step 2: N_1x is convex for each $x \in Z$.

Indeed, if $h_1, h_2 \in N_1x$ then there exist $v_1, v_2 \in S_{F_1,x_t,x'}$ such that for $i = 1, 2$, we have

$$\begin{aligned} h_i(t) = C(t)\phi(0) + S(t)[y_0 - f(0,\phi)] + \textstyle\int_0^t C(t-s)f(s,x_s)ds \\ + \textstyle\int_0^t S(t-s)v_i(s)ds + \int_0^t S(t-\eta)Bu(\eta)d\eta, \end{aligned}$$

were u is defined as in (8.3.1) with v replaced by v_i. Then it is an easy matter to see that, for $0 \leq k \leq 1$,

$$\begin{aligned} (kh_1 + (1-k)h_2)(t) = C(t)\phi(0) + S(t)[y_0 - f(0,\phi)] + \textstyle\int_0^t C(t-s)f(s,x_s)ds \\ + \textstyle\int_0^t S(t-s)(kv_1 + (1-k)v_2)(s)ds + \int_0^t S(t-\eta)Bu(\eta)d\eta, \end{aligned}$$

where u is defined as in (8.3.1) with $v = kv_1 + (1-k)v_2$.

Since $S_{F,x_t,x'}$ is convex as F is convex, we have $v = kv_1 + (1-k)v_2 \in S_{F,x_t,x'}$ and hence $kh_1 + (1-k)h_2 \in N_1x$.

Step 3: $N_1(U_q)$ is bounded in Z for each $q \in N$, where U_q is a neighborhood of 0 in Z.

We have to show that there exists a positive constant l such that for any $x \in U_q$ and $h \in N_1x$ such that $\|h\|_Z \leq l$. In other words, we have to bound the sup-norm of both h and h'. We can write

$$\begin{aligned} h(t) = C(t)\phi(0) + S(t)[y_0 - f(0,\phi)] + \textstyle\int_0^t C(t-s)f(s,x_s)ds \\ + \textstyle\int_0^t S(t-s)v(s)ds + \int_0^t S(t-\eta)Bu(\eta)d\eta, \end{aligned}$$

and therefore

$$\begin{aligned} h'(t) &= AS(t)\phi(0) + C(t)[y_0 - f(0,\phi)] + f(t,x_t) + \int_0^t AS(t-s)f(s,x_s)ds \\ &\quad + \int_0^t C(t-\eta)BW^{-1}\Big[x_1 - C(m)\phi(0) - S(m)[y_0 - f(0,\phi)] \\ &\quad - \int_0^m C(b-s)f(s,x_s)ds - \int_0^m S(m-s)v(s)ds\Big](\eta)d\eta \\ &\quad + \int_0^t C(t-s)v(s)ds, \end{aligned}$$

where u is defined as in (8.3.1) and $v \in S_{F,x_t,x'}$.

The assumptions will give uniform estimates for v and x which in turn can be used to obtain the required bounds for h and h' for every $x \in U_q$ and $h \in N_1x$.

Step 4: $N_1(U_q)$ is equi-continuous, for each $q \in N$. That is the family $\{h \in N_1x : x \in U_q\}$ is equi-continuous.

Let $U_q = \{x \in Z, \|x\| \leq q\}$ for some $q \geq 1$. Let $x \in U_q$, $h \in N_1x$ and $t_1, t_2 \in J_m$ such that $0 < t_1 < t_2 \leq m$. Then

$$\begin{aligned} &|h(t_1) - h(t_2)| \\ &\quad \leq |[C(t_1) - C(t_2)]\phi(0)| + |[S(t_1) - S(t_2)][y_0 - f(0,\phi)]| \\ &\qquad + \Big|\int_0^{t_1}[C(t_1-s) - C(t_2-s)]f(s,x_s)\Big| + \Big|\int_{t_1}^{t_2} C(t_2-s)f(s,x_s)ds\Big| \\ &\qquad + \Big|\int_0^{t_1}[S(t_1-\eta) - S(t_2-\eta)]BW^{-1}\Big[x_1 - C(m)\phi(0) - S(m)[y_0 - f(0,\phi)] \end{aligned}$$

$$-\int_0^m C(m-s)f(s,x_s)+\int_0^b S(m-s)v(s)\Big](\eta)d\eta|$$

$$+|\int_{t_1}^{t_2} S(t_2-\eta)BW^{-1}\Big[x_1-C(m)\phi(0)\Big]-S(m)[y_0-f(0,\phi)]$$

$$-\int_0^m C(m-s)f(s,x_s)+\int_0^b S(m-s)v(s)\Big](\eta)d\eta|$$

$$+|\int_0^{t_1}[S(t_1-s)-S(t_2-s)]v(s)|+|\int_{t_1}^{t_2} S(t_2-s)v(s)ds|$$

Now using the bounds on x, v and the given assumptions, by a routine calculation, we obtain a positive constant $L>0$ such that

$$\begin{aligned}|h(t_1)-h(t_2)| \;\le\; & L\{|C(t_1)-C(t_2)|+|S(t_1)-S(t_2)|\}\\ & +L\left\{\int_0^{t_1}|C(t_1-s)-C(t_2-s)|ds+\int_{t_1}^{t_2}|C(t_2-s)|ds\right\}\\ & +L\left\{\int_0^{t_1}|S(t_1-s)-S(t_2-s)|ds+\int_{t_1}^{t_2}|S(t_2-s)|ds\right\}\end{aligned}$$

In an analogous way, one can also obtain a similar estimate for $|h'(t_1)-h'(t_2)|$

Note that $C(t)$ and $S(t)$ are uniformly continuous in the uniform operator topology. Thus the above estimates implies the required equi-continuity. This also proves the relative compactness of $N_1(Uq)$. Now it remains to prove the u.s.c of N_1. By our discussion in Section 8.1, it is enough to prove that N_1 has a closed graph. We do this in the next step using Lemma 8.2.2.

Step 5: Let $h_n\in N_1x_n$ and $h_n\longrightarrow h^*$, $x_n\longrightarrow x^*$. We must show that $h^*\in N_1x^*$. By definition, there exists $v_n\in S_{F,x_{nt},x'_n}$ such that

$$\begin{aligned}h_n(t)&=C(t)\phi(0)+S(t)[x_0-f(0,\phi)]+\textstyle\int_0^t C(t-s)f(s,x_{ns})ds\\ &\quad+\textstyle\int_0^t S(t-s)v_n(s)ds+\int_0^t S(t-\eta)Bu_n(\eta)d\eta,\end{aligned}\tag{8.3.5}$$

where u_n is defined as in (8.3.1), where x is replaced by x_n. The difficulty is that we do not have the convergence of v_n and hence that of u_n. In fact, we cannot expect the convergence of v_n and the existence of v^* (to be defined later) has to be achieved by a suitable selection. First, we separate the part of v_n from u_n. Write $u_n=\bar{u}_n+\tilde{u}_n$, where

$$\bar{u}_n(t)=W^{-1}\left[x_1-C(m)\phi(0)-S(m)[y(0)-f(0,\phi)]-\int_0^m C(m-s)f(s,x_{ns})\right](t)$$

and

$$\bar{u}_n(t) = -W^{-1}\left[\int_0^m S(m-s)v_n(s)ds\right](t).$$

Thus, we get from (8.3.5) that

$$\begin{aligned}\bar{h}_n(t) : &= h_n(t) - C(t)\phi(0) - S(t)[y_0 - f(0,\phi)] - \int_0^t C(t-s)f(s,x_{ns})ds \\ &\quad - \int_0^t S(t-\eta)B\bar{u}_n(\eta)]d\eta \\ &= \int_0^t S(t-\eta)B\tilde{u}_n(\eta)d\eta + \int_0^t S(t-s)v_n(s)ds. \qquad (8.3.6)\end{aligned}$$

Note that the LHS of the above equation do not contain v_n. In order to apply Lemma 8.2.2, define $\Gamma : L^1(J_m, X) \to C(J_m, X)$ by

$$\Gamma(v)(t) := -\int_0^t S(t-s)BW^{-1}\left[\int_0^m S(m-\eta)v(\eta)d\eta\right](s)ds + \int_0^t S(t-s)v(s)ds.$$

Then $\bar{h}_n(t) \in \Gamma(S_{F,x_{nt},x'_n})$ and since h_n and x_n converges, we deduce that $\bar{h}_n$ also converges to $\bar{h}^*$ and is given by

$$\begin{aligned}\bar{h}^*(t) := h^*(t) - C(t)\phi(0) - S(t)[y_0 - f(0,\phi)] - \textstyle\int_0^t C(t-s)f(s,x_s^*)ds \\ - \textstyle\int_0^t S(t-\eta)B\bar{u}(\eta)]d\eta,\end{aligned}$$

where $\bar{u}$ has the same definition as $\bar{u}_n$ with x_n replaced by x^*. Finally from Lemma 8.2.2, there exists $v^* \in \Gamma(S_{F,x_t^*,x^{*\prime}})$ such that

$$\begin{aligned}h^*(t) = C(t)\phi(0) + S(t)[y_0 - f(0,\phi)] + \textstyle\int_0^t C(t-s)f(s,x_s^*)ds \\ + \textstyle\int_0^t S(t-s)v^*(s)ds + \int_0^t S(t-\eta)Bu^*(\eta)d\eta,\end{aligned}$$

where u^* is defined as in (8.3.1), where x is replaced by x^*. Observe that we do not claim the convergence of u_n to u^* and v_n to v^*.

This shows that N_1 has a closed graph. As a consequence of Lemma 2.2.12, we deduce that N_1 has a fixed point in Z. Thus, system (8.1.1) is controllable on J and this completes the proof of the main theorem. ∎

8.4 Example

Consider the following second-order partial differential inclusion:

$$\left.\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}(x,t) - f(t,y_t)\right) &\in y_{xx}(x,t) + u(x,t) + F(t,y_t,\tfrac{\partial y}{\partial t}(x,t)) \\ y(0,t) &= y(\pi,t) = 0 \ \text{ for } \ t>0 \\ y(x,t) &= \phi(x,t), \text{for } -r \le t \le 0 \\ \frac{\partial y}{\partial t}(x,0) &= x_0(y), \ t \in J = [0,\infty) \qquad \text{for } 0<y<\pi \end{aligned}\right\} \tag{8.4.1}$$

Here one can take arbitrary non linear functions f and F satisfying the assumptions (H5)-(H7). Let $X = L^2[0,\pi]$ and $C_r = C([-r,0],X)$ be as in Section 8.1. We use the same notations. Then, for example, one can take $f: J \times C_r \longrightarrow X$ defined by

$$f(t,\phi)(x) = \eta(t,\phi(x,-r)), \ \phi \in C_r, \ x \in (0,\pi)$$

and $F: J \times C_r \times X \longrightarrow 2^X$ be defined by

$$F(t,\phi,w)(x) = \sigma(t,\phi(x,-r),w(x)), \ \phi \in C_r, \ w \in X, \ x \in (0,\pi)$$

with appropriate conditions on η and σ.

Now $u: (0,\pi) \times J \longrightarrow R$ is continuous in t which is the control function. Define $A: X \longrightarrow X$ by

$$Aw = w'', \ w \in D(A)$$

where

$D(A) = \Big\{w \in X : w, w' \text{ are absolutely continuous, } w'' \in X, w(0) = w(\pi) = 0\Big\}$ Then A has the spectral representation

$$Aw = \sum_{n=1}^{\infty} -n^2(w,w_n)w_n, \ w \in D(A),$$

where, $w_n(s) = \sqrt{\frac{2}{\pi}}\sin ns, n = 1,2,3,......$ is the orthogonal set of eigen functions of A. Further, it can be shown that A is the infinitesimal generator of a strongly continuous Cosine family $C(t)$, $t \in R$, defined on X which is given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt(w,w_n)w_n, w \in X.$$

The associated Sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{1}{n}\sin nt(w,w_n)w_n, \ w \in X.$$

The control operator $B : L^2(J, X) \longrightarrow X$ is defined by

$$(Bu)(t)(x) = u(x, t), \quad x \in (0, \pi),$$

which satisfies the condition $(H4)$. Now the PDE (8.4.1) can be represented in form (8.1.1). Hence, by Section 8.3, the system (8.4.1) is controllable.