CHAPTER-2

PRELIMINARIES

Chapter 2

PRELIMINARIES

2.1 Basic Concepts of Control Theory

In this chapter we provide some basics of control theory and necessary tools of functional analysis which will be used in the sequel for the controllability analysis of nonlinear systems.

Kalman [80] introduced the concept of controllability for the linear system (2.1.1) and was subsequently extended to nonlinear systems dominated by controllable linear parts by Davison and Kunze [52], Mirza and Womack [102], Quinn and Carmichael [117]etc., by using the techniques of fixed point theory. In our investigation, controllability properties of the nonlinear system depend more on the properties of the linear part. So we consider first a finite dimensional linear system represented by the differential equation

$$\left.\begin{array}{l} x'(t) = A(t)x(t) + B(t)u(t), \quad 0 \leq t_0 < t \leq t_1 < \infty \\ x(t_0) = x_0 \end{array}\right\} \tag{2.1.1}$$

where, for each $t \in [t_0, t_1], x(t) \in R^n$ is called the state of the system, $u(t) \in R^m$ is called the control vector and $u \in L^2([t_0, t_1], R^m); A(t), B(t)$ are matrices of dimensions $n \times n$ and $n \times m$, respectively. Assume that the elements $a_{ik}(t)$ of $A(t)(i, k = 1, 2, ..., n)$ are absolutely integrable functions of $t \in [t_0, t_1]$ and elements $b_{il}(t)$ of $B(t)(i = 1, 2, ..., n; l = 1, 2, ..., m)$ are piecewise continuous functions of $t \in [t_0, t_1]$. Throughout this thesis J denotes the time interval either $[t_0, t_1]$ or $[0, T]$ or $[0, b]$, depending upon the context. Here we take $J = [t_0, t_1]$.

DEFINITION 2.1.1 *An $n \times n$ matrix function $\Phi(t, t_0)$ is said to be transition matrix of homogeneous linear part of (2.1.1) if it satisfies the following:*

$$\frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I. \tag{2.1.2}$$

Example: The matrix function given by

$$\Phi(t, t_0) = \begin{bmatrix} cos(t - t_0) & -sin(t - t_0) \\ sin(t - t_0) & cos(t - t_0) \end{bmatrix}$$

is a transition matrix for the system

$$\frac{d}{dt}x(t) = Ax(t)$$

where,

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which is called the generator of the transition matrix $\Phi(t, t_0)$. The transition matrix has the following properties:

1. $\Phi(t, t) = I$ identity matrix of order n, for all $t \in J$
2. $\Phi(t, t_0) = \Phi(t, \tau)\Phi(\tau, t_0), \quad t \leq \tau \leq t_0$
3. $\frac{\partial}{\partial t}(\Phi(t, \tau)x_0) = A(t)(\Phi(t, \tau)x_0), \quad x_0 \in R^n$.
4. $\Phi(t, s)$ is strongly continuous in t, s for $t_0 \leq s \leq t \leq t_1$.

   If $\Phi_1(t), \Phi_2(t), \Phi_3(t), \ldots\ldots.., \Phi_n(t)$ are linearly independent solution of the homogeneous system (2.1.1) and $\Phi(t)$ is the matrix whose columns are $\Phi_1(t)$, $\Phi_2(t)$, $\Phi_3(t), \ldots\ldots.., \Phi_n(t)$, then it can be shown easily that the transition matrix $\Phi(t, \tau)$ satisfies

5. $\Phi(t, \tau) = \Phi(t)[\Phi(\tau)]^{-1}$

Using the variation of constant formula and Theorem 1 of Brockett [39], we have the following theorem concerning the solution of system (2.1.1).

THEOREM 2.1.2 *The sequence of matrices M_k defined recursively by*

$$M_0 = I, \quad M_k = I + \int_{t_0}^{t} A(\tau) M_{k-1}(\tau) d\tau \tag{2.1.3}$$

converges uniformly on J. Moreover, if the limit function is denoted by $\Phi(t, t_0)$ then

$$\frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0) \text{ and } \Phi(t_0, t_0) = I \tag{2.1.4}$$

and the solution of (2.1.1) which passes through x_0 at $t = t_0$ is given by

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau \tag{2.1.5}$$

From Theorem 2.1.2, it follows that the explicit expression for $\Phi(t, t_0)$ is given by the Peano-Baker series (refer Brockett [39])

$$\Phi(t, t_0) = I + \int_{t_0}^{t} A(\tau_1)d\tau_1 + \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2)d\tau_2 \, d\tau_1 + ... \tag{2.1.6}$$

If A is a real constant $n \times n$ matrix, then the Peano-Baker series (2.1.6) becomes

$$\Phi(t, t_0) = I + A(t - t_0) + \frac{A^2(t - t_0)}{2!} + ... = e^{A(t-t_0)} \tag{2.1.7}$$

Though a variety of definitions are available for controllability in the literature, our definition is as follows (refer Russell [120]).

DEFINITION 2.1.3 *The system (2.1.1) is said to be controllable over $[t_0, t_1]$ if for each pair of vectors $x_0, x_1 \in \mathrm{R}^n$ there exists a control $u \in L^2([t_0, t_1], \mathrm{R}^m)$ such that the solution of (2.1.1) with $x(t_0) = x_0$ also satisfies $x(t_1) = x_1$.*
That is,

$$x_1 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

The following definitions are useful in Chapter 5.

Let $\mathcal{T}$ be the set of n-vector valued functions z defined on $J = [t_0, T]$ such that $z(t_0) = x_0, z(T) = x_1$ and z is differentiable almost everywhere.

DEFINITION 2.1.4 *The system (2.1.1) is said to be **T-controllable** if for any $z \in \mathcal{T}$, there exists a control $u \in L^2(J, R^m)$ such that the corresponding solution $x(\cdot)$ of (2.1.1) satisfies $x(t) = z(t)$.*

DEFINITION 2.1.5 *The system (2.1.1) is* ***totally controllable*** *on $J = [t_0, T]$ if for all sub intervals $[t_i, t_f]$ of $[t_0, T]$ the system (2.1.1) is completely controllable.*

Clearly, T- controllability $\Longrightarrow$ Total controllability $\Longrightarrow$ Complete controllability.

REMARK 2.1.6 *The control u which steers x_0 to x_1 need not be unique and in general it depends on x_0 and x_1. The controllability defined above is known in literature, as global controllability. If x_0 and x_1 are required only to belong to a subset $D \subset \mathrm{R}^n$, then the resulting controllability is said to be local controllability.*

Let $\mathcal{C} = L^2(J, \mathrm{R}^m) \to \mathrm{R}^n$ be an operator defined by

$$\mathcal{C}u = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau. \tag{2.1.8}$$

It is clear that the system (2.1.1) is controllable if and only if the control operator $\mathcal{C}$ is onto. If $\mathcal{C}^*$ denotes the adjoint of $\mathcal{C}$ then $\mathcal{C}$ is onto (if and only if $\mathcal{C}^*$ is one-one) if and only if $\mathcal{C}\mathcal{C}^*$ is positive definite. The operator $\mathcal{C}\mathcal{C}^*$ is known as controllability Grammian and is denoted by $W(t_0, t_1)$:

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B^*(\tau)\Phi^*(t_1, \tau)d\tau \tag{2.1.9}$$

By Definition 2.1.3, the system (2.1.1) is globally controllable if $(x_1 - \Phi(t_1, t_0)x_0)$ lies in the Range of $\mathcal{C}$ for for all $x_0, x_1 \in \mathrm{R}^n$. But $(x_1 - \Phi(t_1, t_0)x_0) \in$ Range $(\mathcal{C})$ iff $(x_1 - \Phi(t_1, t_0)x_0) \in$ Range $(\mathcal{C}\mathcal{C}^*)$.

When $W(t_0, t_1)$ is invertible, the control function defined by

$$u(t) = -B^*(t)\Phi^*(t_1, t)W^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)x_0] \tag{2.1.10}$$

steers the system (2.1.1) from $x(t_0) = x_0$ to $x(t_1) = x_1$.

So, we have the following characterization for controllability.

THEOREM 2.1.7 *The linear system (2.1.1) is (globally) controllable if and only if the controllability Grammian $W(t_1, t_0)$ defined in (2.1.9) is nonsingular. That is, there exists a constant $c > 0$ such that*

$$det\ W(t_0, t_1) \geq c$$

DEFINITION 2.1.8 *A bounded linear operator $S : R^n \to L^2(J, R^m)$ is called a steering operator for (2.1.1) if for any $\alpha \in R^n, u = S\alpha$ steers 0 to α.*

An $m \times n$ matrix function $S(t)$ is called a steering function, if the operator S defined by $(S\alpha)(t) = S(t)\alpha$ is a steering operator.

We observe that

1. A bounded linear operator $S : R^n \to L^2(J, R^m)$ is a steering operator if and only if $CS = I$ and
2. an $m \times n$ matrix function $S(t)$ is a steering function if and only if

$$\int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)S(\tau)d\tau = I \tag{2.1.11}$$

We have the following characterization regarding controllability of (2.1.1) in terms of steering operator and steering function.

THEOREM 2.1.9 *The following are equivalent:*

1. *The system (2.1.1) is controllable.*
2. *There exists a steering function for (2.1.1).*
3. *There exists a steering operator for (2.1.1).*

REMARK 2.1.10 *If CC^* is invertible (that is, the controllability Grammian is non-singular) then*

$$S = C^*(CC^*)^{-1} \tag{2.1.12}$$

(the More-Penrose inverse of C) is a steering operator.
In this case

$$S_0(t) = B^*(t)\Phi^*(t_1, t)W^{-1}(t_0, t_1) \tag{2.1.13}$$

is a steering function. Further, $S_0(t)$ is an optimal steering function, (refer Russell [120]).

Consider the finite dimensional nonlinear time varying system, with control, represented by the equation

$$\frac{dx}{dt} = A(t)x + B(t)u + F(t,x), \quad 0 \leq t_0 < t \leq t_1 < \infty \tag{2.1.14}$$

$$x(t_0) = x_0$$

where $A(t)$ and $B(t)$ are as in (2.1.1) and $F : J \times \mathrm{R}^n \to \mathrm{R}^n$ is a nonlinear function satisfying caratheodory conditions (Joshi and Bose [78]). All quantities in (2.1.14) are assumed to be real.

A solution of (2.1.14) is an absolutely continuous function in $L^2([t_0, t_1], \mathrm{R}^n)$ which satisfies (2.1.14) almost everywhere. A solution $x(t)$ exists for (2.1.14) if and only if $x(t)$ satisfies the integral equation

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + \int_{t_0}^{t} \Phi(t,\tau)F(\tau,x(\tau))d\tau \tag{2.1.15}$$

where $\Phi(t,\tau)$ is the transition matrix of the homogeneous linear part. We shall be interested in the global controllability of (2.1.14).
There exists a control u which steers the initial x_0 at time $t = t_0$ to the given final state x_1 at time $t = t_1$ if and only if there exists a solution x of (2.1.14) satisfying

$$x_1 = x(t_1) = \Phi(t_1,t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau + \int_{t_0}^{t_1} \Phi(t_1,\tau)F(\tau,x(\tau))d\tau \tag{2.1.16}$$

That is

$$\left[x_1 - \Phi(t_1,t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1,\tau)F(\tau,x(\tau))d\tau\right] = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau$$

Suppose that the linear part of (2.1.14) is controllable. Thus by Theorem 2.1.9, there exists a steering function $S(t)$ for the linear part of the system (2.1.14) . If there exists x satisfying (2.1.16) then the steering control for (2.1.14) is given by (using Definition 2.1.8).

$$u(t) = S(t)\left[x_1 - \Phi(t_1,t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1,\tau)F(\tau,x(\tau))d\tau\right]. \tag{2.1.17}$$

Thus, the state of the system (2.1.14) is given by

$$\begin{aligned} x(t) &= \Phi(t,t_0)x_0 + \int_{t_0}^{t} \phi(t,\tau)F(\tau,x(\tau))d\tau + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)S(\tau) \\ &\quad \left[x_1 - \Phi(t_1,t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1,s)F(s,x(s))ds\right]d\tau \end{aligned} \tag{2.1.18}$$

Conversely, suppose that (2.1.18) is solvable then $x(t_0) = x_0$ and $x(t_1) = x_1$. This implies that the system (2.1.14) is controllable with the control defined by (2.1.17).

Hence, the controllability of the nonlinear system (2.1.14) is equivalent to the solvability of the coupled equations.

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + \int_{t_0}^{t} \Phi(t,\tau)F(\tau,x(\tau))d\tau \quad (2.1.19)$$

$$u(t) = S(t)\Big[x_1 - \Phi(t_1,t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1,\tau)F(\tau,x(\tau))d\tau\Big] \quad (2.1.20)$$

Let $X_1 = L^2(J, \mathrm{R}^m), X_2 = L^2(J, \mathrm{R}^n)$. Define operators $K, N : X_2 \to X_2, H : X_1 \to X_2$ and $L : X_2 \to X_1$ as follows

$$(Kx)(t) = \int_{t_0}^{t} \Phi(t,\tau)x(\tau)d\tau; \qquad (Nx)(t) = F(t,x(t));$$

$$(Hu)(t) = \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau; \quad (Lx)(t) = S(t)\int_{t_0}^{t_1} \Phi(t_1,\tau)x(\tau)d\tau.$$

Clearly K, H and L are continuous linear operators and N is a nonlinear operator, called Nemytskii operator, refer Joshi and Bose [78]. Using these definitions, the coupled equations can be written as a coupled operator equations

$$x = Hu + KNx + w_1.$$

$$u = u_1 - LNx,$$

where, $w_1 = \Phi(t,t_0)x_0$ and $u_1 = S(t)[x_1 - \Phi(t_1,t_0)x_0]$

Now we consider the linear infinite dimensional system in Banach space described by the equation

$$\left.\begin{array}{l} x'(t) = Ax(t) + B(t)u(t), \;\; 0 \le t_0 < t \le t_1 < \infty \\ x(0) = x_0 \end{array}\right\} \quad (2.1.21)$$

where the state $x(t)$ takes values in a Banach space X with the norm $\|.\|$ for each $t \in J = [0,T]$, control function $u(.)$ is given in $L^2(J,U)$, a Banach space of admissible control functions, with U as a Banach space. Here A is the infinitesimal generator of a strongly continuous semigroup $T(t); t \ge 0$ in a Banach space X.

If A is a matrix then $T(t) = e^{A(t-s)}$ reduces to the transition matrix.

DEFINITION 2.1.11 *A strongly continuous family* $\{T(t)\}_{t\ge0}$ *of bounded operators in a Banach space* X *is called a* ***semigroup*** *generated by* A *if* *(i)* $T(t+s)x = T(t)T(s)x, \ \ x \in X$ *and* $t, s \geq 0$,

(ii) $T(0)x = x, \ \ x \in X$,

(iii) $t \mapsto T(t)x$ *is continuous for* $t \geq 0, x \in X$,

(iv) $Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t}; \qquad x \in D(A)$.

REMARK 2.1.12 *In Definition (2.1.11), the condition (iv) gives the generator of the semigroup in terms of an operator A [137].*

EXAMPLE 2.1.13 *Consider the one-dimensional heat equation on* $\Omega = (0,1)$

$$\left.\begin{array}{l} \dfrac{\partial y}{\partial t} = \dfrac{\partial^2 y}{\partial x^2} \ \ in \ \Omega \times (0,T) \\ y(x,0) = y_0(x) \ \ in \ \Omega \\ y(0,t) = 0 = y(1,t) \ \ in \ (0,T). \end{array}\right\}$$

The above system can be associated with the evolution equation $\frac{dy}{dt} = Ay$ on $L^2(0,1)$ where $A : L^2(0,1) \to L^2(0,1)$ by $Ay = y''$, where $\mathcal{D}(A) = \{y \in H \colon w, w'$ are absolutely continuous, $y(0) = y(1) = 0\}$ It is easy to show that A generates semigroup $S(t), t \geq 0$ in $L^2(0,1)$ given by

$$S(t)y = \sum_{n=1}^{\infty} 2 \ \exp(-n^2\pi^2 t) \sin n\pi x \int_0^1 \sin n\pi y(\tau) d\tau, \ \ y \in L^2(0,1).$$

In recent years, more general family of operators have been introduced.

DEFINITION 2.1.14 *A continuous semigroup operator* $T(t) : R^+ \to R^{n \times n}$ *is called n-times integrated semigroups, generated by A if*

$$R(\lambda; A) = \lambda^n \int_0^\infty e^{-\lambda t} S_n(t) dt; \ \ for \ some \ \lambda \in R, \ \ n \in N \tag{2.1.22}$$

where, $R(\lambda, A) = (\lambda - A)^{-1}$ *is called the resolvent set of the generator A. It has been noticed that resolvent* $R(\lambda, A)$ *is given by the Laplace Transform of the semigroup* $T(t)$ *(Heiber [73]),*

$$R(\lambda; A) = \int_0^\infty e^{-\lambda t} T(t) dt$$

We denote n-times integrated semigroups by $S_n(t)$
In 1991, M Heiber ([71]-[73]) replaced the integer n by any real number α.

$$R(\lambda; A) = \lambda^{\alpha} \int_0^{\infty} e^{-\lambda t} S_{\alpha}(t)dt; \ \alpha \in R^{+} \tag{2.1.23}$$

(2.1.23) is the generalization of (2.1.22).
We denote α -times integrated semigroups by $S_{\alpha}(t)$.
If A generates a semigroup $T(x)$, then the α-times integral (or α -fractional order integral) of semigroup $T(t)$ is defined as,

$$S_{\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(s)ds \tag{2.1.24}$$

where, Γ is the Gamma function.

Then the system (2.1.21) has mild solution of the form (Ntouyas and Tsamatos [110])

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds$$

In many cases, it is advantageous to treat the second order abstract differential systems rather than to convert them to first order systems. A useful tools for the study of abstract second order equations is the theory of strongly continuous cosine families. We will make the use of the basic ideas from cosine family theory, ([132], [133]).

We consider the semi linear second order control system

$$\begin{aligned} x''(t) &= Ax(t) + Bu(t); \quad t \in J = [0, T] \\ x_0 &= \phi, x'(0) = y_0 \end{aligned} \tag{2.1.25}$$

where the state $x(t)$ takes the values in the reflexive Banach space X, A is the infinitesimal generator of the strongly continuous cosine family $C(t), t \in R$ of bounded linear operator from U to X, and the control function u is given in $L^2(J, U)$, a Banach space of all admissible control function, with U as a Banach space and $\phi, y_0 \in X$.

DEFINITION 2.1.15 *A strongly continuous operator $C(t) : R \to R^{n \times n}$ is called a cosine function if*

1.

$$C(0) = I, \ I \text{ is the identity matrix on } R; \tag{2.1.26}$$

2.

$$C(t+s) + C(t-s) = 2C(t)C(s) \text{ for all } s, t \in R; \tag{2.1.27}$$

3. *The map $t \mapsto C(t)x$ is strongly continuous in t on R for each fixed $x \in R$.*

The strongly continuous sine family $\{S(t) : t \in R\}$, associated to the strongly continuous cosine family $\{C(t) : t \in R\}$ is defined by

$$S(t)x = \int_0^t C(s)x\, ds, \qquad x \in R, t \in R.$$

Note that the identity (2.1.27) is same as

$$S(t+s) = C(t)S(s) + S(t)C(s);\ t, s \in R \tag{2.1.28}$$

(Refer Yang [137])
Example: The matrix function given by

$$C(t) = \begin{bmatrix} cost & 0 \\ 0 & cost \end{bmatrix}$$

is a cosine function.

DEFINITION 2.1.16 *The generator of the cosine function is a matrix A defined by*

$$Ax = 2 \lim_{t \to 0} \Big[\frac{C(t)x - x}{t^2}\Big]; \qquad x \in R^n \tag{2.1.29}$$

In the above example, $A = -I$ is the generator.
We now give the following equivalent definition of cosine function in terms of the resolvent set of the generator A, given by $\rho(A)$.
If $R(\lambda^2; A) = (\lambda^2 - A)^{-1}$ is the resolvent of A, then we define cosine function $C(t)$ generated by A as follows.

DEFINITION 2.1.17 *A strongly continuous function $C(t) : R \to R^{n \times n}$ is called the cosine function if*

$$R(\lambda^2; A) = \frac{1}{\lambda}\int_0^\infty e^{-\lambda t} C(t) dt; \qquad \lambda \in \rho(A) \tag{2.1.30}$$

and

$$\|C(t)\| \le Me^{\omega t}; \lambda > \omega \tag{2.1.31}$$

where M and ω are constants.

Now we generalize the definition of cosine function as follows:

DEFINITION 2.1.18 *A strongly continuous operator $C(t) : R \to R^{n\times n}$ is called an $\alpha-$ times integrated cosine function if*

$$R(\lambda^2; A) = \lambda^{\alpha-1}\int_0^\infty e^{-\lambda t}C(t)dt; \qquad \alpha \geq 0, \lambda > \omega \tag{2.1.32}$$

when $\alpha = 0$, (2.1.32) reduces to (2.1.30).

THEOREM 2.1.19 *(Yang [137]) An $n\times n$ matrix A generates an $\alpha-$ times integrated cosine function if and only if there exist constant $M, \omega \in R^+$ such that*

$$\left\| \frac{d^n}{d\lambda^n}\frac{R(\lambda^2; A)}{\lambda^{1-\alpha}} \right\| \leq \frac{Mn!}{(\lambda-\omega)^{n+1}}$$

The condition in the Theorem 2.1.19 follows directly from equation (2.1.30).

The following terminology of cosine operator can be used to deal with the second order systems.

Let $B(X)$ denote the Banach space of bounded linear operators on Banach space X into X. We say that one-parameter family $\left\{C(t) : t \in R\right\}$ of bounded linear operators in $B(X)$ is a strongly continuous cosine family if and only if Definition (2.1.15) holds. Assume the following condition on A.

(H1) A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in R$ of bounded linear operators mapping X into itself and the adjoint operator A^* is densely defined i.e. $\overline{D(A^*)} = X^*$ ([38]).

The infinitesimal generator of a strongly continuous cosine family $C(t), t \in R$ is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}, \qquad x \in D(A)$$

where, $D(A) = \{x \in X : C(t)x$ is twice continuously differentiable in $t\}$.

i.e.

$$D(A) = \{x \in X : C(.)x \in C^2(R, X)\}$$

Define $X_1 = \{x \in X : C(t)x$ is once continuously differentiable in $t\}$.

LEMMA 2.1.20 *([132]) Let $(H1)$ hold. Then*

1. *there exist constants* $M_1 \geq 1$ *and* $w \geq 0$ *such that*

$$\|C(t)\| \leq M_1 e^{wt} \quad \text{and} \quad \|S(t) - S(t^*)\| \leq M_1 \| \int_0^{t^*} e^{ws} ds\|; \quad \text{for} \quad t, t^* \in R;$$

2. $S(t)x \subset X_1$ *and* $S(t)X \subset D(A)$, *for* $t \in R$;

3. $\frac{d}{dt}C(t)x = AS(t)x$, *for* $x \in X_1$ *and* $t \in R$;

4. $\frac{d^2}{dt^2}C(t)x = AC(t)x$, *for* $x \in D(A)$ *and* $t \in R$.

LEMMA 2.1.21 *([132]) Let (H1) hold and* $v : R \to X$ *such that* v *is continuously differentiable and* $q(t) = \int_0^t S(t-s)v(s)ds$ *then,*

$$q \in C^2(R, X) \quad for \quad t \in R, q(t) \in D(A),$$
$$q'(t) = \int_0^t C(t-s)v(s)ds \quad and \quad q''(t) = Aq(t) + v(t).$$

For more details on strongly continuous cosine and sine family, refer Goldstein [68] and Travis and Webb ([132], [133]).

The integral representation of the system (2.1.25) can be written as (Ntouyas and Tsamatos [110])

$$x(t) = C(t)\phi(0) + S(t)y_0 + \int_0^t S(t-s)Bu(s)ds$$

Example:

$$\begin{aligned} \frac{\partial}{\partial t}\Big(\frac{\partial z}{\partial t}(y,t)\Big) &= z_{yy}(y,t) + \mu(y,t) \\ z(0,t) &= z(\pi,t) = 0 \\ \frac{\partial z}{\partial t}(y,0) &= z_0(y), \qquad t \in J = [0,T]; \ \text{for} 0 < y < \pi \end{aligned}$$

where, $\mu : (0,\pi) \times J \to (0,\pi)$ is continuous in $t, X = L^2[0,\pi]$
Let $A : X \to X$ be defined by $Aw = w''$, $w \in D(A)$
where,
$D(A) = \{w \in X : w, w' \text{ are absolutely continuous}, w'' \in X, w(0) = w(x) = 0\}$
Then,

$$Aw = \sum_{n=1}^{\infty} -n^2(w, w_n)w_n, \ w \in D(A)$$

where,
$w_n(s) = \sqrt{\frac{2}{n}} sinns, n = 1, 2, 3, ...$; is the orthogonal set of eigen functions of A. Here, A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in R$ in X and is given by [132]

$$C(t)w = \sum_{n=1}^{\infty} cosnt(w, w_n)w_n, w \in X$$

The associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{1}{n} sinnt(w, w_n)w_n, w \in X$$

The controllability of this system was studied by Balachandran and Marshal Anthoni (refer [20]).

The last chapter of the thesis is related to second-order differential neutral inclusion system. The difference between an ordinary differential equation and a differential inclusion is that the right hand side of the differential inclusion is a set instead of a single-valued function in the differential equation. The solution of the differential inclusion is also a set and not a single system trajectory. Any function that satisfies differential inclusion system is a trajectory of the differential inclusion, but not a solution of the differential inclusion.

In the decade 1930-40 such problems as the existence and the properties of the solutions to the differential inclusions have been resolved in the finite dimensional spaces and subsequently it has been generalized to an infinite dimensional spaces.

The study of IVP with nonlocal conditions is of significance since they have applications in problems in physics and other area of mathematics. Some authors have paid attention to the IVP with nonlocal conditions, in the few past years. We refer to Balachandran and Chandrasekaran ([12]), Byszewski ([40], [41]), Ntouyas and Tsamatos ([110]) and Benchohra and Ntouyas ([30]). The work on evolution nonlocal IVP was initiated by Byszewski [40] by using C_0 semigroup and the Banach fixed point theorem. The existence and uniqueness of mild, strong and classical solutions of the first order evolution nonlocal IVP was provided by him. The IVP for second order semilinear equations with nonlocal conditions was studied by Ntouyas and Tsamatos ([110]).

The following inclusion result is useful in Chapter 8.

Benchohra and Ntouyas [33]discussed the controllability results for multi valued semilinear neutral functional differential inclusion in compact interval

$$\frac{d}{dt}[x'(t) - f(t, x_t)] \in Ax(t) + Bu(t) + F(t, x_t), \qquad t \in J = [0, b] \tag{2.1.33}$$

$$x_0 = \phi, x'(0) = x_1 \quad , \tag{2.1.34}$$

where the state $x(t)$ takes values in the reflexive Banach space X with the norm $|.|$, $x_0 \in X$, A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ of bounded linear operator in a Banach space X, $F : J \times C \to 2^X$ is a bounded, closed, convex multi valued map, $f : J \times C \to X$ is a given function, B is a bounded linear operator from U to X and the control $u(.)$ is given in $L^2(J, U)$, a Banach space of admissible control function with U as a Banach space and $\phi \in C$. Here $C = C([-r, 0], X)$ is the Banach space of all continuous functions $\phi : J_0 = [-r, 0] \to X$ endowed with the sup norm

$$\|\phi\| = sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}$$

$J = [0, b]$ is real interval. Also for any continuous function x defined on the interval $J_1 = [-r, b]$ and any $t \in J$, we denote by x_t, the element of $C(J_0, X)$ defined by $x_t(\theta) = x(t + \theta), \theta \in J_0$.
Here, $x_t(.)$ represents the history of the state from time $t - r$, up to the present time t.

2.2 Some Basic Results of Nonlinear Functional Analysis

Let X be a real Banach space and X^* be the dual of X. The strong convergence of a sequence $\{x_n\}$ to x_0 in X is denoted by $x_n \to x_0$ and weak convergence by $x_n \xrightarrow{w} x_0$. For each $x \in X$ and $x^* \in X^*$, let (x^*, x) denote the evaluation of x^* at x and when X is a Hilbert space $\langle ., . \rangle$ denotes its inner product.
Let $T : D(T) \subset X \to X^*$ be any operator. Then

1. T is said to be monotone if $< Tx - Ty, x - y > \geq 0 \quad \forall x, y \in D(T)$.

2. T is called strictly monotone if the above inequality is strict for $x \neq y$.

3. T is called strongly monotone if there exists a constant $\mu > 0$ such that $< Tx - Ty, x - y > \geq \mu \|x - y\|^2 \quad \forall x, y \in D(T)$.

4. T is said to be of type (M) if for any sequence $x_n \in X$ converging to $x_0 \in X$ with Tx_n converging weakly to $y \in X^*$ and $\overline{lim}_{n\to\infty} < Tx_n, x_n - x_0 > \leq 0$, we have $y = Tx_0$.

5. T is said to be coercive, if $lim_{\|x\|} \to \infty \frac{<Tx,x>}{\|x\|} = \infty$.

Observe that T is monotone (strongly monotone) if and only if $\mu(T) \geq 0$ $(\mu(T) > 0)$.

Let Y be another real Banach space and $\{x_n\}$ is a sequence in X then the operator $T : X \to Y$ is called

1. continuous at x_0 if
$$x_n \to x_0 \Rightarrow Tx_n \to Tx_0$$
2. weakly continuous at x_0 if
$$x_n \xrightarrow{w} x_0 \Rightarrow Tx_n \xrightarrow{w} Tx_0$$
3. completely continuous at x_0 if
$$x_n \xrightarrow{w} x_0 \Rightarrow Tx_n \to Tx_0$$

T is called bounded if it maps every bounded sequence $\{x_n\}$ in X into bounded sequence $\{Tx_n\}$ in Y and T is called compact if for any bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ has a converging subsequence in Y.

Let Lip be the set of all operators $T : X \to X$ such that there exists a constant $\alpha > 0$ satisfying $\|Tx - Ty\| \leq \alpha\|x - y\|$ $\forall x, y \in X$. For $T \in$ Lip , we define

$$\|T\|^* = \sup_{x,y\in X; x\neq y} \frac{\|Tx - Ty\|}{\|x - y\|}$$

If $T \in$ Lip with $\|T\|^* = \alpha$, we say that T is Lipschitz continuous with constant α. We note that

1. $T, S \in \text{Lip} \Rightarrow \|TS\|^* \leq \|T\|^*\|S\|^*$
2. $T \in \mathcal{L}(X) \Rightarrow \|T\|^* = \|T\|$

For more details refer Dolezal [55].

The following theorem is employed to prove some lemmas in Chapter 6.

THEOREM 2.2.1 *[55]. Let $K \in \mathcal{M}$ be continuous and $N \in Lip, \mu(N) > 0$. If $(\mu(K) + \mu(N)\|N\|^{*-2}) > 0$, then $[I + KN]$ is invertible with $[I + KN]^{-1} \in Lip$ and*

$$\|[I + KN]^{-1}\|^* \leq \frac{1}{\mu(N)(\mu(K) + \mu(N)\|N\|^{*-2})}$$

THEOREM 2.2.2 [78] Let X be a real Banach space and $T : X \to X^*$, a mapping of type (M). If T is coercive then the range of T is all of X^*.

THEOREM 2.2.3 *[78] Let N be a continuous monotone mapping of a real reflexive Banach space X into X^* and let the monotone linear operator $K : X^* \to X$ satisfy the following condition:*
$\exists$ a constant $d > 0$ such that $\langle x, Kx \rangle \leq d\|Kx\|^2 \ \forall x \in X^*$.
Then the equation $x + KNx = f$ admits a unique solution x for each f in X.

The following definition of Integral contractor will be used in Chapter 3 which will work as a weaker notion of Lipschitz continuity.

Let $C = C([0, T]; L^2(0, 2\pi))$ denote the Banach space of continuous functions on $J = [0, T]$ with values in (L^2) with the standard norm $\|w\|_C = \sup_{0 \leq t \leq T} \|w(t)\|_{L^2(0,2\pi)}$.

DEFINITION 2.2.4 *Suppose $\Gamma : J \times L^2(0, 2\pi) \longrightarrow BL(C)$ is a bounded continuous operator and there exists a positive number γ such that for any $w, y \in C$, we have*

$$\sup_{0 \leq t \leq T} \|f\Big(t, w(x,t) + y(x,t) + \int_0^t \Phi(t-s)(\Gamma(s, w(x,s))y)(x,s)ds\Big) - f(t, w(x,t)) \\ -(\Gamma(t, w(x,t))y)(x,t)\|_{L^2(0,2\pi)} \leq \gamma \|y(x,t)\|_C \tag{2.2.1}$$

Then we say that $f(t, w(x,t))$ has a bounded integral contractor $\{I + \Phi\Gamma\}$ with respect to $\Phi(t-s)$.

For simplicity, we may refer Γ, the integral contractor instead of $\{I + \int \Phi\Gamma\}$.

REMARK 2.2.5 It is remarkable that if $\Gamma \equiv 0$ then condition (2.2.1) reduces to Lipschitz condition i.e.

$$\|f(., w(x,.) + y(x,.)) - f(., w(x,.))\|_C \leq \gamma \|y(x,t)_C\|$$

We need the following fixed-point theorem due to Schaefer [123] in Chapter 4.

THEOREM 2.2.6 *(Schaefer Fixed Point Theorem (see [123])): Let X be a normed linear space. Let $F : X \longrightarrow X$ be a completely continuous operator, that is, F is continuous and image of any bounded set is contained in a compact set, and let*

$$\xi(F) = \left\{ w \in X, w = \lambda F w \qquad \text{for some} \quad 0 < \lambda < 1 \right\}.$$

Then, either $\xi(F)$ is unbounded or F has a fixed point.

The following generalized contraction principle will be used in Chapter 5, Chapter 6 and Chapter 7.

THEOREM 2.2.7 *[78]. Let T be a continuous mapping of a Banach space X into itself such that there exists a positive integer $n \geq 1$ such that $\|T^n x - T^n y\| \leq k\|x - y\| \; \forall \; x, y \in X$ and for some positive constant $k < 1$. Then T has a unique fixed point.*

REMARK 2.2.8 *([42],[63]) When $n = 1$, the above theorem is known as Banach contraction principle. For any arbitrary x_0, the sequence defined by*

$$x_{n+1} = Tx_n + y$$

converges to the unique solution of $x = Tx + y$.
Moreover, $T^{-1} \in Lip$ with $\|T^{-1}\|^* = \frac{1}{(1-k)}$.

The following is known as Grownwall's inequality and it will be frequently used in the thesis.

THEOREM 2.2.9 *[48] Let $a \in L^1(J), a(t), b(t) \geq 0, b$ be an absolutely continuous function on J. If $x \in L^\infty(J)$ satisfies*

$$x(t) \leq b(t) + \int_{t_0}^{t} a(\tau)x(\tau)d\tau$$

then

$$x(t) \leq b(t_0)exp\left(\int_{t_0}^{t} a(\tau)d\tau\right) + \int_{t_0}^{t} b'(\tau)exp\left(\int_{t_0}^{\tau} a(\sigma)d\sigma\right)d\tau$$

The following tools are useful in Chapter 8.

DEFINITION 2.2.10 *A function $f : X \to R$ is said to be* ***convex*** *if its domain $\mathcal{D}(f)$ is a convex set and for every $u, v \in \mathcal{D}(f)$*

$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda))f(v),$$

where, $0 \leq \lambda \leq 1$.

DEFINITION 2.2.11 *A function F is called* ***upper (lower) semi continuous*** *if for any closed (open) subset C of $X, F^{-1}(C)$ is closed (open).*
i.e. F is upper semicontinuous at $x_0 \in X$ if $x_n \to x_0 \Rightarrow F(x_0) \geq \underline{\lim}_{n\to\infty} F(x_n)$, and F is lower semicontinuous at $x_0 \in X$ if $x_n \to x_0 \Rightarrow F(x_0) \leq \underline{\lim}_{n\to\infty} F(x_n)$.
F is called ***continuous*** *if F is both upper and lower semicontinuous and is called the* ***coercive*** *if $F(x) \to \infty$ as $\|x\| \to \infty$.*

Examples: The set-valued map $F_1 : R \mapsto R$ defined by

$$F_1(x) := \begin{cases} [-1, +1] & \text{if } x \neq 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

is lower semicontinuous at zero but not upper semicontinuous at zero.

The set valued map $F_2 : R \mapsto R$ defined by

$$F_2(x) := \begin{cases} \{0\} & \text{if } x \neq 0 \\ [-1, +1] & \text{if } x = 0 \end{cases}$$

is upper semicontinuous at zero but not lower semicontinuous at zero. Refer, Aubin and Frankowska [7].

The following terminology of set-valued analysis will be used in Chapter 8.

Let $(X, \|.\|)$ be a Banach space. A multi valued map $G_1 : X \to 2^X$ is **convex (closed) valued** if $G_1(x)$ is convex (closed) for all $x \in X$. G_1 is bounded on bounded sets if $G_1(B) = U_{x\in B} G_1(x)$ is bounded in X for any bounded set B of X (i.e.$\sup_{x\in B}\{\sup\{\|x\| : x \in G_1(x)\}\} < \infty$).

The multi map G_1 is called **upper semi continuous (u.s.c.)**on X if for each $x_0 \in X$ the set $G_1(x_0)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G_1(x_0)$, there exists an open neighborhood A of x_0 such that $G_1(A) \subseteq B$.

The multi map G_1 is said to be **completely continuous** if $G_1(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multi valued map G_1 is completely continuous with nonempty compact values, then G_1 **is u.s.c. if and only if** G_1 **has a closed graph.** (i.e.$x_n \to x_0, y_n \to y_0, y_n \in G_1(x_n)$ imply $y_0 \in G_1(x_0)$).

G_1 has a fixed point if there is $x \in X$ such that $x \in G_1 x$.
In the following, $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multi valued map $G_1 : J \to BCC(X)$ is said to be measurable, if for each $x \in X$, the distance between x and $G_1(x)$ is a measurable function on J. i.e. for each $x \in X$, the function $Y : J \to R$ defined by

$$Y(t) = d(x, G_1(t)) = \inf\{\|x - z\| : z \in G_1(t)\} \in L^1(J, R).$$

For more details on multi valued map, see ([53],[74]).

LEMMA 2.2.12 *(**Ma Fixed Point Theorem (see [98])**): Let X be a locally convex space and $N_1 : X \to X$ be a compact convex valued, u.s.c. multi valued map such that there exists a closed neighbourhood Up of 0 for which $N_1(Up)$ is a relatively compact set for each $p \in N$. If the set*

$$\Omega := \left\{ x \in X : \lambda x \in N_1(x) \qquad \textit{for some} \quad \lambda > 1 \right\}$$

is bounded , then N_1 has a fixed point.