CHAPTER-3

EXACT CONTROLLABILITY OF NONLINEAR THIRD-ORDER DISPERSION EQUATION

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3.1 Introduction

The controllability problem of famous Korteweg-De Vries (KDV) equation has been studied extensively by the researchers as far as the linear system is concerned. Russell and Zhang [121] discussed the controllability and stabilizability of the third order linear dispersion system on a periodic domain. They discussed the exponential decay rates with distributed controls of restricted form and for the equation with boundary dissipation. Later on, Zhang [143] studied the exact boundary controllability of the KDV equation of distributed parameter system in which the smoothing properties of the KDV equation is used. Recently, Rosier [119] focused on the exact boundary controllability for the linear KDV equation on the half-line, i.e. in the domain $\Omega = (0, +\infty)$. Rosier discussed the exact boundary controllability holds true in $L^2(0, +\infty)$ provided that the solutions are not required to be in $L^\infty(0, T; L^2(0, +\infty))$. Rosier used the tool of Carleman's estimates and an approximation theorem. The purpose of this chapter is to study the exact controllability of the following nonlinear third-order dispersion equation:

$$\frac{\partial w}{\partial t}(x,t) + \frac{\partial^3 w}{\partial x^3}(x,t) = (Gu)(x,t) + f(t, w(x,t)) \tag{3.1.1}$$

in the domain $t \geq 0, 0 \leq x \leq 2\pi$, with periodic boundary conditions

$$\frac{\partial^k w}{\partial x^k}(0,t) = \frac{\partial^k w}{\partial x^k}(2\pi,t), \quad k = 0,1,2. \tag{3.1.2}$$

and initial condition

$$w(x,0) = 0. \tag{3.1.3}$$

Here u is the control function and the operator G is defined by

$$(Gu)(x,t) = g(x)\Big\{u(x,t) - \int_0^{2\pi} g(s)u(s,t)ds\Big\}. \tag{3.1.4}$$

Then G is a bounded linear operator and $g(x)$ is a piece-wise continuous non-negative function on $[0, 2\pi]$ such that

$$[g] \overset{\text{def}}{=} \int_0^{2\pi} g(x)ds = 1 \tag{3.1.5}$$

and $f : [0,\infty] \times R \longrightarrow R$ is a continuous nonlinear function.

DEFINITION 3.1.1 *The system (3.1.1)-(3.1.3) is said to be exactly controllable over a time interval* $[0,T]$, *if for any given* $w_T \in L^2(0,2\pi)$, *there exists a control* $u \in X := L^2((0,T) \times (0,2\pi)) = L^2(0,T; L^2(0,2\pi))$ *such that the corresponding solution* w *of (3.1.1)-(3.1.3) satisfies* $w(.,T) = w_T$.

Russell and Zhang [121] studied the exact controllability of a corresponding linear system (i.e., with $f \equiv 0$ in (3.1.1)-(3.1.3). In their analysis, they considered controls which conserve the quantity $[w(.,t)]$, which corresponds to the "volume" (refer Russell and Zhang [121]). The following is their controllability result for the linear system.

THEOREM 3.1.2 (Russell-Zhang) *Let* $T > 0$ *be given and let* $g \in C^0[0,2\pi]$ *associated with* G *in (3.1.4). Given any final state* $w_T \in L^2(0,2\pi)$ *with* $[w_T] = 0$, *there exists a control* $u \in L^2(0,T; L^2(0,2\pi))$ *such that the solution* w *of*

$$\frac{\partial w}{\partial t}(x,t) + \frac{\partial^3 w}{\partial x^3}(x,t) = (Gu)(x,t) \tag{3.1.6}$$

together with the initial and boundary conditions (3.1.2)-(3.1.3), satisfies the terminal condition $w(.,T) = w_T$ *in* $L^2(0,2\pi)$. *Moreover, there exists a positive constant* C_1 *independent of* w_T *such that*

$$\|w\|_{L^2(0,T;L^2(0,2\pi))} \leq C_1 \|w_T\|_{L^2(0,2\pi)}. \tag{3.1.7}$$

■

The main purpose of this chapter is to obtain sufficient conditions on the perturbed nonlinear term f which will preserve the exact controllability. In our analysis, we employ the theory of monotone operators, Lipschitz continuous operators and the method of integral contractors to obtain controllability results. We first define the solution operator W for the system (3.1.1)-(3.1.3) and study its properties. Let

$$W : L^2(0,T;L^2(0,2\pi)) \longrightarrow L^2(0,T;L^2(0,2\pi))$$

be defined by

$$(Wu)(.,t) = w(.,t), \tag{3.1.8}$$

where $w(.,t)$ is the unique solution of (3.1.1)-(3.1.3) corresponding to the control u.

In Section 3.2, we give three sets of sufficient conditions to guarantee the existence of the solution operator W. The controllability problem of the given system is then reduced to a solvability problem of some suitable operator equation in Section 3.3. The Section 3.4 deals with the main results on exact controllability of the system (3.1.1)-(3.1.3) through the Lipschitz continuity of W, while in Sections 3.5 and 3.6, we study the exact controllability of the system (3.1.1)-(3.1.3) through integral contractor method which is a weaker condition than Lipschitz continuity.

3.2 Existence of the solution operator W

Define an operator A on $L^2(0,2\pi)$ with domain $D(A)$ defined by

$$D(A) = \left\{ w \in H^3(0,2\pi) : \frac{\partial^k w}{\partial x^k}(0) = \frac{\partial^k w}{\partial x^k}(2\pi),\ k = 0,1,2. \right\}$$

such that

$$Aw = -\frac{\partial^3 w}{\partial x^3} \tag{3.2.1}$$

It follows from Lemma 8.5.2 of Pazy [114] that A is the infinitesimal generator of a C_0 group of isometry on $L^2(0,2\pi)$ and denote it by $\{\Phi(t)\}_{t\geq 0}$. Then for all $w \in D(A)$

$$\langle Aw, w\rangle_{L^2(0,2\pi)} = 0. \tag{3.2.2}$$

This follows readily from

$$\langle Aw, w\rangle_{L^2(0,2\pi)} = \langle -w''', w\rangle = \langle w, w'''\rangle = -\langle Aw, w\rangle$$

where, the middle equality is achieved by integration by parts three times. Also, there exists a constant $M > 0$ such that

$$\sup\{\|\Phi(t)\| : t \in [0,T]\} \leq M. \tag{3.2.3}$$

By the variation of constant formula, we can write a mild solution of (3.1.1) - (3.1.3) as

$$w(.,t) = \int_0^t \Phi(t-s)(Gu)(.,s)ds + \int_0^t \Phi(t-s)f(t,w(.,s))ds \tag{3.2.4}$$

Let $X \stackrel{def}{=} L^2\big(0,T;L^2(0,2\pi)\big)$ and the operators $H, K, N : X \longrightarrow X$ defined by

$$\begin{aligned}(Hu)(t) &= \int_0^t \Phi(t-s)(Gu)(.,s)ds && (3.2.5)\\ (Kw)(t) &= \int_0^t \Phi(t-s)w(s)ds && (3.2.6)\\ (Nw)(t) &= f(t,w(t)), && (3.2.7)\end{aligned}$$

where $w(t) = w(.,t)$. By using the above notations and definitions, the equation (3.2.4) can be written as the operator equation:

$$w = Hu + KNw. \tag{3.2.8}$$

We now prove the following lemmas which will show the existence of the solution operator W. We first discuss separately the two situations viz., f is monotone and f is Lipschitz continuous and lastly when f satisfies certain second sub-gradient estimates.

Just like suppressing the x variable in the above equations, we may also suppress t variable unless it is essential. In the following $f(r)$ means $f(.,r)$.

LEMMA 3.2.1 *Suppose that f satisfy the following:*

[f1] There exist a constant $\beta > 0$ such that for all $r, s \in R$

$$\big(f(r) - f(s)\big)(r-s) \geq -\beta|r-s|^2.$$

[f2] There exist constants $a \geq 0$ and $b > 0$ such that, for all $r \in R$

$$|f(r)| \leq a|r| + b.$$

Then the solution operator W is well-defined.

Proof: We first show that the operator K defined by (3.2.6) satisfies $\langle Kw, w\rangle_X \geq 0$ for all $w \in D(A)$. To see this, let $w \in D(A)$ and define

$$h(t) = \int_0^t \Phi(t-s)w(s)ds.$$

Then $h(t) \in D(A)$ and since $\Phi(t)$ is a strongly continuous group, we have that

$$h'(t) = w(t) + A\int_0^t \Phi(t-s)w(s)ds = w(t) + Ah(t).$$

Hence,

$$\begin{aligned}\langle Kw, w\rangle_X &= \textstyle\int_0^T \langle h(t), h'(t) - Ah(t)\rangle_{L^2(0,2\pi)}dt \\ &= \textstyle\int_0^T \langle h(t), h'(t)\rangle_{L^2(0,2\pi)}dt - \int_0^T \langle h(t), Ah(t)\rangle_{L^2(0,2\pi)}dt \\ &= \tfrac{1}{2}\|h(T)\|^2_{L^2(0,2\pi)} \geq 0 \text{ by } (3.2.2).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\langle Nw - Nv, w - v\rangle_X &= \textstyle\int_0^T \langle Nw(t) - (Nv)(t), w(t) - v(t)\rangle_{L^2(0,2\pi)}dt \\ &= \textstyle\int_0^T \int_0^{2\pi} \Big[f(w(x,t)) - f(v(x,t))\Big]\Big[w(x,t) - v(x,t)\Big]dxdt \\ &\geq -\beta \textstyle\int_0^T \int_0^{2\pi} |w(x,t) - v(x,t)|^2 dxdt \\ &= -\beta\|w - v\|^2_X.\end{aligned}$$

Therefore $-N$ is a strongly monotone operator with monotonicity constant β. Also, hypothesis $[f2]$ implies that N satisfies a growth condition. So the lemma follows along the same lines of Lemma 2.2 of George [60]. ∎

LEMMA 3.2.2 *Suppose that* f *satisfies :*
$[f3]$ *There exists a constant* $\alpha > 0$ *such that for all* $r, s, \in R$

$$|f(r) - f(s)| \leq \alpha|r - s|.$$

Then W *is well-defined and continuous.*

Proof: By using $[f3]$ in (3.2.5)-(3.2.7), it can be shown easily that $[KN]^n$ is a contraction for sufficiently large $n \geq 1$. Therefore, by generalized contraction principle the equation (3.2.8) has a unique solution for each given u. This proves the Lemma.∎

The solution operator W is well defined can also be obtained by using sub-gradient estimate of f which we denote by Df. Our next lemma gives conditions on f in terms of its sub-gradient Df.
The sub-gradient $Df(x)$ of f at a point $x \in B_r(0) \stackrel{def}{=} \{x \in R : |x| < r\}$, $r > 0$ is defined as

$$Df(x) = \Big\{p \in R : f(y) - f(x) \geq p(y - x) - o(|y - x|) \text{ as } y \to x\Big\}.$$

This implies that

$$Df(x) = \{p \in R : \liminf_{y \to x} \frac{f(y) - f(x) - p(y-x)}{|y-x|} \geq 0\}. \tag{3.2.9}$$

If $Df(x) = \phi$ or $f(x) = \infty$, then Df does not exists at x. As an example, take $f(x) = |x|$, then f is differentiable for all $x \neq 0$ and in this case

$$f'(x) = Df(x) = \begin{cases} \{1\} & \text{if } x > 0 \\ \{-1\} & \text{if } x < 0. \end{cases}$$

Further $Df(0) = [-1, 1]$. On the other hand, one can easily see that for the function $g(x) = -|x|$, the sub-gradient $Dg(0)$ does not exists (refer [118]).

We can also define a second order sub-gradient $D^2 f$ by using second order approximation as:

$$D^2 f(x) = \left\{ p \in Df(x) : \liminf_{y \to x} \frac{f(y) - f(x) - p(y-x)}{|y-x|^2} > -\infty \right\}. \tag{3.2.10}$$

Obviously,

$$D^2 f(x) \subset Df(x).$$

For $f(x) = -|x|^\alpha$, $1 < \alpha < 2$, we have $f'(x) = Df(0) = \{0\}$, but $D^2 f(0) = \phi$.

LEMMA 3.2.3 *Suppose that for some $r > 0$*

1. $|D^2 f(x)| \leq \alpha$ *at every point* $x \in B_r(0)$, *where* $D^2 f(x)$ *exists.*
2. $f(x_0) < \infty$ *for some* $x_0 \in B_{\frac{r}{4}}(0)$.
3. f *satisfies* $[f2]$ *of Lemma 3.2.1.*

Then the solution operator W is well-defined.

Proof: From Theorem 1 of Redherffer and Walter [118], it follows that f is locally Lipschitz continuous. Because of the local Lipschitz continuity, there exists a unique local solution to the equation (3.2.8) in a maximal interval $[0, t_{\max}], t_{\max} \leq T$. If $t_{\max} < T$, then $\lim_{t \to t_{\max}} \|w(t)\|_{L^2(0,2\pi)} = \infty$ (see Tanabe [131]). In other words, if $\lim_{t \to t_{\max}} \|w(t)\|_{L^2(0,2\pi)} < \infty$, then there exists a unique solution in the interval $[0, T]$. Now, $[f2]$ with an application of Grownwall's inequality implies that $\|w(.)\|_{L^2(0,2\pi)} < \infty$ for each u and therefore, w exists on $[0, T]$. Hence, W is well-defined. ∎

REMARK 3.2.4 *In the above lemma, we do not require differentiability of f. If f is differentiable then $Df(x)$ reduces to $f'(x)$.* ∎

REMARK 3.2.5 *If W is well-defined and f satisfies [f2], then it is a trivial matter to see using Grownwall's inequality that*

$$\|Wu\|_X \leq C_1\|u\|_X + C_2, \tag{3.2.11}$$

where C_1, C_2 are positive constants which can be explicitly determined in terms of $T, a, b, M, \|G\|$. ∎

REMARK 3.2.6 *In case, if f is Lipschitz continuous, then W is also Lipschitz continuous. This also can be seen by the same arguments. So, there exists a constant C_3 such that*

$$\|Wu - Wv\|_X \leq C_3\|u - v\|_X. \tag{3.2.12}$$

∎

3.3 Reduction of Controllability Problem into Solvability Problem

Define an operator $L : X \longrightarrow L^2(0, 2\pi)$ by

$$(Lu)(x) = \int_0^T \Phi(T-s)(Gu)(x,s)ds. \tag{3.3.1}$$

By Theorem 3.1.2 (linear controllability), the bounded linear operator L is onto. Therefore, for every $w_T \in L^2(0, 2\pi)$, there exists a control $u \in X$ such that

$$w_T = Lu. \tag{3.3.2}$$

Let $N(L)$ be the null space of L, then $X = N(L) \oplus [N(L)]^{\perp}$. Thus $L^{\#}$, the pseudo-inverse of L exists and is defined by

$$L^{\#} = \left(L|_{[N(L)]^{\perp}}\right)^{-1} : Y \stackrel{\text{def}}{=} L^2(0, 2\pi) \longrightarrow [N(L)]^{\perp}$$

such that

$$\begin{aligned} LL^{\#} &= I \\ L^{\#}L &= P_T \stackrel{\text{def}}{=} \text{orthogonal projection of } X \text{ on } [N(L)]^{\perp} \\ L^{\#}L &= I \text{ over } [N(L)]^{\perp}. \end{aligned}$$

So one obtains a unique $\mu \in [N(L)]^{\perp}$ such that $L\mu = w_T$. If μ is found, then any $u \in X$ such that $P_T u = \mu$ will yield $Lu = w_T$. Now define $F : X \longrightarrow Y$ by

$$Fu = \int_0^T \Phi(T-s) f\Big((Wu)(.,s)\Big) ds. \tag{3.3.3}$$

Thus, it follows that the system (3.1.1)-(3.1.3) is exactly controllable if for every $w_T \in Y$, there exists a solution $u \in X$ for the equation:

$$w_T = Lu + Fu \tag{3.3.4}$$

Applying $L^{\#}$ to (3.3.4) with u is replaced by μ, we get

$$L^{\#} w_T = \mu + L^{\#} F \mu. \tag{3.3.5}$$

The above discussions lead us to the following Lemma.

LEMMA 3.3.1 *Suppose that the equation (3.3.5) has a solution μ for every $w_T \in Y$, then the system (3.1.1)-(3.1.3) is exactly controllable.* ∎

3.4 Main Results

By using the ideas from the previous sections, we now able to prove our main results.

THEOREM 3.4.1 *Suppose that the non-linear function f satisfies $[f3]$ and the Lipschitz constant α is sufficiently small, then the non-linear system (3.1.1)-(3.1.3) is exactly controllable.*

Proof: By Lemma 3.2.2 and Remark 3.2.6, W is well-defined and Lipschitz continuous with Lipschitz constant C_3. Hence with a simple calculation, one can get F is Lipschitz. In fact

$$\|Fu - Fv\| \leq M\alpha C_3 \|u - v\|.$$

Thus, since $L^\#$ is a bounded linear operator, we observe that $L^\# F$ is a contraction if

$$\alpha < \frac{1}{M\alpha C_3 \|L^\#\|}.$$

Therefore, by contraction principle, equation (3.3.5) has a unique solution. So direct application of Lemma 3.3.1 completes the proof. ∎

When W is well-defined and compact, we obtain the following results where we assume monotonicity condition of f rather than Lipschitz condition. Note that compactness of W can be obtained by many ways (see conditions given in Lemma 4 of Naito and Seidman [106] to assure that W is compact).

THEOREM 3.4.2 *Assume that*

1. *conditions* $[f1], [f2]$ *hold true.*
2. W *is compact.*
3. *the growth constant* a *in* $[f2]$ *is sufficiently small.*

Then the non-linear system (3.1.1)-(3.1.3) is exactly controllable.

Proof: In view of Lemma 3.2.1 and Lemma 3.3.1, we look for the solvability of (3.3.5).
Define an operator $R : [N(L)]^\perp \longrightarrow [N(L)]^\perp$ by

$$R\mu = [I + L^\# F]\mu.$$

Therefore, we have

$$\langle R\mu, \mu\rangle = \|\mu\|^2 + \langle L^\# F\mu, \mu\rangle.$$

We may easily estimate:

$$\|F\mu\| \leq C_1 a\|\mu\| + C_2.$$

Using the Cauchy-Schwartz inequality, we get

$$\langle L^\# F\mu, \mu\rangle \geq -\|L^\#\|\|F\mu\|\|\mu\| \geq -aC_1\|L^\#\|\|\mu\|^2 - C_2\|L^\#\|\|\mu\|.$$

Thus

$$\frac{\langle R\mu, \mu\rangle}{\|\mu\|} \geq (1 - aC_1\|L^\#\|)\|\mu\| - C_2\|L^\#\|.$$

Hence, if a such that $aC_1\|L^{\#}\| < 1$, then it follows that $lim_{\|\mu\|\to\infty}\frac{(R\mu,\mu)}{\|\mu\|} = \infty$. Therefore, R is a coercive operator. Further, the compactness of W implies that $L^{\#}F$ is also compact. Thus R is a compact perturbation of a strongly monotone operator and hence it is of type (M) (see pp 79 of Joshi and Bose [78]). So by Theorem 3.6.9 of [78], the non-linear mapping R is onto. This proves the theorem. ∎

COROLLARY 3.4.3 *If we replace condition (1) of Theorem 3.4.2 by Assumptions (1) and (2) of Lemma 3.2.3, then also conclusions of Theorem 3.4.2 hold true.* ∎

COROLLARY 3.4.4 *As a particular case, Theorem 3.4.2 and Corollary 3.4.3 hold true if f is uniformly bounded, that is if there exists a positive constant $b > 0$ such that $|f(r)| \leq b$, for all $r \in R$.* ∎

In the following section, we assume a weaker notion on the nonlinear function known as **integral contractors**. This notion was developed (see [3]) as a generalization of inverse derivative. We will see that under this condition, the solution operator W is well defined and system (3.1.1)-(3.1.3) is exactly controllable.

3.5 Existence and uniqueness of the operator W by the method of Integral Contractors

The notion of integral contractor was first introduced by Altman [3] and later on it was used by many authors to study the existence and uniqueness of solution of non-linear evolution systems. In simple terms, various methods of solving non-linear equations can be unified by the single concept of contractors.

Here, we would like to weaken Lipschitz continuity of f by the bounded integral contractor and then study the exact controllability of the system (3.1.1)-(3.1.3) as in Section 3.4.

Let $C = C([0,T]; L^2(0,2\pi))$ denote the Banach space of continuous functions on $J = [0,T]$ with values in (L^2) with the standard norm $\|w\|_C = \sup_{0\leq t\leq T}\|w(t)\|_{L^2(0,2\pi)}$. Define the solution operator $W : X \longrightarrow C$ by $(Wu)(t) = w(.,t)$, where $w(.,t)$ is the unique solution of the nonlinear integral equation (3.2.4).

Now for the concept of integral contractors, we may refer Chapter 2.

REMARK 3.5.1 *If Γ is a contractor defined on $J \times L^2(0, 2\pi)$, then it remains as a contractor in $[0, s] \times L^2(0, 2\pi)$, for any s. In other words, by taking $w, y \in C([0, s] \times L^2(0, 2\pi))$ and extending $w(t) = w(s), y(t) = y(s)$ for all $T \geq t \geq s$ we get*

$$\begin{aligned}
&\sup_{0\leq t\leq s} \|f\Big(t, w(t) + y(t) + \textstyle\int_0^t \Phi(t-s)(\Gamma(s, w(s))y)(s)ds\Big) \\
&\qquad -f(t, w(t)) - (\Gamma(t, w(t))y)(t)\|_{L^2(0,2\pi)} \\
&\leq \sup_{0\leq t\leq T} \|f\Big(t, w(t) + y(t) + \textstyle\int_0^t \Phi(t-s)(\Gamma(s, w(s))y)(s)ds\Big) \\
&\qquad -f(t, w(t)) - (\Gamma(t, w(t))y)(t)\|_{L^2(0,2\pi)} \\
&\leq \gamma\|y\|_C \leq \gamma\|y\|_{C([0,s];X)}.
\end{aligned}$$

■

REMARK 3.5.2 *We know that the Lipschitz condition gives the unique solution of the given system (3.1.1)-(3.1.3), but the condition given in the definition of integral contractor may not give the uniqueness of the solution operator W. The uniqueness of W is ensured by the regularity of the integral contractor.* ■

DEFINITION 3.5.3 *A bounded integral contractor Γ is said to be regular if the integral equation*

$$y(t) + \int_0^t \Phi(t-s)(\Gamma(s, w(s))y)(s)ds = z(t) \tag{3.5.1}$$

has a solution y in C for every $w, z \in C$.

We denote $\beta = \sup\Big\{\|\Gamma(t, w(t))\| : t \in J, w \in C\Big\}$. Observe that, if $f(t, w(x, t))$ is Lipschitz continuous uniformly in t, then it has a regular integral contractor $\{I\}$ with $\Gamma \equiv 0$. Refer Altman [3] for other sufficient conditions for the existence of a bounded integral contractor for f.

We now prove the existence and uniqueness theorem by using integral contractors.

THEOREM 3.5.4 *Suppose that (3.2.8) is satisfied and the nonlinear function f has a regular integral contractor Γ. Then, the solution operator $W : X \longrightarrow C$ is well defined and is Lipschitz continuous. That is there is a constant $k > 0$ such that*

$$\|Wu_1 - Wu_2\|_C \leq k\|u_1 - u_2\|_X. \tag{3.5.2}$$

Proof: We use the following iteration procedure to construct the sequences $\{w_n\}$ and $\{y_n\}$ in C. Define for $n = 0, 1, 2 \cdots$

$$\left.\begin{aligned} w_0(t) &= \int_0^t \Phi(t-s)(Gu)(s)ds \\ y_n(t) &= w_n(t) - \int_0^t \Phi(t-s)f(s, w_n(s))ds - w_0(t) \\ w_{n+1}(t) &= w_n(t) - \left[y_n(t) + \int_0^t \Phi(t-s)\Gamma(s, w_n(s))y_n(s)ds\right] \\ &= \int_0^t \Phi(t-s)f(s, w_n(s))ds - \int_0^t \Phi(t-s)\Gamma(s, w_n(s))y_n(s)ds. \end{aligned}\right\} \quad (3.5.3)$$

Substituting for w_{n+1} in y_{n+1}, we can write using the above equation as

$$\begin{aligned} y_{n+1}(t) &= \int_0^t \Phi(t-s)f(s, w_n(s)) - \int_0^t \Phi(t-s)\Gamma(s, w_n(s))y_n(s)ds \\ &\quad - \int_0^t \Phi(t-s)\Big[f(s, w_n(s) - y_n(s) - \int_0^s \Phi(s-\tau)\Gamma(\tau, w_n(\tau))y_n(\tau)d\tau\Big)ds \\ &= -\int_0^t \Phi(t-s)[\big[f(s, w_n(s) - y_n(s) - \int_0^s \Phi(s-\tau)\Gamma(\tau, w_n(\tau))y_n(\tau)d\tau) \\ &\qquad -f(s, w_n(s)) + \Gamma(s, w_n(s))y_n(s)\big]ds. \end{aligned}$$

Applying the Definition 2.2.4 (see Remark 3.5.1) with $w = w_n$ and $y = -y_n$, we get

$$\|y_{n+1}(t)\|^2_{L^2(0,2\pi)} \le M^2\gamma^2 t \sup_{0\le s\le t} \|y_n(s)\|^2_{L^2(0,2\pi)}. \quad (3.5.4)$$

A slightly modified application yields:

$$\begin{aligned} \|y_{n+1}(t)\|^2_{L^2(0,2\pi)} &\le M^2\gamma^2 \int_0^t \sup_{0\le\tau\le s} \|y_n(\tau)\|^2_{L^2(0,2\pi)}ds \\ &\le M^4\gamma^4 \int_0^t s\|y_{n-1}\|^2_{C([0,s]];L^2(0,2\pi))} \\ &\le M^4\gamma^4 \tfrac{t^2}{2}\|y_{n-1}\|^2_{C([0,T]];L^2(0,2\pi))}, \end{aligned}$$

where, the second inequality was obtained by applying (3.5.4) with n replaced by $n-1$. Repeating the above argument successively, we get

$$\|y_{n+1}(t)\|^2_{L^2(0,2\pi)} \le \tfrac{(MT\gamma)^{n+1}}{(n+1)!}\|y_0\|^2_{C([0,T]];L^2(0,2\pi))}.$$

This shows that $y_n(t)$ converges to 0 in C and hence in$X$ as $n \longrightarrow \infty$. We now show that w_n converges to the solution of the system (3.1.1)-(3.1.3).. To see this, we write

$$w_{n+1}(t) - w_n(t) = -y_n(t) - \int_0^t \Phi(t-s)\Gamma(s, w_n(s))y_n(s)ds.$$

One can easily estimate

$$\|w_{n+1} - w_n\|_C \leq k_1 \frac{(MT\gamma)^n}{n!},$$

and thus

$$\|w_{n+m} - w_n\|_C \leq k_2 \sum_{k=n}^{n+m-1} \frac{(MT\gamma)^k}{k!},$$

where k_1, k_2 are arbitrary constants. The right hand side being the tail of a convergent series, we deduce that w_n is Cauchy and hence it converges to, say, w' in C. Now passing to the limit in the second equation in (3.5.3), we get

$$w'(t) = \int_0^t \Phi(t-s)(Gu)(s)ds + \int_0^t \Phi(t-s)f(s, w'(s))ds.$$

Therefore w' is a mild solution of the system (3.1.1)-(3.1.3) in the sense of (3.2.4).

Now the uniqueness can be shown with the help of regularity of the integral contractor. Let w_1 and w_2 be two solutions of (3.1.1)-(3.1.3) with a given Gu. By the regularity condition (3.5.1) with $w = w_1$ and $z = w_2 - w_1$, there exists a $y \in C$ such that

$$y(t) + \int_0^t \Phi(t-s)\Gamma(s, w_1(s))y(s)ds = w_2(t) - w_1(t). \tag{3.5.5}$$

Applying the definition of integral contractor with $w = w_1$ and using the above equation, we get

$$\|f(t, w_2(t)) - f(t, w_1(t)) - \Gamma(t, w_1(t))y(t)\|_C \leq \gamma \|y\|_C. \tag{3.5.6}$$

As w_1 and w_2 are solutions of (3.2.4), the equation (3.5.5) yields:

$$\begin{aligned} y(t) = & \; w_2(t) - w_1(t) - \int_0^t \Phi(t-s)\Gamma(s, w_1(s))y(s)ds \\ = & \int_0^t \Phi(t-s)\big[f(s, w_2(s)) - f(s, w_1(s))\big]ds \\ & - \int_0^t \Phi(t-s)\Gamma(s, w_1(s))y(s)ds \\ = & \int_0^t \Phi(t-s)\big[f(s, w_2(s)) - f(s, w_1(s)) - \Gamma(s, w_1(s))y(s)\big]ds. \end{aligned}$$

Thus, we get

$$\|y(t)\|^2 \leq M^2\gamma^2 \int_0^t \sup_{0 \leq \tau \leq s} \|y(\tau)\|^2 ds.$$

Hence

$$\sup_{0\le\tau\le t}\|y(\tau)\|^2 \le M^2\gamma^2\int_0^t \sup_{0\le\tau\le s}\|y(\tau)\|^2 ds.$$

By Grownwall's inequality, we see that $y(t)\equiv 0$. Thus $w_1 = w_2$, establishing the well-definedness of the solution operator W.

We now prove that solution operator W is Lipschitz continuous. Let $u_1, u_2 \in X$ and w_1 and w_2 be the corresponding solution of (3.2.4); i. e., $Wu_1 = w_1$ and $Wu_2 = w_2$. By the regularity of the integral contractor, there exists $y \in C$ such that

$$(Wu_2)(t) = (Wu_1)(t) + y(t) + \int_0^t \Phi(t-s)\Gamma(s,(Wu_1)(s))y(s)ds. \tag{3.5.7}$$

Thus by the same arguments as earlier, it is easy to get the following estimate:

$$\|Wu_2 - Wu_1\|_C \le k\|y\|_C, \tag{3.5.8}$$

for some constant k. As Wu_1 and Wu_2 are solutions of (3.2.4), we get

$$\begin{aligned}(Wu_2)(t)-(Wu_1)(t) = &\int_0^t \Phi(t-s)\big[f(s,(Wu_2)(s)) - f(s,(Wu_1)(s))\big]ds\\ &+\int_0^t \Phi(t-s)\Big[(Gu_2)(s)-(Gu_1)(s)\Big]ds,\end{aligned}$$

which implies from (3.5.7) that

$$\begin{aligned}y(t) = &\int_0^t \Phi(t-s)\big[f(s,(Wu_2)(s)) - f(s,(Wu_1)(s)) - \Gamma(s,(Wu_1)(s))y(s)\big]ds\\ &+\int_0^t \Phi(t-s)\Big[(Gu_2)(s)-(Gu_1)(s)\Big]ds.\end{aligned}$$

Again applying the definition of contractors, we get

$$\sup_{0\le\tau\le t}\|y(\tau)\|^2_{L^2(0,2\pi)} \le C_1\int_0^t \sup_{0\le\tau\le s}\|y(\tau)\|^2_{L^2(0,2\pi)}ds + C_2\|Gu_2 - Gu_1\|^2_C.$$

By Grownwall's inequality , we have

$$\sup_{0\le\tau\le t}\|y(\tau)\|^2_{L^2(0,2\pi)} \le C_3\|Gu_2 - Gu_1\|^2_C.$$

Thus, (3.5.8) shows that

$$\|Wu_2 - Wu_1\|_C \le C_4\|u_2 - u_1\|_C.$$

Here C_1, C_2, C_3, C_4 are constants. This completes the proof. ∎

3.6 Controllability via Integral Contractor Method

THEOREM 3.6.1 *Suppose that nonlinear function f has a regular bounded integral contractor $\{I + \int \Phi\Gamma\}$ and γ as in Definition (2.2.4), is sufficiently small. Then the nonlinear system (3.1.1)-(3.1.3) is exactly controllable.*

Proof: By Theorem 3.5.4, the solution operator W is well defined and Lipschitz continuous. Therefore W has a integral contractor $\{I + \int \Phi\Gamma\}$. Hence, F defined by the equation (3.3.3) also has an integral contractor. Since $L^{\#}$ is a bounded linear operator, we observe that $L^{\#}F$ has a bounded integral contractor if

$$\|L^{\#}\| \ \|G\|\gamma M^2 T(1 + \beta MT)e^{\gamma MT} < 1$$

Hence equation (3.3.5) has a unique solution by using the contraction principle. Finally the application of Lemma (3.3.1) proves the exact controllability. ∎