

## Chapter 4

# Controllability of Matrix Second Order Systems: A General Operator Approach

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In this chapter we generalize the concept of the previous chapter by taking a general operator approach instead of trigonometric matrix approach. We investigate controllability property of a class of semilinear system described by matrix second order differential equation. We first obtain necessary and sufficient condition for controllability for the linear system and subsequently provide sufficient conditions on the nonlinear function, so that the nonlinear system is also controllable. Our result on linear system generalizes the work of Hughes and Skelton[42] whereas the

controllability result for the nonlinear system is new. We make use of two special type of matrices  $\Phi$  and  $\Psi$ , which behave like Sine and Cosine matrices, to reduce the system into integral formulation. We employ tools of nonlinear analysis like fixed point theorem to obtain controllability results. In Section 4.1, we provide the introduction to the problem, Section 4.2 deals with the controllability of linear system. In Section 4.3, we prove the controllability result of the nonlinear system. A summary of the chapter is given in section 4.4.

## 4.1 Introduction

Controllability of linear and nonlinear systems represented by first order differential equations has been extensively studied by many authors both in finite and infinite dimensional set-up (George and Joshi [46]). Here our aim is to study the controllability of finite dimensional semilinear system described by matrix second order differential equations. We have treated the second order systems directly rather than converting them to first order systems as discussed in chapter 3.

We consider the control system described by the matrix second order nonlinear differential equation

$$\left. \begin{aligned} \frac{d^2 x(t)}{dt^2} + Ax(t) &= B(t)u(t) + f(t, x(t)) \\ x(0) &= x_0, \quad x'(0) = y_0. \end{aligned} \right\} \quad (4.1.1)$$

where, the state  $x(t) \in R^n$ , the control  $u(t) \in R^m$ ,  $A$  is matrix of order  $n \times n$ ,  $B$  is a matrix of order  $n \times m$  and  $f : [0, T] \times R^n \rightarrow R^n$  is a nonlinear function. The initial states  $x_0$  and  $y_0$  are in  $R^n$ . The homogeneous second order system is given by:

$$\left. \begin{aligned} \frac{d^2 x(t)}{dt^2} + Ax(t) &= 0 \\ x(0) &= x_0, \quad x'(0) = y_0. \end{aligned} \right\} \quad (4.1.2)$$

It follows easily that the equation (4.1.2) has a unique solution  $x(\cdot)$  passing through the initial conditions  $x(0) = x_0$  and  $x'(0) = y_0$ .

The following propositions follow easily from standard arguments.

**Proposition 4.1.1.** *If  $\phi_1(t), \phi_2(t), \dots, \phi_n(t)$  are solutions of (4.1.2) with initial conditions  $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$  then their linear combination*

$$\phi(t) = \sum_{i=1}^n \alpha_i \phi_i(t), \quad \alpha_i \in R$$

*is also a solution of (4.1.2) with initial conditions*

$$\phi(0) = \sum_{i=1}^n \alpha_i x_i \quad \text{and} \quad \phi'(0) = \sum_{i=1}^n \alpha_i y_i$$

**Proposition 4.1.2.** (i) *If  $\phi$  is a solution of (4.1.2), then  $\phi'$  is also a solutions of (4.1.2).*

(ii) *Let  $\phi_1(t), \phi_2(t), \dots, \phi_n(t)$  be solution of (4.1.2) on  $[0, T]$  and  $s \in [0, T]$ , then*

$$\{\phi_1(s), \phi_2(s), \dots, \phi_n(s)\}$$

*is a set of linearly independent vectors in  $R^n$ .*

We define two matrices

$$\Phi = [\phi_1(t), \phi_2(t), \dots, \phi_n(t)], \quad 0 \leq t \leq T$$

and

$$\Psi = [\phi'_1(t), \phi'_2(t), \dots, \phi'_n(t)], \quad 0 \leq t \leq T$$

where  $\phi_1, \phi_2, \dots, \phi_n$  are linearly independent solutions of (4.1.2) satisfying  $\phi_i(0) = \phi''_i(0) = e_i$ , where  $\{e_1, e_2, \dots, e_n\}$  is the canonical basis for  $R^n$  and  $\phi'_i(0) = 0$ .

The matrices  $\Phi$  and  $\Psi$  satisfy the following properties:

(1)  $\Phi$  and  $\Psi$  are solutions of the matrix differential equation

$$\frac{d^2 X(t)}{dt^2} + A(t)X(t) = 0.$$

That is,  $\ddot{\Phi}(t) = -A(t)\Phi(t)$  and  $\ddot{\Psi}(t) = -A(t)\Psi(t)$

(2)  $\Phi(0) = I, \quad \dot{\Phi}(0) = 0$

$$(3) \quad \Psi(0) = 0, \quad \dot{\Psi}(0) = I$$

Now we shall consider the controlled linear system:

$$\left. \begin{aligned} \frac{d^2 x(t)}{dt^2} + Ax(t) &= B(t)u(t) \\ x(0) = x_0, \quad x'(0) &= y_0. \end{aligned} \right\} \quad (4.1.3)$$

**Proposition 4.1.3.** *The unique solution of the system (4.1.3) is given by*

$$x(t) = \Phi(t)x_0 + \Psi(t)y_0 + \int_0^t \Psi(t-s)B(s)u(s)ds. \quad (4.1.4)$$

*Proof.* We shall show that equation (4.1.4) satisfies the equation (4.1.3). Differentiating the equation (4.1.4), we have

$$x'(t) = \Phi'(t)x_0 + \Psi'(t)y_0 + \int_0^t \Psi'(t-s)B(s)u(s)ds + \Psi(0)B(t)u(t)$$

But  $\Psi(0) = 0$ , implies that

$$x'(t) = \Phi'(t)x_0 + \Psi'(t)y_0 + \int_0^t \Psi'(t-s)B(s)u(s)ds$$

Differentiating again with respect to  $t$ , we get

$$x''(t) = \Phi''(t)x_0 + \Psi''(t)y_0 + \int_0^t \Psi''(t-s)B(s)u(s)ds + \Psi'(0)B(t)u(t)$$

Since,  $\Phi''(t) = -A(t)\Phi(t)$ ,  $\Psi''(t) = -A(t)\Psi(t)$  and  $\Psi'(0) = I$

$$x''(t) = -A(t)\Phi(t)x_0 - A(t)\Psi(t)y_0 + \int_0^t -A(t)\Psi(t-s)B(s)u(s)ds + B(t)u(t)$$

$$x''(t) = -A(t)[\Phi(t)x_0 + \Psi(t)y_0 + \int_0^t \Psi(t-s)B(s)u(s)ds] + B(t)u(t)$$

From the equation (4.1.4), we have

$$x''(t) = -A(t)x(t) + B(t)u(t)$$

Also,  $x(0) = x_0$  and  $x'(0) = y_0$ . Thus (4.1.4) is the unique solution of the system (4.1.3)  $\square$

Following the same approach the solution of Matrix Second Order System(4.1.1) can be written as

$$x(t) = \Phi(t)x_0 + \Psi(t)y_0 + \int_0^t \Psi(t-s)B(s)u(s)ds + \int_0^t \Psi(t-s)f(s, x(s))ds \quad (4.1.5)$$

## 4.2 Controllability: Linear System

In this section we obtain the necessary and sufficient conditions for the controllability of the linear system(4.1.3). We need the Lemma 3.3.1 to prove the controllability result for the system (4.1.3).

The controllability Grammian of the linear control system (4.1.3) is given by:

$$W(0, T) = \int_0^T \Psi(T-s)B(s)B(s)^*\Psi^*(T-s)ds.$$

The necessary and sufficient condition for the controllability of the linear system (4.1.3) is given in the following theorem.

**Theorem 4.2.1.** *The linear system (4.1.3) is controllable on  $[0, T]$  if and only if  $W(0, T) = \int_0^T \Psi(T-s)B(s)B(s)^*\Psi^*(T-s)ds$  is non-singular.*  $\square$

*Proof.* Let us suppose that

$$W(0, T) = \int_0^T \Psi(T-s)B(s)B(s)^*\Psi^*(T-s)ds$$

is nonsingular. Now we claim that the control

$$u(t) = B(t)^*\Psi^*(T-t)W^{-1}(0, T)(x_1 - \Phi(T)x_0 - \Psi(T)y_0) \quad (4.2.1)$$

transfers the initial state  $x_0$  to the final state  $x_1$ , during  $[0, T]$ . Substituting  $u(t)$  given in (4.2.1) in the solution (4.1.4), we obtain

$$x(t) = \Phi(t)x_0 + \Psi(t)y_0 + \int_0^t \Psi(t-s)B(s)B(s)^*\Psi^*(T-s)W^{-1}(0, T) \\ (x_1 - \Phi(T)x_0 - \Psi(T)y_0)ds$$

At time  $t=T$ , we have

$$\begin{aligned}
x(T) &= \Phi(T)x_0 + \Psi(T)y_0 + \int_0^T \Psi(T-s)B(s)B(s)^*\Psi^*(T-s)W^{-1}(0,T) \\
&\quad (x_1 - \Phi(T)x_0 - \Psi(T)y_0)ds \\
&= \Phi(T)x_0 + \Psi(T)y_0 + \int_0^T \Psi(T-s)B(s)B(s)^*\Psi^*(T-s)dsW^{-1}(0,T) \\
&\quad (x_1 - \Phi(T)x_0 - \Psi(T)y_0) \\
&= \Phi(T)x_0 + \Psi(T)y_0 + W(0,T)W^{-1}(0,T)(x_1 - \Phi(T)x_0 - \Psi(T)y_0) \\
&= \Phi(T)x_0 + \Psi(T)y_0 + (x_1 - \Phi(T)x_0 - \Psi(T)y_0) \\
&= x_1
\end{aligned}$$

Also, at time  $t=0$ ,  $x(0) = \Phi(0)x_0 + \Psi(0)y_0 = x_0$ . Hence the system is controllable. We prove the converse, by contradiction. Suppose that the system (4.1.3) is controllable but  $W(0,T)$  is singular. That is, by Lemma 3.3.1 the rows of  $\Psi(t)B(t)$  are linearly dependent functions on  $[0,T]$ . Hence there exists a nonzero constant  $1 \times n$  row vector  $\alpha$  such that

$$\alpha\Psi(t)B(t) = 0 \quad \forall t \in [0, T].$$

Let us choose  $x(0) = x_0 = 0$ ,  $x'(0) = y_0 = 0$ . Therefore the solution (4.1.4) becomes

$$x(t) = \int_0^t \Psi(T-s)B(s)u(s)ds.$$

Since the system (4.1.3) is controllable on  $[0,T]$ , taking  $x(T) = \alpha^*$ .

$$x(T) = \alpha^* = \int_0^T \Psi(T-s)B(s)u(s)ds$$

Now premultiplying both side by  $\alpha$ , we have

$$\alpha\alpha^* + \int_0^T \alpha\Psi(T-s)B(s)u(s)ds = 0$$

$$\alpha\alpha^* = 0$$

It gives  $\alpha = 0$ . Hence it contradicts our assumption. Hence  $W(0,T)$  is non-singular.  $\square$

Now we investigate the controllability property of the nonlinear system (4.1.1).

### 4.3 Controllability: Nonlinear System

The integral representation of the nonlinear system (4.1.1) is given in the equation (4.1.5). For studying the controllability of the nonlinear system (4.1.1), we assume that the linear system (4.1.3) is controllable and the control function  $u$  is from  $L^2([0, T], R^m)$ . We make use of the following definitions. Recalling from chapter 2, the control operator  $C : L^2([0, T]; R^m) \rightarrow R^n$  of the linear system (4.1.3) is given by

$$Cu = \int_0^T \Psi(T-s)B(s)u(s)ds \quad (4.3.1)$$

Again from chapter 2, an  $m \times n$  matrix function  $P(t)$  with entries in  $L^2([0, T])$  is said to be a steering function for (4.1.3) on  $[0, T]$  if

$$\int_0^T \Psi(t-s)B(s)P(s)ds = I, \quad I \text{ being the identity matrix in } R^n$$

We now give different assumptions to ensure the controllability of the nonlinear system (4.1.1).

#### Assumptions:

[K1]  $\{\int_0^T \|\Psi(t)\|^2 dt\}^{\frac{1}{2}} = k < \infty.$

[B1] Let  $b = \sup_{t_0 \leq t \leq T} \|B(t)\| < \infty.$

[f1] The nonlinear function  $f$  satisfies Caratheodory conditions, that is,  $x \rightarrow f(., x)$  is continuous for almost all  $t$ ,  $t \rightarrow f(t, .)$  is measurable for almost all  $x$ .

[f2] The nonlinear function  $f$  is Lipchitz continuous, that is, there exists a constant  $\alpha > 0$  such that

$$\|f(t, x) - f(t, y)\| \leq \alpha \|x - y\| \quad \forall x, y \in R^n \text{ and } t \in [0, T].$$

Since the linear system (4.1.3) is controllable, then by Theorem 2.1.3 there exists a steering function  $P(t)$  for the linear system. Consider the control  $u(t)$  define by

$$u(t) = P(t)\{x_1 - \Phi(T)x_0 - \Psi(T)y_0 - \int_0^T \Psi(T-s)f(s, x(s))ds\} \quad (4.3.2)$$

where,  $x(t)$  is the unique solution of (4.1.5) with control  $u(t)$ . Now substituting this control  $u(t)$  into equation (4.1.5), we have

$$\begin{aligned} x(t) = & \Phi(t)x_0 + \Psi(t)y_0 + \int_0^t \Psi(T-s)f(s, x(s))ds + \int_0^t \Psi(T-s)B(s)\{P(s) \\ & \{x_1 - \Phi(T)x_0 - \Psi(T)y_0 - \int_0^T \Psi(T-\tau)f(\tau, x(\tau))d\tau\}ds\} \end{aligned} \quad (4.3.3)$$

If the equation (4.3.3) is solvable then  $x(t)$  satisfies  $x(0) = x_0$  and  $x(T) = x_1$ . This implies that the system (4.1.5) is controllable with control  $u(t)$  given by (4.3.2). Hence, controllability of the system (4.1.5) is equivalent to the solvability of the equation (4.3.3). Now applying the Banach contraction principle, we obtain the solvability of the equation (4.3.3).

Now we define a mapping  $F : C([0, T]; R^n) \rightarrow C([0, T]; R^n)$  by

$$\begin{aligned} (Fx)(t) = & \Phi(t)x_0 + \Psi(t)y_0 + \int_0^t \Psi(T-s)f(s, x(s))ds + \int_0^t \Psi(T-s)B(s)\{P(s) \\ & \{x_1 - \Phi(T)x_0 - \Psi(T)y_0 - \int_0^T \Psi(T-\tau)f(\tau, x(\tau))d\tau\}ds\} \end{aligned} \quad (4.3.4)$$

We prove that  $F$  is a contraction in the following Lemma:

**Lemma 4.3.1.** *Suppose that  $B(t)$ ,  $\Psi(t)$  and  $f$  satisfy the assumptions [K1], [B1], [f1] and [f2]. Let the steering operator  $P(t)$  satisfies  $\|P(t)\| \leq p$  and  $\alpha mT(1 + mbpT) < 1$ . Then  $F$  is a contraction on  $C([0, T]; R^n)$ .*

*Proof.* By definition

$$\begin{aligned} & \|Fx - Fy\|_{C([0, T]; R^n)} \\ &= \sup_{t \in [0, T]} \|(Fx)(t) - (Fy)(t)\| \\ &= \sup_{t \in [0, T]} \left\| \int_0^t \Psi(T-s)(f(s, x(s)) - f(s, y(s)))ds + \int_0^t \Psi(T-s) \right. \\ & \quad \left. B(s)P(s)\left\{ \int_0^T \Psi(T-\tau)(f(\tau, x(\tau)) - f(\tau, y(\tau)))d\tau \right\}ds \right\| \end{aligned}$$



$$\begin{aligned}
&\leq \sup_{t \in [0, T]} \int_0^t \|\Psi(T-s)\| \|(f(s, x(s)) - f(s, y(s)))\| ds \\
&\quad + \sup_{t \in [0, T]} \int_0^t \|\Psi(T-s)\| \|B(s)\| \|P(s)\| \\
&\quad \int_0^T \|\Psi(T-\tau)\| \|(f(\tau, x(\tau)) - f(\tau, y(\tau)))\| ds d\tau \\
&\leq \sup_{t \in [0, T]} m \int_0^t \alpha \|x(s) - y(s)\| ds + \sup_{t \in [0, T]} m^2 b p t \int_0^T \alpha \|y(\tau) - x(\tau)\| d\tau \\
&\leq m\alpha \sup_{t \in [0, T]} \int_0^t \|x(s) - y(s)\| ds + m^2 b p T \alpha \int_0^T \sup_{t \in [0, T]} \|y(\tau) - x(\tau)\| d\tau \\
&\leq m\alpha T \|x - y\| + m^2 b p T \alpha T \|x - y\| \\
&\leq m\alpha T (1 + m b p T) \|x - y\| \\
&\leq m\alpha T (1 + m b p T) \|x - y\|
\end{aligned}$$

Since  $\alpha T(1 + m b p T) < 1$ ,  $F$  is a contraction.  $\square$

Now we have the following theorem concerning the controllability of the system (4.1.1).

**Theorem 4.3.1.** *Suppose that  $B(t)$ ,  $\Psi(t)$  and  $f$  satisfy the assumptions [K1], [B1], [f1] and [f2]. Further, the steering operator  $P(t)$  satisfies  $\|P(t)\| \leq p$  and  $\alpha m T(1 + m b p T) < 1$ . Then the system (4.1.1) is controllable.  $\square$*

*Proof.* In the Lemma 4.3.1 we have proved that  $F$ , as defined in the equation (4.3.4), is a contraction. Hence using the Banach contraction principle,  $F$  has a fixed point. Thus the system (4.3.3) is solvable, subsequently the system (4.1.1) is controllable.  $\square$

If the nonlinear function  $f$  is uniformly bounded then by using Schauder's fixed point theorem applied on the operator  $F$  defined in (4.3.4) we can prove the following result.

**Theorem 4.3.2.** *suppose that the linear system is controllable and the nonlinear function  $f(t, x)$  is Lipschitz and uniformly bounded, that is  $\exists \alpha > 0$  such that  $\|f(t, x) - f(t, y)\| \leq \alpha \|x - y\| \quad \forall x, y \in R^n$  and there exists  $M > 0$  such that  $\|f(t, x)\| \leq M \quad \forall x \in R^n$ . Then the nonlinear system is controllable.  $\square$*

## 4.4 Summary

In this chapter we have generalized the methodology, presented in chapter 2, to prove controllability result for both MSOL and MSON in finite dimensional space. Instead of using Sine and Cosine matrices, here we have used general matrices  $\Phi$  and  $\Psi$ . Matrices  $\Phi$  and  $\Psi$ , have similar properties as Sine and Cosine matrices.