Chapter 7

Controllability of Urysohn Integral Inclusion of Volterra Type

Contents

In this chapter we study controllability of a system described by an integral inclusion of Urysohn type with delay. The existence of solution of Urysohn Inclusion is studied in (Angel [1]). In our approach we reduce the controllability problem of the nonlinear system into solvability problem of another integral inclusion. The solvability of this integral inclusion is subsequently established by imposing suitable standard boundedness, convexity and semicontinuity conditions on the set-valued mapping defining the integral inclusion, and by employing Bohnenblust-Karlin extension of Kakutani's fixed point theorem for set-valued mappings. Section 7.1, deals with the general introduction of the problem, in section 7.2, we give the controllability problem in terms of solvability problem. Section 7.3, we study the solvability of the nonlinear integral equation. Section 7.4, deals with the main results. Chapter ends with a summary in section 7.5.

7.1 Introduction

Integral inclusions arise quite naturally in the treatment of many optimal control problems. In recent years a number of papers appeared in the literature concerning integral inclusions, in particular inclusions of Hammerstein type and Urysohn type(refer Rangimchannov[66],Gaidarov[29], Angel[1]). This type of inclusions have been used to model many thermostatic devices (refer Glashoff and sperckels[33], [34]). Here we consider the nonlinear control system described by the following Urysohn integral inclusion on the time interval $[0, T]$, $T > 0$

$$x(t) \in (Hx)(t) + \int_0^t g(t, s, x_s)F(s, x_s)ds + \int_0^t K(t, s)u(s)ds. \tag{7.1.1}$$

where, for each $t \in [0, T]$ the state x(t) is in R^n and the control $u(t) \in R^m$.
For any given real number $0 < r < T$ and for any function $x \in C([-r, T]; R^n)$ and $s \in [0, T]$, we define an element $x_s \in C([-r, 0]; R^n)$ by $x_s(\theta) = x(s+\theta), \quad -r \leq \theta \leq 0$.

The initial conditions are given by

$$x(\theta) = \phi(\theta), -r \leq \theta \leq 0, \tag{7.1.2}$$

for a fixed, $\phi \in C[-r, 0]$.
$H : L^\infty([-r, T]; R^n) \to C([0, T]; R^n)$ is the Urysohn operator defined by

$$(Hx)(t) = \phi(0) + \int_0^T h(t, s, x_s)ds$$

where, $h : [0, T] \times [0, T] \times L^\infty([-r, 0]; R^n) \to R^n$ is a nonlinear function, $g : [0, T] \times [0, T] \times L^\infty([-r, 0]; R^n) \to M_{n \times n}$ is also a nonlinear function, where $M_{n \times n}$ is a space of $n \times n$ matrices. For $(t, s) \in [0, T] \times [0, T]$, K(t,s) is $n \times n$, matrix $F : [0, T] \times L^\infty([-r, 0]; R^n) \to 2^R$ is a set valued mapping.

Chuong(refer [20]) studied a general Urysohn inclusion of Volterra type, without delay and control. The existence for such system was established under much stronger hypothesis on the set-valued mapping. The existence of the solution of (7.1.1)-(7.1.2) without control was established in (Angel [1]). For fixed u the solution of (7.1.1)-(7.1.2) can be defined as follows:

Definition 7.1.1. *A solution of (7.1.1)-(7.1.2) is a function x, defined on $[-r,T]$ with $x(t)=\phi(t), -r\le t\le 0$, where $\phi\in C([-r,0];R^n)$ and $x(.)\in C([0,T];R^n)$ on $[0,T]$, satisfying the following integral equation*

$$x(t)=\phi(0)+\int_0^T h(t,s,x_s)ds+\int_0^t g(t,s,x_s)v(s)ds+\int_0^t K(t,s)u(s)ds$$

for any selection $v\in L^1([0,T];R^n)$ satisfying the inclusion $v(t)\in F(t,x_t)$ almost everywhere on $[0,T]$.

We now define controllability for the system (7.1.1)-(7.1.2) (Rusel [68]).

Definition 7.1.2. *The system (7.1.1)-(7.1.2) is said to be controllable on $[0,T]$ if for any pair of vectors $x_0,x_1\in R^n$, there exists a control $u\in L^2([0,T];R^n)$ such that the solution of (1.1)-(1.2) together with $x(0)=\phi(0)=x_0$ also satisfies $x(T)=x_1$.*

To ensure the existence of solution for (7.1.1)-(7.1.2) the following conditions on h, g, K and F are assumed.

[H] The function $h:[0,T]\times[0,T]\times L^\infty([-r,0];R^n)\to R^n$ satisfies the following conditions:

- **(a)** for each $(t,s)\in[0,T]\times[0,T]$, the map $\phi\to h(t,s,\phi)$ is continuous,
- **(b)** for almost all $t\in[0,T]$,

$$\int_0^T \sup_{\phi\in L^\infty}|h(t,s,\phi)|ds<\infty,$$

- **(c)**

$$\lim_{t'\to t''}\int_0^T \sup_{\phi\in L^\infty}|h(t',s,\phi)-h(t'',s,\phi)|ds=0,$$

- **(d)** $h(0,\cdot,\cdot)=0$.

[G] The function $g : [0,T] \times [0,T] \times L^{\infty}([-r,0];R^n) \to M_{n\times n}$ satisfies the following conditions:

(a) g is bounded,

(b) for each $(t,s) \in [0,T] \times [0,T]$, the map $\phi \to g(t,s,\phi)$ is continuous,

(c) for each $t'' \in [0,T]$ and almost every $s \in [0,T]$

$$\lim_{t' \to t''} [\sup_{\phi \in L^{\infty}} |g(t',s,\phi) - g(t'',s,\phi)|] = 0.$$

[F] For the set-valued mapping $F : [0,T] \times L^{\infty}([-r,0];R^n) \to 2^{R^n}$, the following conditions are assumed:

(a) for all $(t,\phi) \in [0,T] \times L^{\infty}([-r,0];R^n), F(t,\phi)$ is convex. F is upper semi-continuous in the sense of Kurotowski (refer [45]) with respect to ϕ;

(b) for any $(\bar{t},\bar{\phi}) \in [0,T] \times L^{\infty}([-r,0];R^n)$,

$$F(\bar{t},\bar{\phi}) = \bigcap_{\delta>0} cl \bigcup \{F(\bar{t},\phi); ||\phi - \bar{\phi}|| \leq \delta\}$$

where $\phi \in L^{\infty}([-r,0];R^n)$. Since the intersection of closed set is closed, so each of $F(\bar{t},\bar{\phi})$ is closed.

(c) there exists a measurable set-valued function $P : [0,T] \to E^1$, a constant $M > 0$, and for each $\epsilon > 0$, a function $\psi_\epsilon \in L^1([0,T];R^n)$, $\psi_\epsilon(t) > 0$, such that, for given $x \in L^{\infty}([-r,T];R^n)$ and selection $\xi(t) \in F(t,x_t)$, there exists a selection $\eta(t) \in P(t)$, with

1. $\int_0^T \eta(t)dt \leq M$
2. $|\xi(t)| \leq \psi_\epsilon(t) + \epsilon\eta(t)$

[K] for each $(t,s) \in [0,T] \times [0,T]$, $(t,s) \to K(t,s)$ is continuous with $\|K(t,s)\| \leq k(t,s)$ for $k(t,s) \in L^2([0,T] \times [0,T])$

Here the conditions [H-a] and [H-b] are used to establish the complete continuity of the Urysohn operator (see Krasnoselskii, [51]), and the condition [F-c] is used for proving equi - absolute integrability (see Ioffe [6]) condition of the set of selections.

Here we will use operator theory in the analysis of controllability(Joshi and George [46]). So some basic definition regarding control operator are as follows:

Definition 7.1.3. *The control operator $C : L^2([0,t]; R^m) \rightarrow R^n$ of the system (7.1.1)-(7.1.2) be defined by*

$$Cu = \int_0^T K(T,\tau)u(\tau)d\tau \tag{7.1.3}$$

Definition 7.1.4. *A bounded linear operator $S : R^n \rightarrow L^2([0,t]; R^m)$ is said to be steering operator for the associated linear system*

$$x(t) = \int_0^t K(t,s)u(s)ds \tag{7.1.4}$$

if CS=I where, I being the identity operator on R^n

Definition 7.1.5. *An $m \times n$ matrix function P(t) with entries in $L^2([0,T]; R^m)$ is said to be a steering function for (7.1.4)on [0,T] if*

$$\int_0^T K(T,s)P(s)ds = I$$

We note that if the linear system (7.1.4) is controllable then there exists a steering function P(t),(refer [68]).

7.2 Controllability and Feed-Back Formulation

For studying the controllability of (7.1.1)-(7.1.2), we assume that the corresponding linear system (7.1.4) is controllable and let P(t) be a steering function for it. Now the nonlinear system (7.1.1)-(7.1.2)is controllable on [0,T] if and only if there exists a control u which steers a given initial state $\phi(0)$ of the system to a desired final state x_1. That is, there exists a control function u such that

$$\begin{aligned} x_1 = x(T) = \phi(0) + \int_0^T h(T,s,x_s)ds + \int_0^T g(T,s,x_s)v(s)ds \\ + \int_0^T K(T,s)u(s)ds, \end{aligned} \tag{7.2.1}$$

for any selection $v \in L^1([0,T];R^n)$ satisfying the inclusion $v(t) \in F(t,x_t)$ almost everywhere on $[0,T]$. Let us define a control $u(t)$ by

$$u(t) = P(t)[x_1 - \phi(0) - \int_0^T h(T,s,x_s)ds - \int_0^T g(T,s,x_s)v(s)ds], \tag{7.2.2}$$

where $x(.)$ satisfies the nonlinear system (7.1.1)-(7.1.2). Now substituting this control $u(t)$ into the nonlinear integral equation (7.1.1)-(7.1.2), we get

$$x(t) = \phi(0) + \int_0^T h(t,s,x_s)ds + \int_0^t g(t,s,x_s)v(s)ds + \int_0^t K(t,s)P(s)$$

$$[x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau - \int_0^T g(T,\tau,x_\tau)v(\tau)d\tau]ds. \tag{7.2.3}$$

If the equation (7.2.3) is solvable then x(t) satisfies $x(0) = \phi(0)$ and $x(T) = x_1$. This implies that the system (7.1.1)-(7.1.2) is controllable with a control u given by (7.2.2). Hence the controllability of the nonlinear integral inclusion system (7.1.1)-(7.1.2) is equivalent to the solvability of the integral equation (7.2.3) with suitable selection $v(t) \in F(t,x_t)$.

7.3 Solvability

We apply fixed point theorem for establishing solvability of the nonlinear integral equation(7.2.3).We now recast the integral equation (7.2.3) with a selection v as a set-valued mapping and apply fixed point theorem for a set-valued mapping.

We introduce two set-valued mappings Φ and Ψ whose domain S is defined by

$$S = \{x \in L^\infty([-r,T];R^n)|\ x|_{[-r,0]} = \phi,\ x|_{[0,T]} \in C([0,T];R^n)\} \tag{7.3.1}$$

The maps $\Phi : S \to L^1([0,T];R^n)$ and $\Psi : S \to S$, are defined by

$$\Phi(x) = \{v \in L^1([0,T];R^n)|v(t) \in F(t,x_t), a.e., on[0,T]\} \tag{7.3.2}$$

$$\Psi(x) = \{z \in S|z(t) = (Hx)(t) + \int_0^t g(t,s,x_s)v(s)ds + \int_0^t K(t,s)P(s)$$

$$[x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau - \int_0^T g(T,\tau,x_\tau)v(\tau)d\tau]ds$$
$$, \ z|_{[-r,0]} = \phi, v \in \Phi(x)\} \tag{7.3.3}$$

We will employ the following Bohnenblust-Karlin extension of KaKutani's fixed point theorem for set-valued mappings.

Theorem 7.3.1. ***(Bohnenblust-Karlin** [12])* *Let Σ be a non-empty, closed convex subsets of a Banach space $\mathcal{B}$. If $\Gamma : \Sigma \to 2^\Sigma$ is such that*

.

(a) *$\Gamma(a)$ is non-empty and convex for each $a \in \Sigma$,*

(b) *the graph of Γ, $\mathcal{G}(\Gamma) \subset \Sigma \times \Sigma$, is closed,*

(c) *$\cup \{\Gamma(a)/a \in \Sigma\}$ is contained in a sequentially compact set $\mathcal{F} \in B$,*
then the map Γ has a fixed point, that is, there exists a $\sigma_0 \in \Sigma$ such that $\sigma_0 \in \Gamma(\sigma_0)$.

□

We will apply this theorem to the map Ψ defined on the closed convex set $S \subset L^\infty([-r,T];R^n)$.

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In order to apply Theorem 7.3.1, we need to prove that the set $\Psi(S)$ is relatively sequentially compact. This property in turn, depends on the weak relative compactness of $\Phi(S)$ in $L^1([0,T];R^n)$.

Theorem 7.3.2. *(Angel [1]) The set $\Phi(S)$ defined by the relation (7.3.2) is an equi-absolutely integrable set and is weakly compact in $L^1([0,T];R^n)$.* □

We have the following theorem on the relative compactness of the set $\Psi(s)$.

Theorem 7.3.3. *Under the hypotheses (H),(G),(F) and (K), for each $x \in S$, $\Psi(x)$ is a non-empty and the set $\Psi(S)$ defined by the relation (7.3.3) is a relatively sequentially compact subset of $L^\infty([0,T];R^n)$* □

Proof. First we shall show that $\psi(S) \neq \emptyset \ \forall \ x \in S$. For a given $x \in S$ we have

$\phi(x) \neq \emptyset$ (Angel [1]). Hence choosing $v \in \phi(x)$ we define

$$\begin{aligned} y(t) &= \phi(0) + \int_0^T h(t,s,x_s)ds + \int_0^t g(t,s,x_s)v(s)ds + \int_0^t K(t,s)P(s) \\ &\quad \{x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau - \int_0^T g(T,\tau,x_\tau)v(\tau)d\tau\}ds \end{aligned}$$

For any $t', t'' \in [0,T]$, we have

$$\begin{aligned} \mid y(t') - y(t'') \mid &= \mid \int_0^T (h(t',s,x_s) - h(t'',s,x_s))ds + (\int_0^{t'} g(t',s,x_s)v(s)ds - \\ &\quad \int_0^{t''} g(t'',s,x_s)v(s)ds) + \{\int_0^{t'} K(t',s)P(s)ds - \\ &\quad \int_0^{t''} K(t'',s)P(s)ds\}[x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau - \\ &\quad \int_0^T g(T,\tau,x_\tau)v(\tau)d\tau]| \\ &\leq \int_0^T \sup_{\phi \in L^\infty} |h(t',s,\phi) - h(t'',s,\phi)|ds + \int_0^{t'} |g(t',s,x_s) - \\ &\quad g(t'',s,x_s)||v(s)|ds + \int_{t'}^{t''} |g(t'',s,x_s)||v(s)|ds + \\ &\quad \{\int_0^{t'} |K(t',s) - K(t'',s)||P(s)|ds + \\ &\quad \int_{t'}^{t''} |K(t'',s)||P(s)|ds\}\{|x_1| + |\phi(0)| + \\ &\quad \int_0^T |h(T,\tau,x_\tau)|d\tau + \int_0^T |g(T,\tau,x_\tau)||v(\tau)|d\tau\}ds \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

By [H-c], $\exists\ \delta_1 > 0$ such that

$$\begin{aligned} I_1 &= \int_0^T \sup_{\phi \in L^\infty} |h(t',s,\phi) - h(t'',s,\phi)|ds \\ &< \frac{\epsilon}{5} \qquad if\ |t' - t''| < \delta_1 \end{aligned}$$

Using the condition [F-c] for $\delta_2 > 0$

$$
\begin{aligned}
I_2 &= \int_0^{t'} |g(t',s,x_s) - g(t'',s,x_s)||v(s)|ds \\
&\leq \int_0^{t'} \sup_{\phi \in L^\infty} |g(t',s,\phi) - g(t'',s,\phi)|\psi_{\delta_2}(s) + \delta_2\eta(s)|ds \\
&= \int_0^{t'} \sup_{\phi \in L^\infty} |g(t',s,\phi) - g(t'',s,\phi)|\psi_{\delta_1}(s)ds + \\
&\quad \delta_2 \int_0^{t'} \sup_{\phi \in L^\infty} |g(t',s,\phi) - g(t'',s,\phi)|\eta(s)ds \\
&= I_{21} + I_{22}
\end{aligned}
$$

Since g is a bounded function, taking its bound as M_g and $\psi_{\delta_2} \in L^1([0,T];R^n)$. We apply the Lesbesgue dominated convergence theorem. Therefore, for a small $\delta_3 > 0$

$$
\lim_{t' \to t''} \int_0^{t'} \sup_{\phi \in L^\infty} |g(t',s,\phi) - g(t'',s,\phi)|\psi_{\delta_2}(s)ds
$$

$$
= \int_0^{t'} \lim_{t' \to t''} \sup_{\phi \in L^\infty} |g(t',s,\phi) - g(t'',s,\phi)|\psi_{\delta_2}(s)ds
$$

Using [G-c]

$$
\begin{aligned}
I_{21} &= \int_0^{t'} \sup_{\phi \in L^\infty} |g(t',s,\phi) - g(t'',s,\phi)|\psi_{\delta_2}(s)ds \\
&\leq \frac{\epsilon}{5k_1} \int_0^{t'} \psi_{\delta_2}(s)ds \quad for \quad \int_0^{t'} \psi_{\delta_2}(s)ds = k_1 \leq \infty \\
&\leq \frac{\epsilon}{5k_1} k_1 \\
&= \frac{\epsilon}{5} \quad if \;\; |t' - t''| \leq \delta_3 \\
I_{22} &= \delta_2 \int_0^{t'} \sup_{\phi \in L^\infty} |g(t',s,\phi) - g(t'',s,\phi)|\eta(s)ds \\
&\leq \delta_2 2M_g \int_0^{t'} \eta(s)ds \\
&\leq \delta_1 2M_g M \\
&\leq \frac{\epsilon}{10M_gM} 2M_gM \qquad (taking \;\; \delta_1 \leq \frac{\epsilon}{10M_gM}) \\
&= \frac{\epsilon}{5}
\end{aligned}
$$

$$\begin{aligned}
I_3 &= \int_{t'}^{t''} |g(t'', s, x_s)||v(s)|ds \\
&\leq (M_g \int_{t'}^{t''} |v(s)|ds) \\
&\leq M_g \left(\int_{t'}^{t''} |v(s)|ds \right)^{\frac{1}{2}} \left(\int_{t'}^{t''} ds \right)^{\frac{1}{2}} \\
&\leq M_g(k_1 + M)^{\frac{1}{2}}(t'' - t')^{\frac{1}{2}} \quad for\ t'' - t' \leq \delta_4 \\
&\leq M_g(k_1 + M)^{\frac{1}{2}}\delta_4 \\
&\leq \frac{\epsilon}{5} \quad ,taking\ \delta_4 \leq \frac{\epsilon}{5M_g(M+K)^{\frac{1}{2}}}
\end{aligned}$$

The function h satisfies the condition [H], so for any given $t \in [0, T]$, there exists a finite $b = b(\hat{t})$ such that

$$\int_0^T h(\hat{t}, s, x_s)ds \leq \int_0^T \sup_{\phi \in L^\infty} |h(\hat{t}, s, \phi)ds \leq b(\hat{t})$$

$$\begin{aligned}
I_4 &= \{|x_1| + |\phi(0)| + \int_0^T |h(T, \tau, x_\tau)|d\tau + \int_0^T |g(T, \tau, x_\tau)||v(\tau)|d\tau\} \\
&\quad (\int_0^{t'} |K(t', s) - K(t'', s)||P(s)|ds + \int_{t'}^{t''} |K(t'', s)||P(s)|ds) \\
&\leq \{|x_1| + |\phi(0)| + b(T) + M_g(M + k_1)\} \\
&\quad (P \int_0^{t'} |K(t', s) - K(t'', s)|ds + KP \int_{t'}^{t''} ds) \\
&\quad where,\ K\ and\ P\ are\ bounds\ of\ K(t, s)\ and\ P(s)
\end{aligned}$$

$$\begin{aligned}
&\leq R(Pt' \frac{\epsilon}{10PRt'} + KP(t' - t'')) \\
&\leq R(\frac{\epsilon}{10R} + KP\delta_5)\ (if\ |t^{'} - t^{''}| \leq \delta_5\ and\ Taking\ \delta_5 \leq \frac{\epsilon}{10KPR}) \\
&\leq R(\frac{\epsilon}{10R} + KP \frac{\epsilon}{10RKP}) \\
&= \frac{\epsilon}{5}
\end{aligned}$$

Thus, continuity of y, follows by choosing $\delta \leq min(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)$ and so the function z is defined by

$$z(t) = \begin{cases} \phi(t) & , -r \leq t \leq 0 \\ y(t) & , 0 \leq t \leq T \end{cases}$$

is piecewise continuous.

Here the elements of $\Psi(S)$ in the interval $[0,T]$ form an equicontinuos family. Hence Relative sequential compactness will now follow from the equiboundedness of $\Psi(S)$, since then any sequence in $\Psi(S)$ say, $\{z_k\}$, restricted to $[0,T]$, will have a uniformly convergent subsequence by the Arzela - Ascoli theorem. Now to show that $\Psi(S)$ is equibounded, let us consider $y \in \Psi(S)$ on $[0,T]$, for a given $t_0 \in [0,T]$,

$$\begin{aligned}
y(t_0) &= \phi(0) + \int_0^T h(t_0,s,x_s)ds + \int_0^{t_0} g(t_0,s,x_s)v(s)ds + \int_0^{t_0} K(t_0,s)P(s) \\
&\quad \{x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau - \int_0^T g(T,\tau,x_\tau)v(\tau)d\tau\}ds \\
|y(t_0)| &\leq |\phi(0)| + \int_0^T |h(t_0,s,x_s)|ds + \int_0^{t_0} |g(t_0,s,x_s)||v(s)|ds + \int_0^{t_0} |K(t_0,s)| \\
&\quad |P(s)||x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau - \int_0^T g(T,\tau,x_\tau)v(\tau)d\tau|ds \\
&\leq |\phi(0)| + \int_0^T |h(t_0,s,x_s)|ds + M_g(M+k_1) + \\
&\quad (KPt_0)\{|x_1| + |\phi(0)| + \int_0^T |h(T,\tau,x_\tau)|d\tau + M_g(M+k_1) \\
&\leq |\phi(0)| + b(t_0) + M_g(M+k_1) + (KPt_0)\{|x_1| + |\phi(0)| + \\
&\quad h(T) + M_g(M+k_1)\} \\
&< \infty
\end{aligned}$$

Hence y is bounded uniformly on $[0,T]$. It follows that the set $\Psi(S)$ is relatively sequentially compact, since the initial function ϕ is fixed and the restrictions of elements of S to $[0,T]$ are continuous. □

Theorem 7.3.4. *The set $\Psi(x)$ is convex for each $x \in S$.* □

Proof. Let $y^{(1)}, y^{(2)} \in \Psi(x)$. Then there exists $v^{(i)}(t) \in F(t,x_t)$, i=1,2 such that

$$y^{(i)}(t) = \phi(0) + \int_0^T h(t,s,x_s)ds + \int_0^t g(t,s,x_s)v^{(i)}(s)ds + \int_0^t K(s,t)P(s)$$

$$[x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau - \int_0^T g(T,\tau,x_\tau)v^{(i)}(\tau)d\tau]ds.$$

Thus for $0 < \lambda < 1$,

$$\lambda y^{(1)}(t) + (1-\lambda)y^{(2)}(t) = \phi(0) + \int_0^T h(t,s,x_s)ds +$$

$$\int_0^t g(t,s,x_s)(\lambda v^{(1)}(s) + (1-\lambda)v^{(2)}(s))ds + \int_0^t K(s,t)P(s)$$

$$\{x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau - \int_0^T g(T,\tau,x_\tau)(\lambda v^{(1)}(\tau) + (1-\lambda)v^{(2)}(\tau))d\tau\}ds.$$

By the convexity of $F(t,x_t)$ we have $(\lambda v^{(1)}(t) + (1-\lambda)v^{(2)}(t)) \in F(t,x_t)$.
Hence $\Psi(x)$ is convex. □

Now we prove that $\mathcal{G}(\Psi)$ is closed. For proving this, we use the following theorems, which was used in (Angel [2]) and modified by (Cesari [16]).

Theorem 7.3.5. *Let $I = [0,T]$, consider the set-valued mapping, $F : I \times L^\infty \to 2^{E^n}$, and assume that F satisfies the conditions (F-a) and (F-b) with respect to ϕ. Let ξ, ξ_k, x, x_k be functions measurable on I, x, x_k bounded, and let $\xi, \xi_k \in L^1(I; R^n)$. Then if $\xi_k(t) \in F(t,x_t)$ a.e in I and $\xi_k \to \xi$ weakly in $L^1(I;R^n)$, while $x \to x_k$ uniformly on I, then $\xi(t) \in F(t,x_t)$ in I.* □

We now use Theorem(7.3.5) to show that the graph of the map Ψ, defined by the relation (7.3.3), has a closed graph.

Theorem 7.3.6. *Under the assumption (H),(G),(F) and (K) the map $\Psi : S \to 2^S$ has a closed graph. That is, $\{(x,y) \in S \times S \mid y \in \Psi(x)\}$is closed.* □

Proof. Let $\{x_k, y_k\}$ be a sequence of functions such that $y_k \in \Psi(x_k)$ which converges to a limit point (x,y) of $\mathcal{G}(\Psi)$. Thus, $x_k \to x$ and $y_k \to y$ uniformly on [0,T]. By definition of Ψ there exists a sequence v_k, with $v_k \in \Phi(x_k)$, such that

$$y_k(t) = \phi(0) + \int_0^T h(t,s,x_{k_s})ds + \int_0^t g(t,s,x_{k_s})v_k(s)ds + \int_0^t K(t,s)P(s)$$

$$[x_1 - \phi(0) - \int_0^T h(T,\tau,x_{k_\tau})d\tau - \int_0^T g(T,\tau,x_{k_\tau})v_k(\tau)d\tau]ds$$

Without loss of generality we may assume that $v_k \to v$ weakly in $L^1([0,T];R^n)$ and

$v(s) \in F(s, x_s)$. We wish to show y satisfies the equation

$$y(t) = \phi(0) + \int_0^T h(t,s,x_s)ds + \int_0^t g(t,s,x_s)v(s)ds + \int_0^t K(t,s)P(s)$$

$$[x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau - \int_0^T g(T,\tau,x_\tau)v(\tau)d\tau]ds$$

Now considering,

$$|y(t) - \phi(0) - \int_0^T h(t,s,x_s)ds - \int_0^t g(t,s,x_s)v(s)ds - \int_0^t K(t,s)P(s)$$

$$[x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau - \int_0^T g(T,\tau,x_\tau)v(\tau)d\tau]ds|$$

$$= |y(t) - y_k(t) + y_k(t) - \phi(0) - \int_0^T h(t,s,x_s)ds - \int_0^t g(t,s,x_s)v(s)ds - \int_0^t K(t,s)P(s)$$

$$[x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau - \int_0^T g(T,\tau,x_\tau)v(\tau)d\tau]ds|$$

$$= |y(t) - y_k(t) + \phi(0) + \int_0^T h(t,s,x_{k_s})ds + \int_0^t g(t,s,x_{k_s})v_k(s)ds + \int_0^t K(t,s)P(s)$$

$$[x_1 - \phi(0) - \int_0^T h(T,\tau,x_{k_\tau})d\tau - \int_0^T g(T,\tau,x_{k_\tau})v_k(\tau)d\tau]ds$$

$$-\phi(0) - \int_0^T h(t,s,x_s)ds - \int_0^t g(t,s,x_s)v(s)ds - \int_0^t K(t,s)P(s)$$

$$[x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau - \int_0^T g(T,\tau,x_\tau)v(\tau)d\tau]ds|$$

$$\leq |y(t) - y_k(t)| + \int_0^T |h(t,s,x_{k_s}) - h(t,s,x_s)|ds + \int_0^t |g(t,s,x_{k_s})v_k(s) - g(t,s,x_s)v(s)|ds$$

$$+ \int_0^t |K(t,s)P(s)[\int_0^T (h(T,\tau,x_\tau) - h(T,\tau,x_{k_\tau}))d\tau +$$

$$\int_0^T g(T,\tau,x_\tau)v(\tau) - g(T,\tau,x_{k_\tau})v_k(\tau)d\tau]ds|$$

$$\leq |y(t) - y_k(t)| + \int_0^T |h(t,s,x_{k_s}) - h(t,s,x_s)|ds + \int_0^t |g(t,s,x_s)||v_k(s) - v(s)|ds$$

$$+ \int_0^t |g(t,s,x_{k_s}) - g(t,s,x_s)||v_k(s)|ds + \int_0^t |K(t,s)P(s)|$$

$$|[\int_0^T (h(T,\tau,x_\tau) - h(T,\tau,x_{k_\tau}))d\tau + \int_0^T g(T,\tau,x_\tau)v(\tau) - g(T,\tau,x_{k_\tau})v_k(\tau)d\tau]|ds$$

Here we need to show that the relation holds pointwise. So for a fixed t_0 we consider each terms separately.

$$|y(t_0) - y_k(t_0)| \leq \frac{\epsilon}{5}$$

since $y_k \longrightarrow y$ uniformly. From [H] each element of the sequence of functions $s \longrightarrow |h(t_0, s, x_k)|$ k=1,2,..... is bounded above by the integrable function $s \longrightarrow \sup|h(t_0), s, \phi|$. Since $x_k \longrightarrow x$ uniformly we have from [H]that $h(t_0, s, x_{k_s}) \longrightarrow h(t_0, s, x_s)$ pointwise a.e. in [0,T] and so

$$\lim_{k\to\infty} \int_0^T h(t_0, s, x_{k_s})ds = \int_0^T h(t_0, s, x_s)ds$$

Also,

$$\int_0^{t_0} |g(t, s, x_s)||v_k(s) - v(s)|ds \leq \frac{\epsilon}{5}$$

Applying Egorov's theorem and condition [G]

$$\int_0^{t_0} |g(t_0, s, x_{k_s}) - g(t_0, s, x_s)||v_k(s)|ds$$

can be made less than $\frac{\epsilon}{5}$. Using the continuity and boundedness of K, P and the conditions [H] ,[G] and [F] for the following terms, we get

$$\int_0^t |K(t,s)P(s)|\ |[\int_0^T (h(T,\tau,x_\tau) - h(T,\tau,x_{k_\tau}))d\tau$$

$$+ \int_0^T g(T,\tau,x_\tau)v(\tau) - g(T,\tau,x_{k_\tau})v_k(\tau)d\tau]|ds \leq \frac{\epsilon}{5}$$

Hence for a given $\epsilon > 0$

$$|y(t) - \phi(0) - \int_0^T h(t,s,x_s)ds + \int_0^t g(t,s,x_s)v(s)ds - \int_0^t K(t,s)P(s)$$

$$[x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau - \int_0^T g(T,\tau,x_\tau)v(\tau)d\tau]ds| \leq \epsilon$$

Hence, $(x, y) \in \mathcal{G}$ and the graph of Ψ is closed. □

With this theorem all of the hypothesis of the fixed point theorem are satisfied. And now we consider the main controllability theorem:

7.4 The main Result

Theorem 7.4.1. *Under the assumption [H]-[B], the nonlinear system described by the integral inclusion (7.2.1) is controllable.* □

Proof. We have proved in Theorem 7.3.2, Theorem 7.3.3, Theorem 7.3.4 and Theorem 7.3.6 that under the assumptions [H]-[B] the map $\psi : S \to 2^S$ satisfies all the hypotheses of the Bohnenblust-Karlin extension of KaKutani's fixed point theorem. Hence Ψ has a fixed point in S. Let $x \in S$ be the fixed point of the mapping ψ defined by the relation (7.3.3) that is $x \in \psi(x)$. Therefore, for a selection $v \in \phi(x)$ such that $v(t) \in F(t, x_t)$ a.e, we have

$$x(t) = \phi(0) + \int_0^T h(t,s,x_s)ds + \int_0^t g(t,s,x_s)v(s)ds$$

$$+ \int_0^t K(s,t)B(s)P(s)[x_1 - \phi(0) - \int_0^T h(T,\tau,x_\tau)d\tau$$

$$- \int_0^T g(T,\tau,x_\tau)v(\tau)d\tau]ds.$$

Obviously $x(0) = x_0$ and $x(T) = x_1$. Hence the system is controllable. □

We conclude this section with an example, which demonstrates our result.

Example 7.4.1. *Let us consider the integral inclusion:*

$$x(t) \in \int_0^1 [\frac{sin(s^2)sin(t^2)}{3 + arctanx(s)} + \frac{cos(sx(s))}{3\sqrt{1+t}}F(s,x(s))]ds + \int_0^t e^{t-s}u(s)ds$$

where r=0, m=n=1, and $F : [0,T] \times R \to 2^R$ *is the set-valued map defined by*

$$F(t,x) = \begin{cases} u, & with\ |u| \leq t + |x|,\ if -1 - \frac{1}{\sqrt{t}} \leq x \leq 1 + \frac{1}{\sqrt{t}},\ t \neq 0, \\ u, & with\ |u| \leq |x|,\ t = 0, \\ 0, & otherwise. \end{cases}$$

Here F has a closed graph and convex values, also the growth condition [F-c] is satisfied for the set-valued map F, since any selection $\xi(t) \in F(t, x(t))$ satisfies

$$|\xi(t)| \leq t + |x(t)| \leq t + 1 + \frac{1}{\sqrt{(t)}}$$

It follows that F(t,x(t) is integrally bounded. Now

$$h(t, s, x) = \frac{sin(s^2)sin(t^2)}{3 + arctanx(s)}$$

satisfies

$$|h(t, s, x)| \leq 1$$

and $h(0, \cdot, \cdot) = 0$ while

$$g(t, s, x) = \frac{cos(sx(s))}{3\sqrt{1+t}}$$

is bounded. Also h, g and $K(t, s) = e^{t-s}$ satisfies all the conditions [H], [G] and [K]. Since the linear system is obviously controllable, applying Theorem 7.4.1 we have the above integral inclusion is controllable.

7.5 Summary

In this chapter controllability result for a system described by an integral inclusion of Urysohn type with delay has been proved. The controllability problem was transformed into a solvability problem. Bohnenblust-Karlin extension of Kakutani's fixed point theorem for set-valued mappings was applied to prove the solvability. A numerical example has been given at the end.