Chapter 2

Preliminaries

Contents

In this chapter we review some basics of control theory and some results of mathematical analysis, differential equations and linear algebra which will be used for the controllability analysis of linear and nonlinear systems.

2.1 Control Theory

Kalman [49] introduced the concept of controllability for the linear system (2.1.1) and was subsequently extended to nonlinear systems, dominated by controllable linear parts, by Davison and Kunze [21], Mirza and Womack [61], Quinn and Carmichael [65] etc., by using the techniques of fixed point theory.

In our investigation, controllability properties of the nonlinear system depend more on the properties of the linear part. So we first consider a finite dimensional linear system represented by the differential equation

$$\left.\begin{array}{l} x'(t) = A(t)x(t) + B(t)u(t), \ 0 \leq t_0 < t \leq t_1 < \infty \\ x(t_0) = x_0 \end{array}\right\} \tag{2.1.1}$$

where, for each $t \in [t_0, t_1], x(t) \in R^n$ is called the state of the system, $u(t) \in R^m$ is called the control vector and $u \in L^2([t_0, t_1], R^m); A(t), B(t)$ are matrices of dimensions $n \times n$ and $n \times m$, respectively. Assume that the elements $a_{ik}(t)$ of $A(t)(i, k = 1, 2, ..., n)$ are absolutely integrable functions of $t \in [t_0, t_1]$ and elements $b_{il}(t)$ of $B(t)(i = 1, 2, ..., n; l = 1, 2, ..., m)$ are piecewise continuous functions of $t \in [t_0, t_1]$.

Consider the homogeneous linear system

$$\left.\begin{array}{l} \frac{dx(t)}{dt} = A(t)x(t) \ \ t_0 \leq t \leq t_1 \\ x(t_0) = x_0. \end{array}\right\} \tag{2.1.2}$$

where, the state $x(t) \in R^n$, $A(t)$ is matrix of order $n \times n$. It follows easily that the equation (2.1.2) has a unique solution $x(\cdot)$ passing through the initial condition $x(t_0) = x_0$.

Proposition 2.1.1. *If $\phi_1(t), \phi_2(t), ..., \phi_n(t)$ are solutions of (2.1.2) with initial conditions $x_1, x_2, ..., x_n$ then their linear combination*

$$\phi(t) = \sum_{i=1}^{n} \alpha_i \phi_i(t), \qquad \alpha_i \in R$$

is also a solution of (2.1.2) with initial condition

$$\phi(t_0) = \sum_{i=1}^{n} \alpha_i x_i$$

Proposition 2.1.2. *Let $\phi_1(t), \phi_2(t), ..., \phi_m(t)$ be solution of (2.1.2) on $[t_0, t_1]$ and $s \in [t_0, t_1]$, then*

$$\{\phi_1(\cdot), \phi_2(\cdot), ..., \phi_m(\cdot)\}$$

are linearly independent solutions if and only if

$$\{\phi_1(s), \phi_2(s), ..., \phi_m(s)\}$$

is linearly independent set of vectors in R^n.

An $n \times n$ matrix function $\Phi(t, t_0)$ is said to be transition matrix of homogeneous equation(2.1.2) if it satisfies the following:

$$\begin{aligned} \frac{d}{dt}\Phi(t, t_0) &= A(t)\Phi(t, t_0) \\ \Phi(t_0, t_0) &= I. \end{aligned}$$

Example:2.1.1 The matrix function given by

$$\Phi(t, t_0) = \begin{bmatrix} cos(t - t_0) & -sin(t - t_0) \\ sin(t - t_0) & cos(t - t_0) \end{bmatrix}$$

is the transition matrix for the system

$$\frac{d}{dt}x(t) = Ax(t)$$

$$x(t_0) = x_0$$

where,

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

A is called the generator of the transition matrix $\Phi(t, t_0)$. The transition matrix has the following properties:

1. $\Phi(t, t) = I$, identity matrix of order n, for all $t \in [t_0, t_1]$
2. $\Phi(t, t_0) = \Phi(t, \tau)\Phi(\tau, t_0), \quad t \leq \tau \leq t_0$
3. $\frac{\partial}{\partial t}(\Phi(t, \tau)) = A(t)\Phi(t, \tau)$
4. If $\Phi_1(t), \Phi_2(t), \Phi_3(t)$ ...., $\Phi_n(t)$ are linearly independent solutions of the homogeneous system corresponding to (2.1.1) and $\Phi(t)$ is the matrix whose columns are $\Phi_1(t)$, $\Phi_2(t)$, $\Phi_3(t)$, ..., $\Phi_n(t)$, then it can be shown easily that the transition matrix $\Phi(t, \tau)$ satisfies $\Phi(t, \tau) = \Phi(t)[\Phi(\tau)]^{-1}$.

Using the variation of constant formula and Theorem 1 of Brockett [14], we have the following theorem concerning the solution of system (2.1.1).

Theorem 2.1.1. *The sequence of matrices M_k defined recursively by*

$$M_0 = I, \quad M_k = I + \int_{t_0}^{t} A(\tau)M_{k-1}(\tau)d\tau$$

converges uniformly on $[t_0, t_1]$. Moreover, if the limit function is denoted by $\Phi(t, t_0)$ then

$$\frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0) \text{ and } \Phi(t_0, t_0) = I$$

and the solution of (2.1.1) which passes through x_0 at $t = t_0$ is given by

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

□

From Theorem 2.1.1, it follows that the explicit expression for $\Phi(t, t_0)$ is given by the Peano-Baker series (refer Brockett [14])

$$\Phi(t, t_0) = I + \int_{t_0}^{t} A(\tau_1)d\tau_1 + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)d\tau_2 \, d\tau_1 + ... \tag{2.1.3}$$

If A is a real constant $n \times n$ matrix, then the Peano-Baker series (2.1.3) becomes

$$\Phi(t, t_0) = I + A(t - t_0) + \frac{A^2(t - t_0)}{2!} + ... = e^{A(t-t_0)}$$

A variety of definitions are available for controllability in the literature. We define controllability as follows (refer Russell [68]).

Definition 2.1.1. *The system (2.1.1) is said to be* ***controllable*** *over $[t_0, t_1]$ if for each pair of vectors $x_0, x_1 \in \mathrm{R}^n$ there exists a control $u \in L^2([t_0, t_1], \mathrm{R}^m)$ such that the solution of (2.1.1) with $x(t_0) = x_0$ also satisfies $x(t_1) = x_1$. That is,*

$$x_1 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

The above definition of controllability is refered as definition of complete controllability.

Remark 2.1.1. *The control u which steers x_0 to x_1 need not be unique and in general* *it depends on x_0 and x_1. The controllability defined above as global controllability. If x_0 and x_1 are required only to belong to $D \subset \mathrm{R}^n$, then the resulting controllability is said to be local controllability.*

Let $\mathcal{C} : L^2(J, \mathrm{R}^m) \to \mathrm{R}^n$, J is the time interval of the form $[t_0, t_1]$ or $[0, T]$, be a **control operator** defined by

$$\mathcal{C}u = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau. \tag{2.1.4}$$

The following statements regarding the linear system (2.1.1) are equivalent:

1. The system (2.1.1) is controllable.
2. The control operator $\mathcal{C}$ is onto.
3. The adjoint operator $\mathcal{C}^*$ of $\mathcal{C}$ is one-one.
4. The matrix $\mathcal{C}\mathcal{C}^*$ is positive definite.

The operator $\mathcal{C}\mathcal{C}^*$ is known as **controllability Grammian** and is denoted by $W(t_0, t_1)$. Thus we have

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B^*(\tau)\Phi^*(t_1, \tau)d\tau \tag{2.1.5}$$

By Definition 2.1.1, the system (2.1.1) is globally controllable if $(x_1 - \Phi(t_1, t_0)x_0)$ lies in the Range of $\mathcal{C}$ for all $x_0, x_1 \in \mathrm{R}^n$. But $(x_1 - \Phi(t_1, t_0)x_0) \in$ Range $(\mathcal{C})$ iff $(x_1 - \Phi(t_1, t_0)x_0) \in$ Range $(\mathcal{C}\mathcal{C}^*)$= Range $(W(t_0, t_1)$.

When $W(t_0, t_1)$ is invertible, the control function defined by

$$u(t) = B^*(t)\Phi^*(t_1, t)W^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)x_0] \tag{2.1.6}$$

steers the system (2.1.1) from $x(t_0) = x_0$ to $x(t_1) = x_1$.

So, we have the following characterization for controllability.

Theorem 2.1.2. *The linear system (2.1.1) is (globally) controllable if and only if the controllability Grammian $W(t_1, t_0)$ defined in (2.1.5) is nonsingular. That is, there exists a constant $c > 0$ such that*

$$det\ W(t_0, t_1) \geq c$$

□

Definition 2.1.2. *A bounded linear operator $P : R^n \to L^2(J, R^m)$ is called a* ***steering operator*** *for (2.1.1) if for any $\alpha \in R^n, u = P\alpha$ steers 0 to α.*

An $m \times n$ matrix function $P(t)$ is called a steering function, if the operator P defined by $(P\alpha)(t) = P(t)\alpha$ is a steering operator.

We observe that

1. A bounded linear operator $P : R^n \to L^2(J, R^m)$ is a steering operator if and only if $CP = I$ and
2. An $m \times n$ matrix function $P(t)$ is a steering function if and only if

$$\int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)P(\tau)d\tau = I \tag{2.1.7}$$

We have the following characterization regarding controllability of (2.1.1) in terms of steering operator and steering function.

Theorem 2.1.3. *The following are equivalent:*

1. *The system (2.1.1) is controllable.*
2. *There exists a steering function for (2.1.1).*
3. *There exists a steering operator for (2.1.1).*

□

Remark 2.1.2. *If CC^* is invertible (that is, the controllability Grammian is non-singular) then*

$$P = C^*(CC^*)^{-1}$$

(the More-Penrose inverse of C) is a steering operator. In this case

$$P(t) = B^*(t)\Phi^*(t_1, t)W^{-1}(t_0, t_1)$$

is a steering function. Further, $P(t)$ is an optimal steering function, (refer Russell [68]).

Now we consider the finite dimensional nonlinear time varying system, with control, represented by the equation

$$\left.\begin{array}{l}\frac{dx}{dt} = A(t)x + B(t)u + f(t,x), \quad 0 \le t_0 < t \le t_1 < \infty \\ x(t_0) = x_0\end{array}\right\} \tag{2.1.8}$$

where $A(t)$ and $B(t)$ are as in (2.1.1) and $f : J \times \mathrm{R}^n \to \mathrm{R}^n$ is a nonlinear function satisfying Caratheodory conditions, that is, f is measurable with respect to t for all x and continuous with respect to x for almost all $t \in [t_0, t_1]$ (Joshi and Bose [47]). All quantities in (2.1.8) are assumed to be real.

A solution of (2.1.8) is an absolutely continuous function in $L^2([t_0, t_1], \mathrm{R}^n)$ which satisfies (2.1.8) almost everywhere. A solution $x(t)$ exists for (2.1.8) if and only if $x(t)$ satisfies the integral equation

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + \int_{t_0}^{t} \Phi(t,\tau)f(\tau, x(\tau))d\tau$$

where $\Phi(t,\tau)$ is the transition matrix of the homogeneous linear part. We shall be interested in the global controllability of (2.1.8).

There exists a control u which steers the initial x_0 at time $t = t_0$ to the given final state x_1 at time $t = t_1$ if and only if there exists a solution x of (2.1.8) satisfying

$$x_1 = x(t_1) = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau + \int_{t_0}^{t_1} \Phi(t_1,\tau)f(\tau, x(\tau))d\tau \tag{2.1.9}$$

That is

$$\left[x_1 - \Phi(t_1, t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1, \tau) f(\tau, x(\tau)) d\tau\right] = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau) d\tau$$

Suppose that the linear part of (2.1.8) is controllable. Thus by Theorem 2.1.3, there exists a steering function $P(t)$ for the linear part of the system (2.1.8). If there exists x satisfying (2.1.9) then the steering control for (2.1.8) is given by (using Definition 2.1.2)

$$u(t) = P(t)\Big[x_1 - \Phi(t_1, t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1, \tau) f(\tau, x(\tau)) d\tau\Big]. \qquad (2.1.10)$$

Thus, the state of the system (2.1.8) is given by

$$\begin{aligned} x(t) &= \Phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, \tau) f(\tau, x(\tau)) d\tau + \int_{t_0}^{t} \Phi(t, \tau) B(\tau) P(\tau) \\ &\quad \Big[x_1 - \Phi(t_1, t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1, s) f(s, x(s)) ds\Big] d\tau \qquad (2.1.11) \end{aligned}$$

Suppose that (2.1.11) is solvable, then we have $x(t_0) = x_0$ and $x(t_1) = x_1$. This implies that the system (2.1.8) is controllable with the control defined by (2.1.10).

Hence, the controllability of the nonlinear system (2.1.8) is equivalent to the solvability of the coupled equations:

$$\begin{aligned} x(t) &= \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau) B(\tau) u(\tau) d\tau + \int_{t_0}^{t} \Phi(t, \tau) f(\tau, x(\tau)) d\tau \\ u(t) &= P(t)\Big[x_1 - \Phi(t_1, t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1, \tau) F(\tau, x(\tau)) d\tau\Big]. \end{aligned}$$

Let $X_1 = L^2(J, \mathrm{R}^m), X_2 = L^2(J, \mathrm{R}^n)$. Define operators $K, N : X_2 \to X_2, H : X_1 \to X_2$ and $L : X_2 \to X_1$ as follows

$$\begin{aligned} (Kx)(t) &= \textstyle\int_{t_0}^{t} \Phi(t, \tau) x(\tau) d\tau; & (Nx)(t) &= f(t, x(t)); \\ (Hu)(t) &= \textstyle\int_{t_0}^{t} \Phi(t, \tau) B(\tau) u(\tau) d\tau; & (Lx)(t) &= P(t) \textstyle\int_{t_0}^{t_1} \Phi(t_1, \tau) x(\tau) d\tau. \end{aligned}$$

Clearly K, H and L are continuous linear operators and N is a nonlinear operator, called Nemytskii operator (refer Joshi and Bose [47]). Using these definitions, the coupled equations can be written as a coupled operator equations

$$x = Hu + KNx + w_1.$$

$$u = u_1 - LNx,$$

where, $w_1 = \Phi(t, t_0)x_0$ and $u_1 = P(t)[x_1 - \Phi(t_1, t_0)x_0]$. Thus, controllability of the nonlinear system (2.1.8) is equivalent to the solvability of the above operator equation. Now we consider the linear infinite dimensional system in Banach space described by the equation

$$\left.\begin{array}{l} x'(t) = Ax(t) + B(t)u(t), \;\; 0 \leq t_0 < t \leq t_1 < \infty \\ x(0) = x_0 \end{array}\right\} \tag{2.1.12}$$

where, the state $x(t)$ takes values in a Banach space X for each $t \in J = [t_0, t_1]$, control function $u(.)$ is given in $L^2(J, U)$, a Banach space of admissible control functions, with U as a Banach space. Here A is the infinitesimal generator of a strongly continuous semigroup $T(t), t \geq 0$ in the Banach space X.

If X is a finite dimentsional space and A is a matrix then $T(t-s) = e^{A(t-s)}$ reduces to the transition matrix.

Definition 2.1.3. *A strongly continuous family $\{T(t)\}_{t \geq 0}$ of bounded operators in a Banach space X is called a* ***semigroup*** *generated by A if*

(i) $T(t+s)x = T(t)T(s)x, \;\; x \in X$ *and* $t, s \geq 0$,

(ii) $T(0)x = x, \;\; x \in X$,

(iii)$t \mapsto T(t)x$ *is continuous for* $t \geq 0, x \in X$,

(iv) $Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} = \frac{d^+ T(t)x}{dt}|_{t=0}, \;\; x \in D(A)$, *where*
$D(A) = \{x \in X : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{exists}\}$.

Remark 2.1.3. *In Definition 2.1.3, the condition (iv) gives the generator of the semigroup in terms of an operator A (refer [80]).*

Example:2.1.2 Consider the one-dimensional heat equation on $\Omega = (0,1)$

$$\left.\begin{aligned} &\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \text{ in } \Omega \times (0,T) \\ &y(x,0) = y_0(x) \text{ in } \Omega \\ &y(0,t) = 0 = y(1,t) \text{ in } (0,T). \end{aligned}\right\}$$

The above system can be associated with the evolution equation

$$\frac{dy}{dt} = Ay \text{ on } H = L^2(0,1)$$

where $A : H \to H$ such that $Ay = y''$

$$\mathcal{D}(A) = \{y \in H : y, y' \text{ are absolutely continuous}, \; y(0) = y(1) = 0\}$$

H is a Hilbert space. It is easy to show that A generates semigroup $S(t), t \geq 0$ in $L^2(0,1)$ given by

$$S(t)y = \sum_{n=1}^{\infty} 2 \, \exp(-n^2\pi^2 t) \sin n\pi x \int_0^1 \sin n\pi y(\tau) d\tau, \quad y \in L^2(0,1).$$

2.2 Some Results from Analysis

This section deals with some basic results from mathematical analysis and differential equations, including some definitions, lemmas and theorems which will be of frequent use in the subsequent chapters.

Definition 2.2.1. *Let X be a real Banach space and X^* be the dual of X. Let $T : D(T) \subset X \to X^*$ be any operator and Y be another real Banach space and $\{x_n\}$ is a sequence in X then the operator $T : X \to Y$ is called* ***continuous*** *at x_0 if*

$$x_n \to x_0 \Rightarrow Tx_n \to Tx_0$$

Definition 2.2.2. *T, as defined in Definition 2.2.1, is called* ***bounded*** *if it maps every bounded sequence $\{x_n\}$ in X into bounded sequence $\{Tx_n\}$ in Y and T is called* ***compact*** *if for any bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ has a converging subsequence in Y.*

Definition 2.2.3. *$T : X \to Y$ is* ***relatively compact*** *if for any bounded subset B of X, $\overline{T(B)}$ lies in compact subset of Y.*

Definition 2.2.4. *Let Lip be the set of all operators $T : X \to X$ such that there exists a constant $\alpha > 0$ satisfying $\|Tx - Ty\| \leq \alpha\|x - y\|$ $\forall x, y \in X$. For $T \in$ Lip , we define*

$$\|T\|^* = \sup_{x,y \in X; x \neq y} \frac{\|Tx - Ty\|}{\|x - y\|}$$

If $T \in$ Lip with $\|T\|^* = \alpha$, we say that T is ***Lipschitz continuous*** *with constant α. If $\alpha < 1$ then we say that T is a contraction.*

We note that :

1. $T, S \in \text{Lip} \Rightarrow \|TS\|^* \leq \|T\|^*\|S\|^*$
2. $T \in BL(X) \Rightarrow \|T\|^* = \|T\|$

For more details refer Dolezal [26].

The following Fixed point theorem will be used to prove the controllability results of different nonlinear systems.

Theorem 2.2.1. ***(Banach contraction Principle,*** *Limaye [55])*
Let X be a Banach space and $T : X \to X$ be a contraction on X. Then T has precisely one fixed point, and the fixed point can be computed by the iterative scheme $x_{n+1} = Tx_n$, x_0 being any arbitrary initial guess. □

Chapter 3 and Chapter 4 deal with second order system in finite dimension. Since in many cases, it is advantageous to treat the second order system rather than to convert them to first order system. We treat this system directly, rather than converting it into first order system. In order to derive results to handle second order system without converting it into first order one, we use Sine and Cosine Matrices. Commonly trigonometric matrices are computed using Taylor series expansion. However, these expansions are valied in a small neighborhood. Hence to have global approximation values we use rational approximation.

Definition 2.2.5. *A* ***rational approximation*** *to f(x) on [a,b] is the quotient of two polynomials $P_N(x)$ and $Q_M(x)$ of degrees N and M, respectively. We use the* *notation* $R_{N,M}(x)$ *to denote this quotient:*

$$f(x) \approx R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)} \text{ for } a \leq x \leq b.$$

Definition 2.2.6. ***Páde approximation*** *is a rational approximation in which it requires that f(x) and its derivative be continuous at x = 0.*

Páde approximation is used to compute the Sine and Cosine Matrices and which will be useful to find the solution of the second order system discussed in chapter 3.

In order to study the controllability of linear second order matrix differential equation, we will require the following results.

Lemma 2.2.1. (Chen[17]) *Let* f_i, *for* $i = 1, 2, ..., n$, *be* $1 \times p$ *complex vector valued continuous functions defined on* $[t_1, t_2]$. *Let* F *be the* $n \times p$ *matrix with* f_i *as its* i^{th} *row. Define*

$$W(t_1, t_2) = \int_{t_1}^{t_2} F(t)F^*(t)dt$$

Then $f_1, f_2, ..., f_n$ *are linearly independent on* $[t_1, t_2]$ *if and only if the* $n \times n$ *constant matrix* $W(t_1, t_2)$ *is nonsingular.*

Lemma 2.2.2. (Chen [17]) *Assume that for each* i, f_i *is analytic on* $[t_1, t_2]$. *Let* F *be the* $n \times p$ *matrix with* f_i *as its* i^{th} *row, and let* $F^{(k)}$ *be the* k^{th} *derivative of* F. *Let* t_0 *be any fixed point in* $[t_1, t_2]$. *Then the* f_i *are linearly independent on* $[t_1, t_2]$ *if and only if*

$$Rank[F(t_0) : F^{(1)}(t_0) : .... : F^{(n-1)}(t_0) : .....] = n.$$

In chapter 5, we investigate the controllability property of a impulsive control system governed by the nonlinear integro-differential equation:

$$\left.\begin{array}{l} x'(t) = Ax(t) + f(t, x(t), Tx(t), Sx(t)) + B(t)u(t), \quad 0 < t < T, \quad t \neq t_k \\ x(0) = x_0, \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, ..., m. \end{array}\right\} \quad (2.2.1)$$

in a Banach space X.

Where, $f \in C([0,T] \times X \times X \times X, X)$, A is infinitesimal generator of a C_0 semigroup $\{G(t)|_{t \geq 0}\}$ with impulsive condition (refer Pazy, [64]) and $B(t)$ is a bounded linear operator from U to X and the control function $u(\cdot)$ is in $L^2([0,T];U)$, U is another Banach space. T and S are operators defined by

$$Tx(t) = \int_0^t K(t,s)x(s)ds, \quad K \in C[D, R^+]$$

$$Sx(t) = \int_0^T H(t,s)x(s)ds, \quad H \in C[D_0, R^+]$$

where

$$D = \{(t,s) \in R^2 : 0 \leq s \leq t \leq T\}$$

$$D_0 = \{(t,s) \in R^2 : 0 \leq t, s \leq T\}$$

$x_0 \in X$ is the initial condition. $0 < t_1 < t_2 < t_3 < ... < t_m < T, I_k : X \to X$ is an impulsive function, $k = 1, 2, ..., m$. $\Delta x(t_k)$ denotes the jump of $x(t)$ at $t = t_k$, that is,

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-)$$

where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively.

Definitions and theorems below are useful in proving controllability result of the impulsive systems.

Definition 2.2.7. ***Space of Piecewise continuous functions*** *is denoted by $PC([0,T],X)$ and is defined as the space of all functions $x : [0,T] \to X$, such that $x(t)$ is continuous at $t \neq t_k$ and left continuous at $t = t_k$ and the right limit $x(t_k^+)$ exists for $k = 1, 2, ..., m$.*

Evidently, $PC([0,T],X)$ is a Banach space with norm (refer Guo and Liu [40])

$$||x||_{PC} = \sup_{t\in[0,T]} ||x(t)||.$$

Definition 2.2.8. *A function $x(\cdot) \in PC([0,T],X)$ is a **mild solution** of equation (2.2.1) if it satisfies*

$$x(t) = G(t)x_0 + \int_0^t G(t-s)f(s,x(s),Tx(s),Sx(s))ds +$$

$$\sum_{0<t_k<t} G(t-t_k)I_k(x(t_k)) + \int_0^t G(t-s)B(s)u(s)ds, \quad 0 \leq t \leq T \tag{2.2.2}$$

Definition 2.2.9. *A **classical solution** of equation (2.2.1) is a function* $x(\cdot) \in PC([0,T],X) \bigcap C^1((0,T)\backslash\{t_1,t_2,...,t_p\},X)$, $x(t) \in D(A)$ *for* $t \in (0,T)\backslash\{t_1,t_2,...,t_p\}$, *which satisfies equation (2.2.1) on* $[0,T]$.

Let us consider a metric d on a set T , let

$$B(T) = \{x : T \to X, \sup_{t\in T} |x(t)| < \infty\}$$

and

$$C(T) = \{x \in B(T) : \ x \text{ continuous on } T\}$$

and $B(T)$ is complete in the sup metric

$$d(x,y) = \sup_{t\in T} |x(t) - y(t)| \quad , x, y \in B(T)$$

Definition 2.2.10. *A subset E of C(T) is bounded in the sup metric if and only if there is some* $\alpha > 0$ *such that* $|x(t)| \leq \alpha$ *for all* $x \in E$ *and all* $t \in T$, *then the functions in E are **uniformly bounded** on T.*

Definition 2.2.11. *A subset E of C(T) is said to be **equicontinuous** at* $t \in T$ *if for every* $\epsilon > 0$, *there is some* $\delta > 0$ *such that* $|x(t) - x(s)| < \epsilon$ *for all* $x \in E$ *and* $s \in T$ *with* $d(s,t) < \delta$, *where* δ *may depend on t, but not* $x \in E$.

Theorem 2.2.2. *(**Arzela-Ascoli** [55])*
Let T be a compact metric space and $E \subset C(T)$. *Suppose that E is bounded as well as equicontinuous at each* $t \in T$. *Then*

(a) ***Ascoli-1883***
E is uniformly bounded on T. In fact E is totally bounded in the sup metric on C(T).

(b) ***Arzela - 1889***
Every sequence in E contains a uniformly convergent subsequence.

□

Theorem 2.2.3. *(**Schauder Fixed Point Theorem** [19])*
Let E be a nonempty convex closed subset of a normed linear space X and let F be

a relatively compact subset of E. Then every continuous mapping of E into F has a fixed point. □

Chapter 6 studies controllability analysis of first order system using spectral theory. The following theorem will be useful in proving the controllability of the linear system.

Theorem 2.2.4. *Let A be a real symmetric matrix and $\lambda_1, \lambda_2, ..., \lambda_k$ be distinct eigenvalues of A. Let $u_i \in R^n$ be non-zero vector such that*

$$Au_i = \lambda_i u_i, \quad 1 \leq i \leq k.$$

Then $\{u_1, u_2, ..., u_k\}$ forms an orthonormal set. □

Theorem 2.2.5. *Let A be an $n \times n$ real symmetric matrix such that all its eigen values are distinct. Then there exists an orthogonal matrix P such that*

$$P^{-1}AP = D$$

where D is a diagonal matrix with diagonal entries being the eigen values of A. □

The last chapter of the thesis is related to inclusion systems. The difference between an ordinary differential equation and a differential inclusion is that the right hand side of the differential inclusion is a **set** instead of a single-valued function in the differential equation. The solution of the differential inclusion is also a set and not a single system trajectory. Any function that satisfies differential inclusion system is a trajectory of the differential inclusion, but not a solution of the differential inclusion.

In the decade 1930-40 the existence and the properties of the solution to the differential inclusions in the finite dimensional spaces have been studied and subsequently it has been generalized to infinite dimensional spaces.

The following tools will be used in the analysis of inclusion systems.

Definition 2.2.12. *A function $f : X \to R$ is said to be* ***convex*** *if its domain $D(f)$ is a convex set and for every $u, v \in D(f)$*

$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda))f(v),$$

where, $0 \leq \lambda \leq 1$.

Definition 2.2.13. *Let $(X, \|.\|)$ be a Banach space. A multi valued map $G_1 : X \to 2^X$ is* **convex (closed) valued** *if $G_1(x)$ is convex (closed) for all $x \in X$. G_1 is bounded on bounded sets if $G_1(B) = U_{x\in B}G_1(x)$ is bounded in X for any bounded set B of X (i.e.$sup_{x\in B}\{sup\{\|x\| : x \in G_1(x)\}\} < \infty$).*

Definition 2.2.14. *The multi-map G_1 is called* **upper semi continuous (u.s.c.)** *on X if for each $x_0 \in X$ the set $G_1(x_0)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G_1(x_0)$, there exists an open neighborhood A of x_0 such that $G_1(A) \subseteq B$.*

Definition 2.2.15. *The multi-map G_1 is called* **lower semi continuous (l.s.c.) at** *$x \in D(G_1)$ if and only if for any $y \in G_1(x)$ and for any sequence of elements $x_n \in D(G_1)$ converging to x, there exists a sequence of elements $y_n \in G_1(x_n)$ converging to y. It is said to be* **lower semi continuous (l.s.c.)** *if it is lower semi continuous at every point $x \in D(G_1)$.*

Examples: Let $X = R$. The set-valued map $F_1 : R \mapsto R$ defined by

$$F_1(x) := \begin{cases} [-1, +1] & \text{if } x \neq 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

is lower semi continuous at zero but not upper semi continuous at zero.

The set-valued map $F_2 : R \mapsto R$ defined by

$$F_2(x) := \begin{cases} \{0\} & \text{if } x \neq 0 \\ [-1, +1] & \text{if } x = 0 \end{cases}$$

is upper semi continuous at zero but not lower semi continuous at zero. Refer, Aubin and Frankowska [6].

Let $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

Definition 2.2.16. *A multi-valued map $G_1 : J \to BCC(X)$ is said to be measurable, if for each $x \in X$, the distance between x and $G_1(x)$ is a measurable function on J. i.e. for each $x \in X$, the function $Y : J \to R$ defined by*

$$Y(t) = d(x, G_1(t)) = inf\{\|x - z\| : z \in G_1(t)\} \in L^1(J, R).$$

For more details on multi-valued map, see ([22],[43]).

Theorem 2.2.6. ***(Lebesgue Dominated Convergence Theorem** (refer Royden[67]))*
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $L[a,b]$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ almost everwhere.}$$

Suppose there exits $g \in L[a,b]$ such that $|f_n(x)| \leq g(x)$ almost everywhere. Then $f \in L[a,b]$ and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

□

Theorem 2.2.7. ***(Bohnenblust-Karlin** [12])*
Let Σ be a non-empty, closed convex subsets of a Banach space $\mathcal{B}$. If $\Gamma : \Sigma \to 2^{\Sigma}$ is such that

(a) *$\Gamma(a)$ is non-empty and convex for each $a \in \Sigma$,*

(b) *the graph of Γ, $\mathcal{G}(\Gamma) \subset \Sigma \times \Sigma$, is closed,*

(c) *$\cup \{\Gamma(a)/a \in \Sigma\}$ is contained in a sequentially compact set $\mathcal{F} \in B$, then the map Γ has a fixed point, that is, there exists a $\sigma_0 \in \Sigma$ such that $\sigma_0 \in \Gamma(\sigma_0)$.*

□