Chapter 1

Introduction

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1.1 Introduction

In the present thesis, we investigate controllability and stability problems of discrete-time systems. Since more and more digital devices are being used for information processing and control purposes in a variety of system applications, including industrial processes, power networks, biological systems and communication networks, study of discrete-time systems is essential. This trend is mainly due to the availability of low cost digital computers. So, for those applications where digital devices are used, it is reasonable to model the system in discrete-time. In addition, there are other application areas, e.g. econometric systems, monetary systems, inventory systems, environmental systems where the underlying models are inherently discrete-time and here discrete-time approaches to analysis and control are the most appropriate.

In order to deal with these two situations, there has been a lot of interest in developing techniques which allow us to do study the basic properties of the discrete-time systems. The behavior of discrete-time systems can be described in terms of difference equations. During 90's only the difference equations have started receiving the attention they deserve.

Controllability and Stability are the two basic properties of dynamical systems. A system is said to be controllable if it is possible to steer a system from a given initial state to an arbitrary final state in finite number of time steps. Kalman (refer [18], [19]) introduced the concept of controllability of finite-dimensional linear systems in 1960's and subsequently this concept was extended to nonlinear systems by Krabs (see [58], [59], [60], [61]), Klamka [26], Chen and Narendra [27]. The classical theory of controllability for discrete-time systems in finite dimensional space was extended for linear abstract system defined on infinite dimensional spaces by Sasu [3] and Phat [57] etc.

Problems of optimal control have received a great deal of attention from control engineers. In designing an optimal control system, we need to find a rule for determining the present control decision subject to certain constraints, so as to minimize some measure of the deviation from ideal behavior. Many authors (see Belbas [48], [49] and Gaishun and Dymkov [11]) have contributed on this problem.

The control theory of linear system is almost saturated in the literature. Though there has been many results available for the nonlinear systems, many problems are still open for nonlinear system. Furthermore, the computational algorithm for the steering control is important for engineering systems, which is not easily available in the literature. Development of powerful tools in difference equations, linear algebra and functional analysis resulted in the enrichment of control theory considerably.

In the present thesis, we investigate controllability of nonlinear systems by using some tools from functional analysis such as fixed point theory, inverse function theorem, implicit function theorem etc. Along with controllability results, we made attempt to obtain a computational procedure for the actual computation of steering control for nonlinear system and we prove that the steering controller which we have considered is well-defined (see [38]).

Volterra systems are often appear in population dynamics. We study the controllability of discrete Volterra systems both linear and nonlinear using inverse function theorem and implicit function theorem. Our results are computational in nature. We have also investigated quadratic optimal control for discrete linear Volterra system by the conventional minimization method using Lagrange multipliers.

We are often interested in the qualitative behavior of solutions without actually computing them. Realizing that most of the problems that arise in practice are nonlinear and mostly unsolvable, this investigation is of vital importance to scientists, engineers and even to applied mathematicians. Also for a given difference system, one of the pioneer problems is the study of ultimate behavior of its solutions. i.e the study of asymptotic behavior of discrete systems. There exists a huge literature devoted to this problem.

In 2006, Czornik [6] and Xue [63] obtained new necessary and sufficient conditions for asymptotic stability of null solution of linear difference system. We extended these results to nonlinear difference systems. Also Pinto [31], introduced the concept of (h, k) dichotomy and proved the asymptotic relationship between solution of ordinary linear difference system and its nonlinear perturbed system. We extend this result to more general discrete Volterra system using the concept of ordinary dichotomy. Many authors like Elaydi [9], Kolmanovskii [56], Eloe [36], Medina (refer [44], [45], [46]), Baker (see [51], [52]) have greatly contributed in this area.

The problems that we deal in the thesis are as follows:

I. Steering Control of Semi-linear Discrete Dynamical System

Krabs [61] studied the controllability of a general difference system of the form

$$x(t+1) = f(x(t), u(t))$$

and also obtained controller that steers a given initial state to a desired final state for the linear system

$$x(t+1) = A(t)x(t) + B(t)u(t), \ \ x(0) = x_0 \ \ , t \in N_0 = \{0, 1, 2, ...\}. \tag{1.1.1}$$

Here in this problem, we consider a semi-linear system of difference equation of the form

$$x(t+1) = A(t)x(t) + B(t)u(t) + f(t, x(t)), \ \ x(0) = x_0 \ \ , t \in N_0 \tag{1.1.2}$$

Here, $(A(t))_{t \in N_0}$ and $(B(t))_{t \in N_0}$ are sequences of real $n \times n$ and $n \times m$ matrices respectively, $(x(t))_{t\in N_0}$ and $(u(t))_{t\in N_0}$ are sequences of state vectors in R^n and control vectors in R^m, respectively. $f(.,.) : N_0 \times R^n \to R^n$ is a nonlinear function satisfying Lipschitz condition with respect to the second argument. We give the computational scheme for the steering control for (1.1.2). For $t \in N_0$, we define a controller

$$u(t) := B(t)^*\Phi(N,t+1)^*W_r(0,N)^{-1}[x_1 - \Phi(N,0)x_0 - \sum_{j=0}^{N-1}\Phi(N,j+1)f(j,x(j))] \tag{1.1.3}$$

where $W_r(0,N)$ is called reachability Grammian defined by

$$W_r(0,N) := \sum_{j=0}^{N-1}\Phi(N,j+1)B(j)B(j)^*\Phi(N,j+1)^*. \tag{1.1.4}$$

here $\Phi(m,n)$ is a fundamental matrix of linear homogeneous system

$$x(t+1) = A(t)x(t).$$

We claim that this control is well defined and steers the semi-linear system (1.1.2) from x_0 to x_1 under the following conditions (refer [38]).

Conditions :

[L] The linear system (1.1.1) is controllable.

[N] The nonlinear function $f(t,x)$ is Lipschitz continuous with respect to x,

$$i.e.\ \exists\ \alpha > 0\ \ \| f(t,x) - f(t,y) \| \leq \alpha\ \| x - y \|, \forall x, y \in R^n$$

Under the same assumptions, we also prove that controllability and reachability of the system (1.1.2) are equivalent. Numerical example for the computation of steering control of system (1.1.2) is also provided.

II. Controllability of Discrete Volterra Systems

Gaishun and Dymkov [11] studied the controllability of the Volterra linear discrete system

$$x(t+1) = \sum_{i=0}^{t} A(i)x(t-i) + Bu(t),\ \ t \in N_0 = \{0,1,2,...\}$$

by a method based on the representation of the Volterra operator generated by the equation in the ring of formal power series. In this problem, we study controllability of a non-autonomous linear Volterra system of the form :

$$\Sigma_L : x(t+1) = \sum_{i=0}^{t} A(i)x(t-i) + B(t)u(t), \;\; t \in N_0 \tag{1.1.5}$$

and a semi-linear discrete Volterra system of the form

$$\Sigma_N : x(t+1) = \sum_{i=0}^{t} A(i)x(t-i) + B(t)u(t) + f(x(t), u(t)), \;\; t \in N_0 \tag{1.1.6}$$

using a different approach and in much more straightforward manner. Here, $(A(t))_{t\in N_0}$ and $(B(t))_{t\in N_0}$ are sequences of real $n \times n$ and $n \times m$ - matrices, respectively, and $(x(t))_{t\in N_0}$ and $(u(t))_{t\in N_0}$ are sequences of state vectors in R^n and control vectors in R^m, respectively. $f(.,.) : R^n \times R^m \to R^n$ is a nonlinear function of state and control variables. It follows easily that for a given control sequence $\{u(t)\}_{t\in N_0}$ and initial state $x(t_0) = x_0$, there exists a unique solution to the linear system Σ_L.

We make the following definitions to obtain solution of Σ_L. Define the set of linear operators $Q_t : R^n \to R^n$, $t \in N_0$ by

$$Q_0 = I, \quad Q_{t+1} = \sum_{i=0}^{t} A(i)Q_{t-i}, \;\; t \in N_0 \tag{1.1.7}$$

Hence the solution of (1.1.5) is given by

$$x(t) = Q_t x_0 + \sum_{i=0}^{t-1} Q_i B(t-i-1)u(t-i-1) \tag{1.1.8}$$

The controllability Grammian for the linear Volterra system (1.1.5) is given by

$$W(0, N) = \sum_{i=1}^{N} Q_{i-1}B(N-i)B^*(N-i)Q_{i-1}^* \tag{1.1.9}$$

We establish controllability result for the linear Volterra system Σ_L. We give two different conditions, for the global controllability of (1.1.5), namely

(i) condition using controllability Grammian and

(ii) Kalman type rank condition.

Also using the notion of "higher order" functions, inverse function theorem and implicit function theorem, we prove local controllability of the semi-linear system (1.1.6). Numerical examples are provided to substantiate our results.

III. Asymptotic Stability of Nonlinear Discrete Dynamical System Involving (sp) Matrix

We consider the following discrete dynamical system

$$x(t+1) = g(t, x(t)) \quad t \in N_0 = \{0, 1, 2, ...\} \tag{1.1.10}$$

where $g : N_0 \times \Omega \to \Omega$, $\Omega \subset R^n$, is a continuous nonlinear function satisfying $g(t, 0) = 0 \ \forall t$. We take g in the form

$$g(t, x(t)) = Ax(t) + f(t, x(t))$$

where $x(t) \in \Omega$, $A \in s = \{A = (a_{ij})_{n\times n} : a_{ij} \geq 0, \Sigma_{j=1}^{n} a_{ij} \leq 1, \forall i = 1, 2, ..., n\}$ is a (sp) matrix, and the function $f : N_0 \times \Omega \to \Omega$ satisfies the inequality

$$\| f(t, x(t)) \| \leq a(t) \| x(t) \|, t \in N_0$$

where $\sum a(t)$ is a convergent series of positive numbers. We prove that the null solution of the system is exponentially stable. It is well known that if the spectral radius of the jacobian $Dg(0)$ of system (1.1.10) is strictly smaller than 1, then the null solution is exponentially stable. In order to check this condition, we have to compute the eigenvalues of the jacobian, which is a difficult task for higher dimensional systems. Checking if a matrix is (sp) can be easily done even for higher dimensional matrices, using a simple algorithm (described in the definition of the (sp) matrix in the subsequent chapter). Therefore, the method proposed here is very efficient for numerical computations, as it avoids the evaluation of the eigenvalues of the Jacobian. Recently Xue and Guo [63] studied asymptotic stability of null solution of

$$x(t+1) = Ax(t) \quad t = 0, 1, 2, ... \tag{1.1.11}$$

by introducing the notion of (sp) matrix. In [63], authors proved that the zero solution of (1.1.11) is asymptotically stable if and only if A is a (sp) matrix. We provide sufficient conditions on the nonlinear function f to ensure that the null solution of the perturbed system (1.1.10) is not only asymptotically stable but exponentially stable also (refer [37]). Exponential stability is much stronger property than asymptotic stability. Numerical examples are given to support our results.

IV. Accurate Solution Estimate and Asymptotic Behavior of Nonlinear Discrete System

In this problem, we deal with the nonlinear nonautonomous discrete dynamical system of the form

$$x(t+1) = A(t)x(t) + f(t, x(t)), \;\; t \in N_0 = \{0, 1, 2, ...\} \tag{1.1.12}$$

We first derive accurate estimate for the norm of solution of this system. This give us stability condition and bound for the region of attraction of the stationary solution. Medina and Gil [43], derived accurate estimates for the norms of solutions of such system by using the approach based on "freezing" method for difference equations and on recent estimates for the powers of a constant matrix.

We also give sufficient conditions for the asymptotic stability of the null solution of the above system. Our approach is based on the concept of generalized subradius for the coefficient matrices. In (Czornik 2005 [5]), the ideas of generalized spectral subradius and the joint spectral subradius are introduced and shown the relationship between generalized spectral radii and the stability of discrete time-varying linear system. Numerical example showing asymptotic behavior of the null solution is also given to support our result.

V. Asymptotic Equivalence of Discrete Volterra Systems

The problem of the asymptotic relationship between the solutions of a linear Volterra difference equation and its perturbed equation is studied by Criscii [34], Morchalo [13], Choi [53], Cuevas and Pinto [4], Medina [44] by means of the direct Lyapunov method, comparision theorems and by taking certain conditions on nonlinear function f and resolvent matrix associated with (1.1.15). Pinto [31] studied the asymptotic equivalence between the solutions of linear difference system

$$x(t+1) = A(t)x(t), t \in N_0 \tag{1.1.13}$$

and its perturbed equation

$$y(t+1) = A(t)y(t) + f(t, y(t)) \tag{1.1.14}$$

under dichotomic situations of system (1.1.13). In this problem, we consider the linear Volterra system of the type

$$x(t+1) = A(t)x(t) + \sum_{r=t_0}^{t} B(t-r)x(r), \quad x(t_0) = x_0, \quad t \in N(t_0) = \{t_0, t_0+1, ...\}, t_0 \in N_0 \tag{1.1.15}$$

and its perturbation

$$y(t+1) = A(t)y(t) + \sum_{r=t_0}^{t} B(t-r)y(r) + f(t, y(t)), \quad y(t_0) = x_0, \quad t \in N(t_0) \tag{1.1.16}$$

where $A(t)$ is a $n \times n$ nonsingular matrix function, $B(t)$ is a $n \times n$ matrix function and $f : N(t_0) \times R^n \to R^n$ is a continuous nonlinear function. It can be proved that

$$x(t, t_0, x_0) = X(t)x_0 \tag{1.1.17}$$

is a unique solution of equation (1.1.15) with $x(t_0) = x_0$. where $X(t)$ is a $n \times n$ matrix, called the fundamental matrix of system (1.1.15) and satisfies the following equation

$$X(t+1) = A(t)X(t) + \sum_{r=t_0}^{t} B(t-r)X(r)$$

We use Banach's fixed point theorem and dichotomy property of the linear system to obtain the asymptotic equivalence between the solutions of linear difference system (1.1.15) and (1.1.16). Our result generalizes the theorem proved by Pinto [31] under the following hypothesis.

[A] The linear system (1.1.15) has an ordinary dichotomy on $N(t_0)$

[B] $f : N(t_0) \times R^n \to R^n$ is a continuous function such that

$$\| f(t,x) - f(t,y) \| \leq \mu(t) \| x - y \|$$

where $\mu \in l^1([t_0, \infty])$ and $\sum_{s=t_0}^{\infty} \| f(s,0) \| < \infty$.

Furthermore, let P be the projection matrix used in the definition of dichotomy. Under asymptotic condition $X(t)P \to 0$ as $t \to \infty$, we prove that

$$y(t) = x(t) + o(1) \;\; as \;\; t \to \infty$$

VI. Optimal Control of Discrete Volterra System - A Classical Approach

The problem of optimal control for discrete Volterra system is highly interesting and many authors Gaishun and Dymkov [11], Belbas ([48], [49]) have contributed in this directions. Recently Belbas [48] has studied optimal control problem of Volterra equations with impulses. Here we adapt independent approach of classical minimization technique called method of Lagrangian multipliers to find the optimal control of the following linear Volterra system.

$$x(t+1) = \sum_{i=0}^{t} A(i)x(t-i) + Bu(t), \;\; t \in N_0 \tag{1.1.18}$$

where $A_i, i = 0, 1, ...t$'s are $n \times n$ nonsingular matrices and B is $n \times r$ matrix. We consider a quadratic performance index for the finite time process $(0 \leq t \leq N)$ as

$$J = \frac{1}{2}x^*(N)Sx(N) + \frac{1}{2}\sum_{t=0}^{N-1}[x^*(t)Qx(t) + u^*(t)Ru(t)] \tag{1.1.19}$$

where S, Q are $n \times n$ positive definite or positive semidefinite Hermitian matrices (or real symmetric matrices), R is an $r \times r$ positive definite Hermitian or real symmetric matrix.

We find a controller which minimizes J as given by equation (1.1.19), when it is subjected to the constraint equation specified by (1.1.18) and when initial condition on state vector is specified as

$$x(0) = c. \tag{1.1.20}$$

For computation of steering control and other numerical examples, we use Matlab software.

1.2 Layout of the Thesis

The thesis is organized as follows:

Chapter 1 deals with a general introduction to the thesis.

Chapter 2 focuses on the necessary concepts of control theory and analysis which will be used subsequently in the thesis.

In Chapter 3, we investigate the controllability property of a class of semi-linear non-autonomous system described by the difference equation

$$x(t+1) = A(t)x(t) + B(t)u(t) + f(t, x(t)) \;\; t \in N_0$$

under the assumption that its linear part is controllable and the nonlinear function f satisfies a Lipschitz condition. We also give an algorithm to compute steering control for the system.

In Chapter 4, a necessary and sufficient condition is established for controllability of discrete-time linear Volterra systems. Local controllability result for a semi-linear discrete Volterra system is also proved. Numerical examples are provided to illustrate our results.

In Chapter 5, we give sufficient conditions for exponential stability of null solution of a nonlinear autonomous discrete dynamical system by using the concept of (sp) matrix and taking some growth condition on nonlinear function.

We also derive accurate estimate for the norm of solution of discrete system. This gives us stability condition and bound for the region of attraction of the stationary solution. We also give sufficient conditions for the asymptotic stability of the null solution of the system using the approach based on the concept of generalized subradius for the coefficient matrices.

The problem of the asymptotic relationship between the solutions of a linear Volterra difference equation and its nonlinear perturbation is discussed using the concept of discrete dichotomy.

Chapter 6, concludes the thesis with the optimal control problem of liner discrete Volterra system using the classical optimization technique of Lagrange multipliers.

Numerical computation is carried out in MATLAB to support our results with examples. MATLAB programs are included in the Appendix.