Chapter 2

Preliminaries

Contents

Since the thesis deals with the controllability and Stability of nonlinear discrete dynamical systems, we devote this chapter to a review of the most important ingredients of discrete-time systems. The behavior of discrete-time systems can be described in terms of difference equations. So the main purpose of this chapter is to establish a conceptual framework, introduce notational conventions and give a survey of the basic facts of difference equations. The starting point of the chapter (i.e Section 2.1) shows importance of studying discrete systems. In Section 2.2, state space description of discrete-time system is given. Section 2.3 discusses the linearization of nonlinear system. In Section 2.4, initial value problem of discrete-time systems is introduced and shown the existence and uniqueness of its solutions. The concept of fundamental matrix along with its properties are given in this section and the solutions of discrete-time systems using fundamental matrix and using variation of constants method are also given. In Section 2.5, we introduce the discrete Volterra systems and its solution using resolvent matrix. Section 2.6 deals with the stability theory. Firstly, the various notions of stability are introduced followed by the known results in this area. Discrete Gronwall inequality is a very useful tool which is also given in this section.
In this section we also introduce the concept of (sp) matrix and its application in stability of linear discrete-time system is given. Also the definition of generalized subradius is given and shown the relationship between generalized spectral radii and stability of discrete time-varying linear system. In this section, we also define the notion of ordinary dichotomy and discuss its applications in showing the asymptotic equivalence of linear discrete system and its perturbed system. Section 2.7 incorporates controllability and reachability concepts.

The notion of optimal control is also discussed in this section. Some tools of analysis namely higher order functions, inverse function theorem, implicit function theorem, Banach's fixed point theorem are given in Section 2.8. In what follows, throughout we shall assume that the functions appearing in the nonlinear systems under study are continuous with respect to the dependent variables.

2.1 Why to Study Discrete Dynamical System?

We need difference equation models when either the independent variable is discrete or it is mathematically convenient to treat it as a discrete variable. There are lots of applications areas where discrete-time system is preferred. In Genetics, the genetic characteristics change from generation to generation and the variable representing generation is a discrete variable. In Economics, the price changes are considered from year to year or from month to month or from week to week or from day to day. In every case, time variable is discretized. In Population dynamics, we consider the changes in population from one age-group to another and the variable representing age-group is a discrete variable. Discrete-time systems can be classified into two types.

- **Inherently discrete-time systems**, such as digital computers, digital filters, monetary systems and inventory systems, population dynamics. In such systems it makes sense to consider the system at discrete instants of time only, and what happens in between is irrelevant.

  e.g. If we study saving bank account transactions or the process of repaying loans in bank, we obviously arrive at the discrete-time system.

  1. Saving bank account

  Let the scalar quantity $x(n)$ be the balance of a saving bank account at the beginning of the n-th month, and let r be the monthly interest rate. Also let the scalar quantity $u(n)$ be the total of deposits and withdrawals during the n-th month. Assuming that the interest is computed monthly on the basis of the balance at the beginning of the month, the sequence $x(n),\ n = 0, 1, 2, ...$ satisfies the linear difference equation

  $$x(n+1) = (1+r)x(n) + u(n), \quad x(0) = x_0 \tag{2.1.1}$$

  Where x_0 is the initial balance. This equation describes a linear time invariant discrete-time system.

2. Amortization

Amortization is a process by which a loan is repaid by a sequence of periodic payments, each of which is part payment of interest and part payment to reduce the outstanding principal.

Let $p(n)$ represent the outstanding principal after the n^{th} payment $g(n)$. Suppose that interest charges compounded at the rate r per payment period.

The formulation of model here is based on the fact that the outstanding principal $p(n+1)$after the $(n+1)^{st}$payment is equal to the outstanding principal $p(n)$ after the n^{th}payment plus the interest rate $rp(n)$ incurred during the $(n+1)^{st}$ period minus the n^{th} payment $g(n)$. Hence

$$p(n+1) = (1+r)p(n) - g(n), \;\; p(0) = p_0 \tag{2.1.2}$$

Where p_0 is the initial debt.

- **Discrete-time systems that result from considering continuous-time systems at discrete instants of time only.** This may be done for reasons of convenience (e.g. when analyzing a continuous-time system on digital computer), or may arise naturally when the continuous-time system is interconnected with inherently discrete-time systems (such as digital controller).

In recent years there has been a rapid increase in the use of digital controllers in control systems. They are used for achieving optimal performance for example, in the form of maximum productivity, maximum profit, minimum cost or minimum energy use. The application of computer control has made possible "intelligent" motion in industrial robots, the optimization of fuel economy in automobiles, and refinements in the operation of household appliances and machines such as microwave ovens and sewing machines.

The current trend towards digital rather than analog control of dynamic systems is mainly due to the availability of low-cost digital computers. Also many problems in natural sciences and engineering fields are formulated into scalar differential equation or a vector differential equations. In solving these differential equations by use of digital computer, we reduce differential equations to difference equations by using any discretization techniques.

For example, if we are given differential equation of the form

$$\frac{dx}{dt} = f(t, x(t)), \;\; x(t_0) = x_0,$$

then by using Euler's method we can obtain difference equation as follows.

$$\begin{aligned}
\frac{x(t+h) - x(t)}{h} &= f(t, x(t)) \\
i.e. \quad x(t+h) &= x(t) + hf(t, x(t)) \\
i.e. \quad x(t+1) &= x(t) + f(t, x(t))
\end{aligned}$$

So we conclude that there are lots of applications where we arrive at difference equations and hence its study is essential.

2.2 State Space Description of Discrete-time System

Many systems can be described by a set of simultaneous difference equations of the form

$$x(t+1) = f(t, x(t)), \;\; t \in N_0 \triangleq \{0, 1, 2, ...\} \tag{2.2.1}$$

where $f : N_0 \times \Omega \to R^n, \Omega \subset R^n$, is continuously differentiable at equilibrium point x^*.

Its specialized form is given by

$$x(t+1) = f(x(t)), \;\; t \in N_0 \tag{2.2.2}$$

System (2.2.1) is known as nonlinear nonautonomous system of difference equations and system (2.2.2) is known as nonlinear autonomous system of difference equations. In following section we discuss the linearization method to linearize these nonlinear systems (see Elaydi [8], page no. 197).

Remark 2.2.1. *It is noted that a point x^* in R^n is called an equilibrium point of (2.2.1) if $f(t, x^*) = x^*$ for all $t \geq t_0$. In most of the literature x^* is assumed to be the origin 0 and is called the zero solution.*

2.3 Linearization of Nonlinear Systems

Consider system (2.2.1). Let us write $f = (f_1,\ f_2\ ,...,\ f_n)^T$. The Jacobian matrix of f is defined as

$$\frac{\partial f(t,x)}{\partial x}\Big|_{x=0} = \frac{\partial f(t,0)}{\partial x} = \begin{pmatrix} \frac{\partial f_1(t,0)}{\partial x_1} & \frac{\partial f_1(t,0)}{\partial x_2} & \cdots & \frac{\partial f_1(t,0)}{\partial x_n} \\ \frac{\partial f_2(t,0)}{\partial x_1} & \frac{\partial f_2(t,0)}{\partial x_2} & \cdots & \frac{\partial f_2(t,0)}{\partial x_n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial f_n(t,0)}{\partial x_1} & \frac{\partial f_n(t,0)}{\partial x_2} & \cdots & \frac{\partial f_n(t,0)}{\partial x_n} \end{pmatrix}$$

For simplicity $\frac{\partial f(t,0)}{\partial x}$ is denoted by $f'(t,0)$. Let

$$\frac{\partial f(t,0)}{\partial x} = A(t)$$

and

$$g(t,x) = f(t,x) - A(t)x(t)$$

Then system (2.2.1) may be written in the form of

$$x(t+1) = A(t)x(t) + g(t,x(t)) \tag{2.3.1}$$

having its linear component

$$x(t+1) = A(t)x(t) \tag{2.3.2}$$

where $(A(t))_{t\in N_0}$ is a sequences of real $n \times n$ matrices, $(x(t))_{t\in N_0}$ is a sequences of state vectors in R^n, $g(t,x(t)) : N_0 \times \Omega \to R^n,\ \ \Omega \subset R^n$ is a nonlinear function represents the perturbation due to noise, inaccuracy in measurements or other outside disturbances. Here it is assumed that $g(t,x(t)) = o(x)$ as $\| x \| \to 0$.

i.e. if given $\epsilon > 0$, there is $\delta > 0$ such that

$$\| g(t,x) \| \leq \epsilon \| x \| \ \ whenever \ \ \| x \| < \delta \ and \ t \in N_0$$

An important special case of equation (2.3.1) is the autonomous system given by (2.2.2) which may be written as

$$x(t+1) = Ax(t) + g(x(t)) \tag{2.3.3}$$

having its linear component

$$x(t+1) = Ax(t) \tag{2.3.4}$$

where $A = f'(0)$ is the Jacobian matrix of f at 0 and $g(x) = f(x) - Ax$. Since f is differentiable at 0, we can write $g(x) = o(x) \ \ as \ \ \| x \| \rightarrow 0$. That is,

$$\lim_{\|x\| \rightarrow 0} \frac{\| g(x) \|}{\| x \|} = 0$$

2.4 Existence and Uniqueness of Solutions

2.4.1 Solutions of Linear Systems

Note that linear system (2.3.2) with initial condition $x(t_0) = x_0$ becomes initial value problem and its solution is guaranteed by the following theorem.

Theorem 2.4.1. *(refer Elaydi [8]) For each $x_0 \in R^n$ and $t_0 \in N_0$, there exists a unique solution $x(t, t_0, x_0)$ of (2.3.2) with $x(t_0) = x_0$, and the solution can be expressed as*

$$x(t) = \left[\Pi_{i=t_0}^{t-1} A(i)\right] x_0 \tag{2.4.1}$$

where,

$$\begin{aligned} \Pi_{i=t_0}^{t-1} A(i) &= A(t-1)A(t-2)...A(t_0), \ \ if \ \ t > t_0 \\ &= I \qquad\qquad\qquad\qquad\qquad , if \ \ t = t_0 \end{aligned}$$

and similarly the solution of linear autonomous system (2.3.4) with $x(t_0) = x_0$ is given by

$$x(t) = A^{t-t_0} x_0 \tag{2.4.2}$$

The notion of a fundamental matrix is very crucial in the theory of linear systems.

2.4.2 Fundamental Matrices

Let $\Phi(t)$ be a matrix whose columns are solutions of equation (2.3.2). i.e. if $x_1(t), x_2(t), ..., x_n(t)$ are solutions of (2.3.2), we write

$$\Phi(t) = [x_1(t)\ x_2(t)\ ...\ x_n(t)]$$

Definition 2.4.1. *If $\Phi(t)$ is a matrix that is nonsingular for all $t \geq t_0$ and satisfies equation*

$$\Phi(t+1) = A(t)\Phi(t),$$

then it is said to be a fundamental matrix for system (2.3.2).

Note that if $\Phi(t)$ is a fundamental matrix, and C is any nonsingular matrix, then $\Phi(t)C$ is also a fundamental matrix. Thus there are infinitely many fundamental matrices for system (2.3.2). It can be easily shown that

$$\Phi(t) = \Pi_{i=t_0}^{t-1} A(i), \ \ with\ \ \Phi(t_0) = I \tag{2.4.3}$$

is a fundamental matrix of (2.3.2). In the autonomous case we get

$$\Phi(t) = A^{t-t_0}. \tag{2.4.4}$$

Definition 2.4.2. *If $\Phi(t)$ is a fundamental matrix then $\Phi(t)\Phi^{-1}(t_0)$ is also a fundamental matrix. This special fundamental matrix is referred as state transition matrix or principal fundamental matrix.*

We may write in general the state transition matrix as $\Phi(n,m) = \Phi(n)\Phi^{-1}(m)$. This special fundamental matrix has some properties which are listed below (Elaydi [8]).

Properties :

1. $\Phi^{-1}(n,m) = \Phi(m,n)$
2. $\Phi(n,m) = \Phi(n,r)\Phi(r,m)$
3. $\Phi(n,m) = \Pi_{i=m}^{n-1} A(i)$

2.4.3 Solution of Nonlinear Systems

It can be easily verified by using the variation of constant formula (see Elaydi [8]) that the solution of (2.3.1) is given by following theorem.

Theorem 2.4.2. *The unique solution of the initial value problem*

$$x(t+1) = A(t)x(t) + g(t, x(t)), \;\; x(t_0) = x_0 \tag{2.4.5}$$

is given by

$$x(t) = \Phi(t, t_0)x_0 + \sum_{r=t_0}^{t-1} \Phi(t, r+1)g(r, x(r)) \tag{2.4.6}$$

or more explicitly by

$$x(t) = \left[\Pi_{i=t_0}^{t-1} A(i)\right] x_0 + \sum_{r=t_0}^{t-1} \left[\Pi_{i=r+1}^{t-1} A(i)\right] g(r, x(r)) \tag{2.4.7}$$

Corollary 2.4.1. *For autonomous systems (that is, when A is a constant matrix), the solution of (2.3.3) with $x(t_0) = x_0$ is given by*

$$x(t) = A^{t-t_0}x_0 + \sum_{r=t_0}^{t-1} A^{t-r-1} g(x(r)) \tag{2.4.8}$$

2.5 Discrete Volterra Systems

Volterra equations with discrete time arise mainly in the process of modeling of some real phenomena or by applying a numerical method to a Volterra integral equations. Volterra difference equation of the form

$$x(t+1) = A(t)x(t) + \sum_{r=0}^{t} B(t, r)x(r), \;\; x(0) = x_0 \tag{2.5.1}$$

may be considered as the discrete analogue of the famous Volterra integrodifferential equation

$$x^{'}(t) = A(t)x(t) + \int_0^t B(t, r)x(r)dr \tag{2.5.2}$$

Consider the perturbed equation of (2.5.1)

$$x(t+1) = A(t)x(t) + \sum_{r=0}^{t} B(t,r)x(r) + f(t), \;\; x(0) = x_0 \tag{2.5.3}$$

where $x(t) \in R^n$, $A(t)$, $B(t,r)$ are $n \times n$ matrix functions on N_0 and $N_0 \times N_0$ respectively. $f(t)$ is an $n-$vector function on N_0. Define the resolvent matrix $R(t,m)$ of equation (2.5.1) as the unique solution of the matrix equation

$$R(t+1,m) = A(t)R(t,m) + \sum_{r=0}^{t} B(t,r)R(t,m), \;\; t \geq m \tag{2.5.4}$$

with $R(m,m) = I$ for $0 \leq m \leq t$.

Elaydi [7] showed that equation (2.5.3) has a unique solution $x(t)$ which can be expressed as

$$x(t) = x(t,0,x_0) = R(t,0)x_0 + \sum_{r=0}^{t-1} R(t,r+1)f(r) \tag{2.5.5}$$

In case $A(t) = A$ constant matrix (with A nonsingular) and $B(t,r) = B(t-r)$, equation (2.5.3) is given by

$$x(t+1) = Ax(t) + \sum_{r=0}^{t} B(t-r)x(r) + f(t), \;\; x(0) = x_0 \tag{2.5.6}$$

This is known as Volterra difference equations of convolution type. The unique solution of (2.5.6) is given by

$$x(t,0,t_0) = X(t)x_0 + \sum_{r=0}^{t-1} X(t-r-1)f(r) \tag{2.5.7}$$

where $X(t)$ is called the fundamental matrix of system

$$x(t+1) = Ax(t) + \sum_{r=0}^{t} B(t-r)x(r) \tag{2.5.8}$$

and satisfies the matrix equation

$$X(t+1) = AX(t) + \sum_{r=0}^{t} B(t-r)X(r)$$

Notice that $X(0) = I$ and $x(t, 0, x_0) = X(t)x_0$ is the unique solution of equation (2.5.8) with $x(0, 0, x_0) = x_0$. In Chapters 4, 5 and 6, such type of Volterra systems are considered.

2.6 Stability Analysis

Often we are interested in the methods and suitable criterion that describe the nature and behavior of solutions of difference systems without actually constructing or approximating solutions. Realizing that most of the problems that arise in practice are nonlinear and mostly unsolvable, this investigation is of vital importance. Also for a given difference system, one of the pioneer problems is the study of ultimate behavior of its solutions (i.e. asymptotic behavior of discrete systems).

First we recall some concepts about the stability of discrete dynamical systems (see Agarawal [1], Elaydi [8]).
Consider a system of difference equations

$$x(t+1) = f(t, x(t)), \;\; x(t_0) = x_0 \;\; , t \in N_0 \triangleq \{0, 1, 2, ...\} \tag{2.6.1}$$

Let the solution $x(t) = x(t, t_0, x_0)$ of (2.6.1) exist for all $t \in N_0$. For this solution we shall define various concepts of stability.

Definition 2.6.1. *The solution $x(t)$ of system (2.6.1) is said to be*

1. ***Stable** if, for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon, t_0)$ such that, for any solution $\bar{x}(t) = x(t, t_0, \bar{x}_0)$ of (2.6.1), the inequality $\| \bar{x}_0 - x_0 \| < \delta$ implies $\| \bar{x}(t) - x(t) \| < \epsilon$ for all $t \in N_0$, **uniformly stable** if, δ may be chosen independent of t_0, **unstable** if it is not stable.*

2. ***Attractive** if, there exists a $\delta = \delta(t_0)$ such that, for any solution $\bar{x}(t) = x(t, t_0, \bar{x}_0)$ of (2.6.1), the inequality $\| \bar{x}_0 - x_0 \| < \delta$ implies $\| \bar{x}(t) - x(t) \| \to$*

*0 as $t \to \infty$, **uniformly attractive** if, the choice of δ is independent of t_0.*

3. ***Asymptotically Stable** if, it is stable and attractive and **uniformly asymptotically stable** if, it is uniformly stable and uniformly attractive.*
4. ***Exponentially Stable** if, there exist $\delta > 0$, $M > 0$ and $\eta \in (0,1)$ such that the inequality $\| \bar{x}_0 - x_0 \| < \delta$ implies $\| \bar{x}(t) - x(t) \| \leq M \| \bar{x}_0 - x_0 \| \eta^{t-t_0}$*
5. ***Bounded** if, for some positive constant M, $\| x(t, t_0, x_0) \| \leq M$ for all $t \geq t_0$ where M may depend on each solution.*

Remark 2.6.1. *For the linear autonomous systems, the notions uniformly stable, uniformly asymptotically stable and uniformly attractive are equivalent to stable, asymptotically stable and attractive respectively. Also note that exponential stability implies asymptotic stability which in turn implies stability.*

2.6.1 Stability of Linear Systems

Consider the linear nonautonomous (time-variant) system given by

$$x(t+1) = A(t)x(t) \tag{2.6.2}$$

and linear autonomous system (time-invariant) given by

$$x(t+1) = Ax(t) \tag{2.6.3}$$

For the stability of linear systems, we have the following well known results (see Agarwal [1], Elaydi [8]) which provide the necessary and sufficient conditions in terms of their fundamental matrices.

Theorem 2.6.1. *(Elaydi [8]) Let $\Phi(t, t_0)$ be the principal fundamental matrix of (2.6.2). Then the system (2.6.2) is*

1. *stable if and only if there exists a positive constant M such that*

$$\| \Phi(t, t_0) \| \leq M, \quad for \ all \ t \in N_0$$

2. *uniformly stable if and only if there exists positive constant M such that*

$$\| \Phi(t,r) \| = \| \Phi(t,t_0)\Phi^{-1}(r,t_0) \| \leq M, \ for\ all\ t_0 \leq r \leq t \in N_0$$

3. *asymptotically stable if and only if*

$$\| \Phi(t,t_0) \| \rightarrow 0, \ as\ t \rightarrow \infty$$

4. *uniformly asymptotically stable if and only if, there exist positive constants M and $\eta \in (0,1)$ such that*

$$\| \Phi(t,r) \| \leq M\eta^{t-r}, \ for\ all\ t_0 \leq r \leq t \in N_0$$

Also we have the following simple but powerful criteria for uniform stability and uniform asymptotic stability.

Theorem 2.6.2. *(see Elaydi [8])*

1. *If $\sum_{i=1}^{n} | a_{ij}(t) | \leq 1,\ 1 \leq j \leq n,\ t \geq t_0$, then the zero solution of system (2.6.2) is uniformly stable.*

2. *If $\sum_{i=1}^{n} | a_{ij}(t) | \leq 1 - \nu$ for some $\nu > 0,\ 1 \leq j \leq n,\ t \geq t_0$, then the zero solution of (2.6.2) is uniformly asymptotically stable.*

Theorem 2.6.3. *(see Elaydi [8]) The following statements hold true.*

1. *The zero solution of equation (2.6.3) is stable if and only if $\rho(A) \leq 1$ and the eigenvalues of unit modules are semisimple.*

2. *The zero solution of equation (2.6.3) is asymptotically stable if and only if $\rho(A) < 1$.*

2.6.2 Stability of Nonlinear Systems

We now consider the nonlinear nonautonomous (time-variant) system given by

$$x(t+1) = A(t)x(t) + g(t,x(t)), \ t \in N_0 \tag{2.6.4}$$

and nonlinear autonomous system (time-invariant) given by

$$x(t+1) = Ax(t) + g(x(t)), \ \ t \in N_0 \tag{2.6.5}$$

which are considered as the perturbed systems of (2.6.2) and (2.6.3) respectively. We recall that these systems arise after linearization of equations (2.2.1) and (2.2.2) respectively. There are sufficient conditions on the nonlinear perturbed function $g(t, x)$ so that the stability properties and asymptotic properties of the unperturbed systems are maintained for perturbed systems.

The following results are well known for nonlinear systems.

Theorem 2.6.4. *(refer Elaydi [8]) Assume that $g(t,x) = o(\parallel x \parallel)$ as $\parallel x \parallel \to 0$. If the zero solution of the linear system (2.6.2) is uniformly asymptotically stable, then the zero solution of the nonlinear system (2.6.4) is exponentially stable.*

Corollary 2.6.1. *(refer Elaydi [8]) If $\parallel f'(0) \parallel < 1$, then the zero solution of (2.2.2) is exponentially stable.*

A well-known result of Perron which dates back to 1929 (see Gordon [50]), (Ortega [14] , page 270) and (LaSalle [15] , Theorem 9.14) states that (2.6.5) is asymptotically stable provided that spectral radius $\rho(A) < 1$ and $g(x) = o(\parallel x \parallel)$. where

$$\rho(A) = \max\{| \lambda |: \lambda \ is \ an \ eigen \ value \ of \ A\}$$

is the spectral radius of A.

Also we have the following theorems regarding instability of systems (2.6.5).

Theorem 2.6.5. *(refer Elaydi [8]) The following statements hold true.*

1. *If $\rho(A) = 1$, and $g(x) = o(x)$ as $\parallel x \parallel \to 0$, then the zero solution of (2.6.5) may be stable or unstable..*

2. *If $\rho(A) > 1$, and $g(x) = o(x)$ as $\parallel x \parallel \to 0$, then the zero solution of (2.6.5) is unstable.*

Remark 2.6.2. *It is possible that* $\| A \| \geq 1$ *but* $\rho(A) < 1$. *For example for* $A = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$, $\| A \|_2 = \sqrt{\rho(A^T A)} = 1.2071 > 1$,

$$\| A \|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} | a_{ij} | = \frac{3}{2},$$

$$\| A \|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} | a_{ij} | = \frac{3}{2}.$$

Obviously, $\rho(A) = \frac{1}{2} < 1$. *Hence we conclude that with the above matrix* A, *the zero solution of the system* $x(t+1) = Ax(t) + g(x(t))$ *is exponentially stable provided that* $g(x) = o(x)$ *as* $\| x \| \to 0$.

Following lemma is called the Discrete Gronwall inequality and it is extensively useful in difference equations.

Discrete Gronwall Inequality(refer Elaydi [8])

Lemma 2.6.1. *Let* $z(t)$ *and* $h(t)$ *be two sequences of real numbers,* $t \geq t_0 \geq 0$ *and* $h(t) \geq 0$. *If*

$$z(t) \leq M\big[z(t_0) + \sum_{j=t_0}^{t-1} h(j) z(j)\big], \quad for \ some \ M > 0,$$

then

$$z(t) \leq z(t_0) \Pi_{j=t_0}^{t-1} \big[(1 + Mh(j))\big], \quad t \geq t_0$$

or

$$z(t) \leq z(t_0) exp\big[\sum_{j=t_0}^{t-1} Mh(j)\big], \quad t \geq t_0$$

In the following we describe special types of matrices and its applications in the stability analysis.

2.6.3 (sp) Matrices

Recently Xue and Guo [63] introduced a notion of (sp) matrix and proved its usefulness in the study of asymptotic stability of null solution of linear system

$$x(t+1) = Ax(t)$$

Definition 2.6.2. *((sp) **Matrix**) We call*

$$A \in s = \{A = (a_{ij})_{n\times n} : a_{ij} \geq 0, \sum_{j=1}^{n} a_{ij} \leq 1, \forall i = 1,2,...,n\}$$

a (sp) matrix if there exists $m \in N = \{1,2,...\}$ *and a sequence of subscript sets*

$\{I_1^{(k)}\}$, $\{I_2^{(k)}\}$, $k = 0,1,...,m$ *from* $I = \{1,2,...,n\}$ *such that*

$I = I_1^{(0)} \cup I_2^{(0)}$, $I_1^{(0)} = \{i : \sum_{j=1}^{n} a_{ij} < 1\}$, $I_2^{(0)} = \{i : \sum_{j=1}^{n} a_{ij} = 1\}$,

$I_1^{(k)} = \{i \in I_2^{(k-1)} : \ \exists j \in I_1^{(k-1)} \text{ such that } a_{ij} \neq 0\}$,

$I_2^{(k)} = \{i \in I_2^{(k-1)} : \forall j \in I_1^{(k-1)} \text{ such that } a_{ij} = 0\}, k = 1,2,...,m-1$,

$I_1^{(m)} = I_2^{(m-1)}, I_2^{(m)} = \phi$,

where $I_1^{(k)}$ *and* $I_2^{(k)}$, $k = 0,1,2,...,m-1$ *are nonempty or* $I_2^{(0)} = \phi$.

Example 2.6.1. *The matrix* $A = \begin{pmatrix} 0 & 0 & \frac{4}{7} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ \frac{2}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ *is a (sp) matrix with* $m = 2$

and subscript sets are
$I_1^{(0)} = \{1\}$, $I_2^{(0)} = \{2,3,4\}$,
$I_1^{(1)} = \{2,3\}$, $I_2^{(1)} = \{4\}$,
$I_1^{(2)} = \{4\}$, $I_2^{(2)} = \phi$

Example 2.6.2. *Let* $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{5} & 0 & 0 \\ \frac{3}{5} & 0 & 0 & 0 \end{pmatrix}$ *Then* $A \in s$, *but* A *is not a (sp) matrix because*
$I_1^{(0)} = \{2,3,4\}$, $I_2^{(0)} = \{1\}$, $I_1^{(1)} = \phi$.

Xue and Guo [63] proved the following result.

Theorem 2.6.6. *Let* $A \in s$. *Then the zero solution of (2.6.3) is asymptotically stable if and only if* A *is a (sp) matrix.*

We extended this result for the nonlinear system (2.6.5) (refer [37]).

2.6.4 Generalized Subradius

Czornik [5] introduced the ideas of generalized spectral subradius and the joint spectral subradius and shown the relationship between generalized spectral radii and the stability of discrete time-varying linear system. Let Σ denote a nonempty set of all real $n \times n$ matrices. For $m \geq 1$, Σ^m is the set of all products of matrices in Σ of length m,

$$\Sigma^m = \{A_1 A_2 ... A_m : A_i \in \Sigma, \;\; i = 1, 2, ..., m\}$$

Denote by $\rho(A)$ the spectral radius and by $\| A \|$ a matrix norm of the matrix A. Let $A \in \Sigma^m$.

Definition 2.6.3. *The* ***generalized spectral subradius*** *of* Σ *is defined as*

$$\tilde{\rho}_*(\Sigma) = \inf_{m \geq 1} (\inf_{A \in \Sigma^m} \rho(A))^{\frac{1}{m}}.$$

Definition 2.6.4. *The* ***joint spectral subradius*** *of* Σ *is defined as*

$$\hat{\rho}_*(\Sigma) = \inf_{m \geq 1} (\inf_{A \in \Sigma^m} \| A \|)^{\frac{1}{m}}.$$

Czornik [5] proved that the generalized spectral subradius and joint spectral subradius are equal for any nonempty set Σ and the common value of $\tilde{\rho}_*(\Sigma)$ and $\hat{\rho}_*(\Sigma)$ is called the **generalized subradius** of Σ and will be denoted by $\rho_*(\Sigma)$. Also in [5], the relationship between the generalized subradius and asymptotic behavior of null solution is established.

Theorem 2.6.7. *(refer Czornik [5]) Consider a discrete time-varying linear system*

$$x(t+1) = A(t)x(t), \quad x(0) = x_0,$$

where A is a sequence of matrices taken from Σ. Then there exists a sequence A such that for any $x_0 \in R^n$, we have $\lim_{t\to\infty} x(t) = 0$ if and only if $\rho_*(\Sigma) < 1$.

We extend this result for nonlinear system (2.6.4).

2.6.5 Dichotomy

The problem of the asymptotic relationship between the solutions of a linear difference equation and the corresponding nonlinearly perturbed equation is studied by means of dichotomic behavior of linear difference system.
Two systems of differential or difference equations are said to be asymptotically equivalent if, corresponding to each solution of one system, there exists a solution of the other system such that the difference between these two solutions tends to zero.
Consider the nonlinear perturbed system

$$y(t+1) = A(t)y(t) + g(t, y(t)), \; t \in N_0 \tag{2.6.6}$$

along with the associated unperturbed system

$$x(t+1) = A(t)x(t) \tag{2.6.7}$$

where $A(t)$ is an invertible $n \times n$ matrix function on N_0 and $g(t, y)$ is a function from $N_0 \times R^n \to R^n$ which is continuous in y. Let $\Phi(t)$ be the fundamental matrix of system (2.6.7).
Elaydi [8] defined the ordinary dichotomy and provided some applications. Before we define ordinary dichotomy, let us first define the following.

Definition 2.6.5. *(refer Agarwal [1]) A matrix P is said to be a* ***projection*** *if, $P^2 = P$. If P is a projection, then so is $(I - P)$. Two such projections whose sum is I and hence whose product is 0, are said to be supplementary.*

Definition 2.6.6. *(Elaydi [8]) The linear system (2.6.7) has an* ***ordinary dichotomy*** *if there exists a projection matrix P and a positive constant M such that*

$$\begin{aligned} \| \Phi(t)P\Phi^{-1}(s) \| &\leq M \ for \ t \geq s \geq t_0 \\ \| \Phi(t)(I-P)\Phi^{-1}(s) \| &\leq M \ for \ s \geq t \geq t_0. \end{aligned}$$

where $\Phi(t)$ be the fundamental matrix of the system (2.6.7).

The following results are useful.

Theorem 2.6.8. *(see Elaydi [8])Suppose that system (2.6.7) possesses an ordinary dichotomy. If, in addition,*

$$\sum_{j=t_0}^{\infty} \| g(j,0) \| < \infty \tag{2.6.8}$$

and

$$\| g(t,x) - g(t,y) \| \leq \gamma(t) \| x - y \| \tag{2.6.9}$$

where $\gamma(t) \in l^1([t_0,\infty))$, then for each bounded solution $x(t)$ of equation (2.6.7), there corresponds a bounded solution $y(t)$ of equation (2.6.6) and vice versa. Furthermore, $y(t)$ is given by the formula

$$y(t) = x(t) + \sum_{j=t_0}^{t-1} \Phi(t)P\Phi^{-1}(j+1)g(j,y(j)) - \sum_{j=t}^{\infty} \Phi(t)(I-P)\Phi^{-1}(j+1)g(j,y(j))$$

Theorem 2.6.9. *(refer Elaydi [8]) Let all the assumptions of Theorem 2.6.8 hold. If $\Phi(t)P \to 0$ as $t \to \infty$, then for each bounded solution $x(t)$ of (2.6.7) there corresponds a bounded solution $y(t)$ of (2.6.6) such that*

$$y(t) = x(t) + o(1).$$

Pinto [31] introduced concept of (h,k) dichotomy and proved result similar to above theorems under certain hypothesis.

Definition 2.6.7. *The linear system (2.6.7) has an (h, k) dichotomy iff there exists a projection P and a positive constant c such that*

$$\| \Phi(t)P\Phi^{-1}(s) \| \leq ch(t)h(s)^{-1} \ \ for \ \ t \geq s \geq t_0$$

$$\| \Phi(t)(I-P)\Phi^{-1}(s) \| \leq ck(t)^{-1}k(s) \ \ for \ \ s \geq t \geq t_0.$$

where $h(t)$ and $k(t)$ are two positive sequences defined on N_0. They proved following result.

Theorem 2.6.10. *(Pinto [31]) Let following assumptions hold true.*

(A) *The linear system (2.6.7) has an (h, k) dichotomy satisfying the compensation law:*

$$h(t)h(s)^{-1}k(t)k(s)^{-1} \leq c_1, \ \ t \geq s \geq t_0$$

where c_1 is a positive constant.

(B) *$g : N_0 \times R^n \to R^n$ is a continuous function such that*

$$\| g(t, y_1) - g(t, y_2) \| \leq \gamma(t) \| y_1 - y_2 \|, \ \ for \ \ (t, y_i) \in N_0 \times R^n, \ \ (i = \{1, 2\})$$

for a nonnegative function γ satisfying $\beta.\gamma \in l^1(N_0)$ and $h(t)^{-1}g(t, 0) \in l^1(N_0)$, where $\beta(t) = h(t-1)h(t)^{-1}$. Then there exists a one-to-one and bicontinuous correspondence between the solutions of (2.6.7) and (2.6.6). Moreover, if $h(t)^{-1}\Phi(t)P \to 0$, as $t \to \infty$, then we get

$$y(t) = x(t) + o(1), \ \ as \ \ t \to \infty$$

We use these concepts in proving asymptotic equivalence of solution of linear Volterra system and its nonlinear perturbation.

2.7 Controllability Analysis

Controllability is one of the fundamental concepts in mathematical control theory. This is a qualitative property of dynamical control systems. Systematic study of controllability was started at the beginning of sixties, when the theory of controllability based on the description in the form of state space for both time-invariant and time-varying linear control systems was worked out by Kalman ([19]).

Roughly speaking, a system is controllable if it is possible to steer dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In the literature there are many different definitions of controllability, which strongly depend on the class of dynamical control systems. Now consider a system of linear difference equations of the type

$$x(t+1) = Ax(t) + Bu(t), \;\; x(0) = x_0, \;\; t \in N_0 \tag{2.7.1}$$

Here, A and B are of real $n \times n$ and $n \times m$ matrices, respectively, and $(x(t))_{t \in N_0}$ and $(u(t))_{t \in N_0}$ are sequences of state vectors in R^n and control vectors in R^m, respectively. First we define various notions of controllability.

2.7.1 Various Notions and Basic Results of Controllability

Definition 2.7.1. ***(Complete Controllability)*** *System (2.7.1) is said to be completely controllable or simply controllable if for any $t_0 \in N_0$, any initial state $x(t_0) = x_0$ and any given final state x_f, there exists a finite time $N > t_0$ and a control $u(t)$, $t_0 < t \leq N$ such that $x(N) = x_f$.*

Definition 2.7.2. ***(Controllability to Origin)*** *A system (2.7.1) is controllable to the origin if for any $t_0 \in N_0$ and $x_0 \in R^n$, there exists a finite time $N > t_0$ and a control $u(t)$, $t_0 < t \leq N$ such that $x(N) = 0$.*

Definition 2.7.3. ***(Local Controllability)*** *A system is locally controllable if there exists a neighborhood Ω of the origin such that, for any $x_0, x_1 \in \Omega$ there is a sequence of inputs $\boldsymbol{u} = (u(0), u(1), ..., u(N-1))$ that steers the system from x_0 to x_1.*

Definition 2.7.4. ***(Reachability)*** *A state $x_1 \in R^n$ is said to be reachable in N time steps, if there exist a sequence of control vectors $u(t) \in R^m$, $t \in N_0$, such that the corresponding solution starting from $x(0) = 0$, also satisfies $x(N) = x_1$. Moreover, the system (2.7.1) is reachable, if every $x_1 \in R^n$ is reachable.*

The controllability matrix W of System (2.7.1) is defined as the $n \times nm$ matrix

$$W = [B, \ AB, \ A^2B, ..., A^{n-1}B]$$

The controllability matrix plays a major role in control theory. The following basic result has been established in the literature.

Theorem 2.7.1. *(Elaydi [8]) System (2.7.1) is completely controllable if and only if rank* $W = n$.

Remark 2.7.1. *Clearly complete controllability is a stronger property than controllability to the origin. The two notions coincide in linear continuous-time systems. However, for the discrete-time system (2.7.1), controllability to the origin does not imply complete controllability unless* A *is nonsingular.*

The following example illustrate the result.

Example 2.7.1. *Consider the control system* $x(t+1) = Ax(t) + Bu(t)$ *with*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Now for

$$x(0) = x_0 = \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix},$$

we have

$$\begin{aligned} x(1) &= Ax_0 + Bu(0) \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(0) \\ &= \begin{pmatrix} x_{02} \\ 0 \end{pmatrix} + \begin{pmatrix} u(0) \\ 0 \end{pmatrix} \end{aligned}$$

So if we pick $u(0) = -x_{02}$, *then we will have* $x(1) = 0$. *Therefore system (2.7.1) is controllable to zero. However, we observe that*

$$rank[B, AB] = rank \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 < 2$$

Thus by Theorem 2.7.1, system (2.7.1) is not completely controllable.

For the time-varying systems of difference equations

$$x(t+1) = A(t)x(t) + B(t)u(t), \;\; x(0) = x_0, \;\; t \in N_0 \tag{2.7.2}$$

Here, $(A(t))_{t\in N_0}$ and $(B(t))_{t\in N_0}$ are sequences of real $n \times n$ and $n \times m$ matrices, respectively, and $(x(t))_{t\in N_0}$ and $(u(t))_{t\in N_0}$ are sequences of state vectors in R^n and control vectors in R^m, respectively. The controllability of system (2.7.2) is studied using reachability Grammian. The reachability Grammian of (2.7.2) is defined as follows;

$$W_r(0,N) := \sum_{j=0}^{N-1} \Phi(N, j+1)B(j)B(j)^*\Phi(N, j+1)^* \tag{2.7.3}$$

where $\Phi(n,m) = A(n)A(n-1)...A(m)$ is a fundamental matrix of

$$x(t+1) = A(t)x(t), \;\; x(0) = x_0, \;\; t \in N_0 \tag{2.7.4}$$

Denote the linear space of control sequences by

$$U_{[0,N]} = \{u \in R^{m(N+1)} : u := [u(0), u(1), ..., u(N)], \;\; with \;\; u(t) \in R^m, \; 0 \le t \le N\}$$

The following propositions are important in studying controllability problems .

Theorem 2.7.2. *(Callier and Desoer [12]) Let $(A(t)), (B(t)), t \in N_0$ be given compatible matrix-sequences. Then the following are equivalent:*

(i) The linear system (2.7.2) is controllable on $[0, N]$.

(ii) $det(W_r(0, N)) \neq 0$, where the reachability Grammian W_r is as defined as in (2.7.3).

Theorem 2.7.3. *If the system (2.7.2) is controllable on $[0, N]$, then for all $x_0, \; x_1 \in R^n$, there exists $u \in U_{[0,N]}$ defined by*

$$u(t) := B(t)^*\Phi(N, t+1)^*W_r(0, N)^{-1}[x_1 - \Phi(N, 0)x_0]$$

that steers the initial state x_0 to the desired final state x_1 in N time-steps.

The extension of this result is given in chapter 3, where we define and prove steering control for the semilinear discrete-time system given by

$$x(t+1) = A(t)x(t) + f(t, x(t)), \;\; x(0) = x_0, \;\; t \in N_0$$

2.7.2 Optimal Control

Problems of optimal control have received a great deal of attention. An optimal control system is a system whose design "optimizes" (minimizes or maximizes) the value of a function chosen as the performance index. In designing an optimal control system, we need to find a rule for determining the present control decision, subject to certain constrains, so as to minimize some measure of the deviation from ideal behavior. That measure is usually provided by the chosen performance index, which is a function whose value we consider to be an indication of how well the actual performance of the system matches the desired performance. In most cases, the behavior of a system is optimized by choosing the control vector $\mathbf{u}(t)$ in such a way that the performance index is minimized or maximized depending on the nature of the performance index chosen. In Ogata [16], the following system is considered

$$x(t+1) = Ax(t) + Bu(t), \;\; x(0) = c, \;\; t \in N_0 \tag{2.7.5}$$

and performance index is chosen as

$$J = \frac{1}{2}x^*(N)Sx(N) + \frac{1}{2}\sum_{t=0}^{N-1}[x^*(t)Qx(t) + u^*(t)Ru(t)] \tag{2.7.6}$$

where S, Q are $n \times n$ positive definite or positive semidefinite Hermitian matrices (or real symmetric matrices). R is an $r \times r$ positive definite Hermitian or real symmetric matrix. By using Lagrange multipliers method, it is proved that the optimal control is given by

$$u(t) = -R^{-1}B^*[P^{-1}(t+1) + BR^{-1}B^*]^{-1}Ax(t) \tag{2.7.7}$$

where $P(t)$ is $n \times n$ hermitian matrix or real symmetric matrix. Equation (2.7.7) gives the closed loop form or feedback form for the control vector $u(t)$. In Chapter

6, we derive open loop control for the linear Volterra system using the method of Lagrange multipliers.

2.8 Some Tools of Analysis

We use the notion of "higher order" functions which is introduced below (refer Chen and Narendra [27])

Definition 2.8.1. ***(Higher order function)*** *A continuously differentiable function $F : R^m \rightarrow R^n$ is called a "higher order" function, if*

1. $F(0) = 0$
2. $\frac{\partial F}{\partial x}/_{x=0} = 0$

We denote the class of higher order functions by H and use the following properties of the higher order functions which can be verified in a straightforward manner.

1. If P is a $n \times n$ constant matrix and $F(.) \in H$, then $PF(.) \in H$.
2. If $F_1, F_2 \in H$, then $F_1 + F_2 \in H$.
3. If $F_1 \in H$ and $F_2(0) = 0$ and is continuously differentiable then the composition $F_1(F_2(.)) \in H$.

We now present in the following specialized forms for inverse function theorem and the implicit function theorem (refer Corwin and Szczarba [28]) which will be used for obtaining controllability result for the semi-linear system.

Theorem 2.8.1. ***(Inverse function Theorem*** *[28]) In some neighborhood $U_1 \subset R^n$ of the origin, let*

$$Px + f(x) = y, \quad x \in U_1, y \in R^n$$

where P is a $n \times n$ nonsingular matrix and $f(.) \in H$. Then there exists an open set $U_2 \subset U_1$ containing the origin such that the set V, defined by

$$V \equiv PU_2 + f(U_2)$$

is open, and for all $x \in U_2$ there exist $g(.) \in H$ such that

$$x = P^{-1}y + g(y), \ \ y \in V$$

Theorem 2.8.2. ***(Implicit function Theorem** [28]) Let U_1 be an open subset of R^{n+k} containing the origin. Let an element of U_1 be denoted by (x, y) with $x \in R^n$ and $y \in R^k$. Let $F : U_1 \to R^n$ be a function defined by $F(x, y) = Px + Qy + f(x, y)$, where P is nonsingular $n \times n$ matrix, Q is any $n \times n$ matrix and $f(.) \in H$. Then there exists an open set $U_2 \subset R^k$ and $g(.) \in H$ containing the origin such that*

$$x = -P^{-1}Qy + g(y), \ \ y \in U_2$$

and satisfies the equation $F(x, y) = 0$.

By the above two theorems, when the underlying nonlinear function is expressed as the sum of linear and higher order functions, the application of inverse function theorem and implicit function theorem becomes a matter of simple manipulation of equations. Note also that the vector y can be a concatenation of several vectors. This leads to the following corollary (see Chen and Narendra [27]).

Corollary 2.8.1. *If $Px + Qy + f(x, y) = z$, $f(.) \in H$, and P is nonsingular, then there exists $\bar{g}(.) \in H$ such that locally $x = P^{-1}(z - Qy) + \bar{g}(y, z)$.*

Before we state Banach's contraction principle, we define following.

Definition 2.8.2. *(see Joshi and Bose [30]) Let X and Y be Banach spaces and F be a mapping from X into Y. F is said to be **Lipschitz** if there exists a real number $k > 0$ such that for all x, $y \in X$ we have*

$$\| Fx - Fy \| \leq k \| x - y \|$$

. F is said to be contraction if $k < 1$ and non-expansive if $k = 1$. F is said to be contractive if for all $x, y \in X$ and $x \neq y$, we have

$$\| Fx - Fy \| \leq \| x - y \| .$$

Remark 2.8.1. *Note that contraction $\Rightarrow$ contractive $\Rightarrow$ non $-$ expansive $\Rightarrow$ Lipschitz and all such mappings are continuous.*

Theorem 2.8.3. ***(Banach contraction Principle)****(refer Limaye [29]) Let X be a Banach space and $F : X \to X$ be a contraction on X. Then F has precisely one fixed point, and the fixed point can be computed by the iterative scheme $x_{t+1} = Fx_t$, starting from arbitrary x_0.*