Chapter 3

Steering Control of Semi-linear Discrete Dynamical System

Contents

In this chapter, the controllability of a class of semi-linear non-autonomous system described by the difference equation

$$x(t+1) = A(t)x(t) + B(t)u(t) + f(t, x(t)), \;\; t \in N_0 \triangleq \{0, 1, 2, ...\}$$

is established under certain assumptions. The steering control for the above system is defined and the existence and uniqueness of the solution of above system under the new steering control is established using Banach's fixed point theorem. Then the equivalence between the notions of controllability and reachability of the semi-linear non-autonomous system described by the above system are established under suitable criterion. Further an algorithm to compute steering control for the above system is also given. This is followed by numerical examples to illustrate the results.

3.1 Introduction

Krabs [61] studied the controllability of a general difference system of the form

$$x(t+1) = f(x(t), u(t)).$$

Further, they have also obtained a controller that steers a given initial state to a desired final state for the linear system (3.1.2). In this chapter, we consider a semi-linear system of difference equation of the form

$$x(t+1) = A(t)x(t) + B(t)u(t) + f(t, x(t)), \;\; x(0) = x_0 \;, t \in N_0 \qquad (3.1.1)$$

and the corresponding linear system :

$$x(t+1) = A(t)x(t) + B(t)u(t), \;\; x(0) = x_0, \;\; t \in N_0 \qquad (3.1.2)$$

Here, $(A(t))_{t \in N_0}$ and $(B(t))_{t \in N_0}$ are sequences of real $n \times n$ and $n \times m$ matrices, respectively, and $(x(t))_{t \in N_0}$ and $(u(t))_{t \in N_0}$ are sequences of state vectors in R^n and control vectors in R^m, respectively, $f(.,.) : N_0 \times R^n \rightarrow R^n$ is a nonlinear function satisfying Lipschitz condition with respect to the second argument.

We introduce a steering controller for system (3.1.1) and prove that it is well-defined and it steers any initial state x_0 of system (3.1.1) to a desired final state x_1 in $N \in N_0$ time steps under certain conditions.

We define the problem of controllability and reachability as follows.

Problem of Controllability : Let $x_0, x_1 \in R^n$ be given arbitrarily. We say that the system is controllable if there exists a sequence of control vectors $(u(t) \in$

R^m, $t \in N_0$), such that for some $N \in N_0$ the solution $(x(t))_{t\in N_0}$ of equation (3.1.1) starting from the initial state $x(0) = x_0$, also satisfies the end condition $x(N) = x_1$.

Problem of Reachability : We say that the state $x_1 \in R^n$ is reachable in N time steps, if there exists a sequence of control vectors $u(t) \in R^m$, $t \in N_0$, such that the corresponding solution starting from $x(0) = 0$, also satisfies $x(N) = x_1$.

We now express the solution of (3.1.1) and (3.1.2) in terms of the state-transition matrix $\Phi(t, t_0)$ associated with the homogeneous linear part of (3.1.2). The state transition matrix $\Phi(t, t_0)$ is given by

$$\Phi(t, t_0) = A(t-1)A(t-2)...A(t_0) \ \forall \ t \geq t_0$$

It can be shown that the solution of (3.1.1) is given by

$$x(t) = \Phi(t, 0)x_0 + \sum_{j=0}^{t-1} \Phi(t, j+1)B(j)u(j) + \sum_{j=0}^{t-1} \Phi(t, j+1)f(j, x(j)) \qquad (3.1.3)$$

and the solution of (3.1.2) is given by

$$x(t) = \Phi(t, 0)x_0 + \sum_{j=0}^{t-1} \Phi(t, j+1)B(j)u(j) \qquad (3.1.4)$$

In this chapter, we give the computational scheme for the steering control. For $t = 0, 1, ..., N-1$, we define a controller

$$u(t) := B(t)^*\Phi(N, t+1)^*W_r(0, N)^{-1}[x_1 - \Phi(N, 0)x_0 - \sum_{j=0}^{N-1} \Phi(N, j+1)f(j, x(j))] \qquad (3.1.5)$$

for the nonlinear system (3.1.1), where $W_r(0, N)$ is the reachability Grammian defined by

$$W_r(0, N) := \sum_{j=0}^{N-1} \Phi(N, j+1)B(j)B(j)^*\Phi(N, j+1)^* \qquad (3.1.6)$$

It is shown that this control is well-defined and steers the nonlinear system (3.1.1) from x_0 to x_1, under the following assumptions.

Assumptions :

[L] The linear system (3.1.2) is controllable.

[N] The nonlinear function $f(t,x)$ is Lipschitz continuous with respect to x. That is, there exists $\alpha > 0$ such that

$$\| f(t,x) - f(t,y) \| \leq \alpha \ \| x - y \|, \ \forall x, y \in R^n$$

Further, let $S_N \equiv S_N(R^n)$ $(N \geq 0)$ be the linear space of terminating sequences $\{x(t)\}_{t=0}^{N}, (x(t) \in R^n)$ and denote by $S_N^\infty \equiv S_N^\infty(R^n)$, the corresponding Banach space with norm $\| \cdot \|_N^\infty$:

$$\| x \|_N^\infty = \sup_{0 \leq t \leq N} \| x(t) \|$$

Denote the linear space of control sequences by

$$U_{[0,N]} = \{u \in R^{m(N+1)} : u := [u(0), u(1), ..., u(N)], \ with \ u(t) \in R^m, \ 0 \leq t \leq N\}$$

3.2 Steering Control for Semi-linear System

Theorem 3.2.1. *If the the linear system is controllable in N time-steps and the control $u(t)$ defined by (3.1.5) is well-defined, then it steers the nonlinear system (3.1.1) from the initial state x_0 to the desired final state x_1 in N time-steps.*

Proof. Since the linear system (3.1.2) is controllable on $[0, N]$, we have by Theorem 2.7.2 that $detW_r(0,N) \neq 0$. If we substitute the control given by (3.1.5) in the solution

$$x(t) = \Phi(t,0)x_0 + \sum_{j=0}^{t-1} \Phi(t,j+1)B(j)u(j) + \sum_{j=0}^{t-1} \Phi(t,j+1)f(j,x(j))$$

we get,

$$\begin{aligned} x(t) &= \Phi(t,0)x_0 \\ &+\sum_{j=0}^{t-1}\Phi(t,j+1)B(j)B(j)^*\Phi(N,j+1)^*W_r(0,N)^{-1} \\ &\left\{x_1-\Phi(N,0)x_0-\sum_{j=0}^{N-1}\Phi(N,j+1)f(j,x(j))\right\} \\ &+\sum_{j=0}^{t-1}\Phi(t,j+1)f(j,x(j)) \end{aligned} \tag{3.2.1}$$

It can be easily verified that at $t=0$, $x(0)=x_0$ and at $t=N, x(N)=x_1$. Thus, the control u defined in (3.1.5) steers the nonlinear system from the given initial state x_0 to the desired final state x_1. □

3.3 Well-definedness of the Steering Control

We now prove that the control defined in (3.1.5) is meaningful. This control u is well-defined if there is a solution to the equation (3.1.1) with this control. We will prove existence and uniqueness of solution of system (3.1.1). We make use of the following notations and definitions: Let $C=\max_{N\geq t\geq j\geq 0}\parallel\Phi(t,j)\parallel$, $M_1=\max_{N\geq j\geq 0}\parallel B(j)\parallel$ and

$M_2=\parallel W_r(0,N)^{-1}\parallel$.

$\beta=C(1+C^2M_1^2M_2(N-1))$

$\eta=\alpha\beta(N-1)$.

Theorem 3.3.1. *Under Assumptions [L],[N] and $\eta<1$ the steering control defined by*

$$u(t)=B(t)^*\Phi(N,t+1)^*W_r(0,N)^{-1}\left[x_1-\Phi(N,0)x_0-\sum_{j=0}^{N-1}\Phi(N,j+1)f(j,x(j))\right]$$

is well-defined.

Proof. We prove this by showing that the nonlinear system with this control has a unique solution. In Theorem 3.2.1, we have shown that this control does the required steering. We show that the following nonlinear equation has a unique solution.

$$\begin{aligned} x(t) &= \Phi(t,0)x_0 + \sum_{j=0}^{t-1} \Phi(t,j+1)B(j)B(t)^*\Phi(N,t+1)^*W_r(0,N)^{-1} \\ &\quad \{x_1 - \Phi(N,0)x_0 - \sum_{j=0}^{N-1} \Phi(N,j+1)f(j,x(j))\} \\ &\quad + \sum_{j=0}^{t-1} \Phi(t,j+1)f(j,x(j)) \end{aligned}$$

To prove the existence of the solution, we define a mapping
$T : S_N^\infty(R^n) \to S_N^\infty(R^n)$ by

$$\begin{aligned} T(x(t)) &= \Phi(t,0)x_0 + \sum_{j=0}^{t-1} \Phi(t,j+1)B(j)B(j)^*\Phi(N,t+1)^*W_r(0,N)^{-1} \\ &\quad \{x_1 - \Phi(N,0)x_0 - \sum_{j=0}^{N-1} \Phi(N,j+1)f(j,x(j))\} \\ &\quad + \sum_{j=0}^{t-1} \Phi(t,j+1)f(j,x(j)),\ t = 0,1,...,N. \end{aligned}$$

Since $S_N^\infty(R^n)$ is a complete Banach space, we show that operator T has a fixed point by using Banach contraction mapping theorem.

Consider

$$\begin{aligned}
\| T(x(t)) - T(\tilde{x}(t)) \| &\leq \sum_{j=0}^{t-1} \| \Phi(t,j+1)\{f(j,x(j)) - f(j,\tilde{x}(j))\} \| \\
&\quad + \sum_{j=0}^{t-1} \| \Phi(t,j+1)B(j)B(j)^{*}\Phi(N,j+1)^{*}W_r(0,N)^{-1} \\
&\quad \sum_{i=0}^{N-1} \Phi(N,i+1)\{f(i,x(i)) - f(i,\tilde{x}(i))\} \| \\
&\leq C\sum_{j=0}^{t-1} \| f(x(j)) - f(\tilde{x}(j)) \| \\
&\quad + C^2 M_1^2 M_2 \sum_{j=0}^{t-1} C \sum_{i=0}^{N-1} \| f(i,\tilde{x}(i)) - f(i,x(i)) \| \\
&\leq \alpha C \sum_{j=0}^{t-1} \| x(j) - \tilde{x}(j) \| \\
&\quad + \alpha C^3 M_1^2 M_2 (t-1) \sum_{i=0}^{N-1} \| \tilde{x}(i) - x(i) \| \\
&\leq \alpha C(1 + C^2 M_1^2 M_2(t-1)) \sum_{j=0}^{N-1} \| x(j) - \tilde{x}(j) \| \\
&\leq \alpha\beta \sum_{j=0}^{N-1} \| x(j) - \tilde{x}(j) \|,
\end{aligned}$$

$$\begin{aligned}
\textit{Thus}, \sup_{0\leq t\leq N} \| T(x(t)) - T(\tilde{x}(t)) \| &\leq \alpha\beta(N-1) \sup_{0\leq t\leq N} \| x(t) - \tilde{x}(t) \| \\
\| T(x) - T(\tilde{x}) \| &\leq \eta \| x - \tilde{x} \| .
\end{aligned}$$

Since $\eta < 1$, T is a contraction. Hence T has a unique fixed point. Therefore, the nonlinear equation is uniquely solvable. This proves that the control defined in (3.1.5) is well-defined. □

We now give the following computational scheme for the steering control for the nonlinear system.

3.4 Computational Scheme of the Steering Control

Theorem 3.4.1. *Under the assumptions of Theorem 3.3.1, the steering control and controlled trajectory of the nonlinear system (3.1.1) driving the system from* $x(0) = x_0$ *to* $x(N) = x_1$ *can be computed by the following iterative scheme:*

$$u^m(t) = B(t)^*\Phi(N, t+1)^* W_r(0, N)^{-1}[x_1 - \Phi(N, 0)x_0 - \sum_{j=0}^{N-1} \Phi(N, j+1) f(j, x^m(j))] \tag{3.4.1}$$

and

$$x^{m+1}(t) = \Phi(t, 0)x_0 + \sum_{j=0}^{t-1} \Phi(t, j+1)B(j)u^m(j) + \sum_{j=0}^{t-1} \Phi(t, j+1) f(j, x^m(j)) \tag{3.4.2}$$

starting with arbitrary $x^0(t)$, $t = 0, 1, 2, ..., N-1$, $m = 0, 1, 2, ...$

Proof. The computational scheme follows directly from Banach contraction principle and from Theorem 3.3.1. □

3.5 Equivalence between Controllability and Reachability

Although for nonlinear systems controllability and reachability notions are not equivalent, we prove in the following theorem that for the semi-linear system (3.1.1), the two notions are equivalent.

Theorem 3.5.1. *The two notions of controllability and reachability are equivalent for the semi-linear system (3.1.1).*

Proof. From definition, it is obvious that for the system (3.1.1), controllability implies reachability. Conversely, let the system (3.1.1) is reachable on $[0, N]$. Thus, the 0 state can be steered to any desired state $\tilde{x}_1$.

Now, for arbitrary $x_0, x_1 \in R^n$, choose

$$\tilde{x}_1 = x_1 - \phi(N,0)x_0.$$

Since there exists $u(t) \in U_{[0,N]}$ that steers $x_0 = 0$ to $\tilde{x}_1$ for some N. Hence

$$\tilde{x}_1 = \Phi(N,0)0 + \sum_{j=0}^{N-1} \Phi(N,j+1)B(j)u(j) + \sum_{j=0}^{N-1} \Phi(N,j+1)f(j,x(j)) \qquad (3.5.1)$$

i.e.

$$\tilde{x}_1 = \sum_{j=0}^{N-1} \Phi(N,j+1)B(j)u(j) + \sum_{j=0}^{N-1} \Phi(N,j+1)f(j,x(j)) \qquad (3.5.2)$$

i.e.

$$x_1 = \Phi(N,0)x_0 + \sum_{j=0}^{N-1} \Phi(N,j+1)B(j)u(j) + \sum_{j=0}^{N-1} \Phi(N,j+1)f(j,x(j)) \qquad (3.5.3)$$

which shows that the same control steers x_0 to x_1. Hence the system (3.1.1) is controllable. □

3.6 Numerical Examples

Example 3.6.1. *Consider the semi-linear system given by the following equation:*

$$x(t+1) = A(t)x(t) + B(t)u(t) + f(t,x(t)) \qquad (3.6.1)$$

where $A(t) = \frac{1}{4}\begin{pmatrix} cos(2t) & 1 \\ t^2 & cos^2(t) \end{pmatrix}$, $B(t) = \begin{pmatrix} .5 \\ .5t \end{pmatrix}$ *and*

$$f(t,x) = \frac{1}{5}\begin{pmatrix} sin^2(x_1(t)) \\ cos^2(x_2(t)) \end{pmatrix}$$

Let $N = 10$. *Here the reachability Grammian can be computed as*

$$W_r(0,10) = \begin{pmatrix} 1.0774 & 0.2760 \\ 0.2760 & 0.2078 \end{pmatrix}$$

and

$$det(W_r(0,10)) = 0.1477 \neq 0.$$

Hence the linear system is controllable, and

$$\begin{aligned} \| f(t,x) - f(t,y) \| &= \frac{1}{5} \left\| \begin{pmatrix} sin^2(x_1) - sin^2(y_1) \\ cos^2(x_2) - cos^2(y_2) \end{pmatrix} \right\| \\ &\leq \frac{2}{5} \| x - y \| \end{aligned}$$

Hence f is Lipschitz with Lipschitz constant $\frac{2}{5}$. We can easily verify the conditions of Theorem 3.3.1 to conclude that the system is controllable. Figure 1 shows the controlled trajectory steering the system from the initial state $x_0 = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ to the final state $x_1 = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$. Note that the computation of data is done using Matlab program $P-1$ given in Appendix.

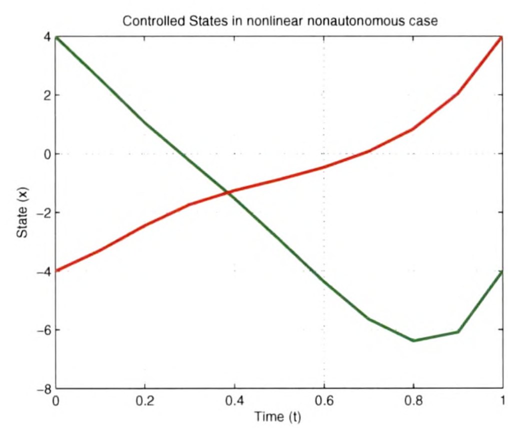

Figure 3.1: Controlled trajectories for nonautonomous nonlinear case

Example 3.6.2. *Consider the nonlinear system given by the following equation:*

$$x(t+1) = Ax(t) + Bu(t) + f(t, x(t)) \tag{3.6.2}$$

where $A = \begin{pmatrix} 1.3511 & 0.0239 \\ 0.1195 & 1.0524 \end{pmatrix}$, $B = \begin{pmatrix} 0.0237 \\ 0.0319 \end{pmatrix}$ and f as defined in above example. We can see in Figure 3.2 that the control steers the initial state $x_0 = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$

to the final state $x_1 = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$ *in 10 time steps. Using Matlab program given in Appendix* $P - 2$, *the transition matrix* $\Phi(t, t_0)$ *and reachability Grammian matrix* $W_r(0, t)$ *with* $t = 10$ *and* $t_0 = 0$ *are computed as follows.*

$$\Phi(10, 0) = \begin{pmatrix} 21.1163 & 1.5201 \\ 7.6004 & 2.1154 \end{pmatrix}$$

and

$$W_r(0, 10) = \begin{pmatrix} 0.3494 & 0.1792 \\ 0.1792 & 0.0942 \end{pmatrix}$$

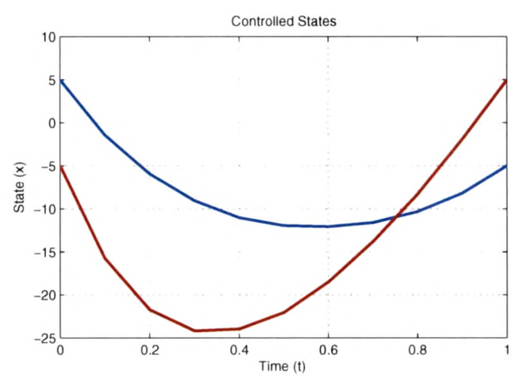

Figure 3.2: Controlled trajectories for Autonomous nonlinear system

3.7 Summary

In this chapter, steering control for the semi-linear discrete-time system is introduced. Using Banach's fixed point theorem its well definedness is proved under the conditions that its linear part is controllable and nonlinear function is Lipschitz. It is also proved that the controllability and reachability are equivalent for the semi-linear system. We have also provided an algorithm for the actual computation of steering control and controlled trajectories for the semi-linear system. At the end we have presented numerical examples for nonlinear systems.