CHAPTER VII

ON ABSOLUTE CONVERGENCE AND ABSOLUTE CESÀRO SUMMABILITY OF DOUBLE ORTHOGONAL SERIES

7.1 Let $\{\phi_{ij}(x,y)\}$ $(i, j = 1, 2, \ldots)$ be a normalized orthogonal system in the rectangle $R\ [a \le x \le b\ ;\ c \le y \le d]$, that is,

$$\iint\limits_R \phi_{ij}(x,y)\, \phi_{kl}(x,y)\, dx\, dy = \begin{cases} 1 & \text{when } i=k,\ j=l \\ 0 & \text{otherwise} \end{cases}$$

Consider the double orthogonal series

$$(7.1.1) \qquad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij}\, \phi_{ij}(x,y)$$

where $\{c_{ij}\}$ is a double sequence of real numbers. The mnth partial sums of (7.1.1) are given by

$$S_{mn}(x,y) = \sum_{k=1}^{m} \sum_{l=1}^{n} c_{kl}\, \phi_{kl}(x,y).$$

Definition 1 : The series (7.1.1) is said to be (C,1,1)-summable to a function s(x,y) if

$$\lim_{m,n \to \infty} \sigma_{mn}(x,y) = S(x,y),$$

where

$$\sigma_{mn}(x,y) = \sum_{k=1}^{m} \sum_{l=1}^{n} \left(1 - \frac{k-1}{m}\right)\left(1 - \frac{l-1}{n}\right) c_{kl}\, \phi_{kl}(x,y).$$

Definition 2 : The series (7.1.1) is said to be absolutely summable* (C,1,1) or summable |C,1,1| if

$$(7.1.2) \qquad \sum_{m=1}^{\infty} \left| \sigma_{m+1,n}(x,y) - \sigma_{mn}(x,y) \right| < \infty$$

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* Sharma [95].

for a fixed n ;

$$(7.1.3) \qquad \sum_{n=2}^{\infty} \left| \sigma_{m,n-1}(x,y) - \sigma_{mn}(x,y) \right| < \infty$$

for a fixed m ;

and

$$(7.1.4) \qquad \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \left| \sigma_{mn}(x,y) - \sigma_{m-1,n}(x,y) - \sigma_{m,n-1}(x,y) + \sigma_{m-1,n-1}(x,y) \right| < \infty .$$

The convergence and Cesàro summability of single orthogonal series have been studied by Alexits[*], Kaczmarz[**], Menchoff[***], Rademacher[+], Tandori[++] and Zygmund[+++] and those of double orthogonal series by Agnew[†], Mitchell[‡], Patel[≠] and Sapre[#]. Patel[§] while studying absolute convergence of single orthogonal series has proved the

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* Alexits [5,6]

** Kaczmarz [39]

*** Menchoff [62]

\+ Rademacher [85]

\++ Tandori [109,113]

\+++ Zygmund [126]

† Agnew [1]

‡ Mitchell [63]

≠ Patel [75]

\# Sapre [91]

§ Patel [74]

following theorem :

<u>Theorem A</u>: If

$$\left(\sum_{k=n}^{\infty} c_k^2\right)^{\frac{1}{2}} = O\left(\frac{1}{n^{\alpha}}\right)$$

where $\alpha > \frac{1}{2}$, then the series

$$(7.1.5) \qquad \sum_{n=0}^{\infty} c_n \phi_n(x)$$

will be absolutely convergent almost everywhere in (a,b).

Sufficient conditions for $|C,1|$-summability of single orthogonal series have been studied by Tandori*. He has proved the following theorem**:

<u>Theorem B</u>: Under the condition

$$\sum_{m=0}^{\infty} \left(\sum_{k=2^m+1}^{2^{m+1}} c_k^2\right)^{\frac{1}{2}} < \infty$$

the orthogonal series (7.1.5) is absolutely $(C,1)$-summable almost everywhere.

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* Tandori [112] .

** Alexits [7] p.124.

In this chapter we generalize the above theorems to double orthogonal series.

Write

$$A_{mn} = \left\{ \sum_{l=2^n+1}^{2^{n+1}} c_{ml}^2 \right\}^{\frac{1}{2}} \quad (m, n = 1, 2, 3, \ldots) \; ;$$

$$A_s^{(p)} = \left\{ \sum_{i=2^s+1}^{2^{s+1}} c_{ip}^2 \right\}^{\frac{1}{2}} \quad \text{for a fixed } p, \; s = 0, 1, 2, \ldots \; ;$$

$$B_t^{(q)} = \left\{ \sum_{j=2^t+1}^{2^{t+1}} c_{qj}^2 \right\}^{\frac{1}{2}} \quad \text{for a fixed } q, \; t = 0, 1, 2, \ldots$$

We prove

<u>THEOREM 7.1</u> If

$$\left(\sum_{k=m}^{\infty} \sum_{l=n}^{\infty} c_{kl}^2 \right)^{\frac{1}{2}} = O\left(\frac{1}{(mn)^{\alpha}} \right) \tag{7.1.0}$$

where $\alpha > \frac{1}{2}$, then the series (7.1.1) is absolutely convergent almost everywhere in $R\left[a,b\,;\,c,d\right]$.

THEOREM 2: The double orthogonal series (7.1.1) is $|C,1,1|$-summable almost everywhere in $R\left[a,b\,;\,c,d\right]$ if

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} A_{mn} < \infty, \tag{7.1.7}$$

$$\sum_{s=0}^{\infty} A_s^{(p)} < \infty \tag{7.1.8}$$

for $p = 0, 1, 2, \ldots$; and

$$\sum_{t=0}^{\infty} B_t^{(q)} < \infty \tag{7.1.9}$$

for $q = 0, 1, 2, \ldots$

7.2 Proof of Theorem 1: By Schwarz's inequality and orthonormality property of the functions $\{\phi_{mn}(x,y)\}$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\iint_R \left|c_{kl}\phi_{kl}(x,y)\right| dx\,dy =$$

$$= O(1)\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}|c_{kl}| .$$

Now

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}|c_{kl}| = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\sum_{m=1}^{k}\sum_{n=1}^{l}\frac{|c_{kl}|}{kl}\right)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\sum_{k=m}^{\infty}\sum_{l=n}^{\infty}\frac{|c_{kl}|}{kl}\right) \leq$$

$$\leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\left(\sum_{k=m}^{\infty}\sum_{l=n}^{\infty}\frac{1}{(kl)^2}\right)\left(\sum_{k=m}^{\infty}\sum_{l=n}^{\infty}c_{kl}^2\right)\right\}^{\frac{1}{2}}$$

$$\leq C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\sqrt{mn}}\left(\sum_{k=m}^{\infty}\sum_{l=n}^{\infty}c_{kl}^2\right)^{\frac{1}{2}} ,$$

where C is an absolute constant,

$$= O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(mn)^{\alpha+\frac{1}{2}}} \qquad \text{from (7.1.6).}$$

Therefore,

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\iint_R |c_{kl}\,\phi_{kl}(x,y)|\,dx\,dy =$$

$$= O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(mn)^{\alpha+\frac{1}{2}}} < \infty .$$

Our result follows by the application of the extension of the Levy's theorem for double series.

7.3 Proof of Theorem 2 : We have

$$\sigma_{m-1,n}(x,y) - \sigma_{mn}(x,y) =$$

$$= \frac{1}{(m-1)n}\sum_{k=1}^{m-1}\sum_{l=1}^{n}\{(m-1)-(k-1)\}\{n-(l-1)\}\,c_{kl}\phi_{kl}(x,y) -$$

$$- \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}(m-k+1)(n-l+1)\,c_{kl}\,\phi_{kl}(x,y) =$$

$$= \sum_{k=1}^{m}\sum_{l=1}^{n}\left\{\frac{m-k}{m-1} - \frac{m-k+1}{m}\right\}\left(\frac{n-l+1}{n}\right)c_{kl}\,\phi_{kl}(x,y)$$

$$= \sum_{k=1}^{m}\sum_{l=1}^{n}\frac{-(k-1)}{m(m-1)}\left(\frac{n-l+1}{n}\right)c_{kl}\,\phi_{kl}(x,y) .$$

Using Schwarz-inequality, orthonormality and $\frac{n-l+1}{n} \leq 1$ for $l = 1, 2, \ldots, n$, we get

$$\iint_R \left| \sigma_{m+1,n}(x,y) - \sigma_{mn}(x,y) \right| dx\,dy =$$

$$= O(1) \left\{ \iint_R \left(\sigma_{m+1,n}(x,y) - \sigma_{mn}(x,y) \right)^2 dx\,dy \right\}^{\frac{1}{2}}$$

$$= O(1) \left\{ \sum_{k=2}^{m} \sum_{l=1}^{n} \frac{k^2}{m^2 (m-1)^2} c_{kl}^2 \right\}^{\frac{1}{2}} .$$

Therefore

$$\sum_{m=2}^{\infty} \iint_R \left| \sigma_{m+1,n}(x,y) - \sigma_{mn}(x,y) \right| dx\,dy =$$

$$= O(1) \sum_{m=2}^{\infty} \frac{1}{(m-1)^2} \left\{ \sum_{k=2}^{m} \sum_{l=1}^{n} k^2 c_{kl}^2 \right\}^{\frac{1}{2}} .$$

Let c_{kp}^2 be the maximum of $c_{k1}^2, c_{k2}^2, \ldots, c_{kn}^2$.

Hence

$$\sum_{m=2}^{\infty} \iint_R \left| \sigma_{m+1,n}(x,y) - \sigma_{mn}(x,y) \right| dx\,dy =$$

$$= O(1)\sum_{m=2}^{\infty}\frac{1}{(m-1)^2}\left\{\sum_{k=2}^{m}k^2 C_{kp}^2\right\}^{\frac{1}{2}}$$

as n is fixed.

Now

$$\left\{\sum_{k=2}^{m}k^2 C_{kp}^2\right\}^{\frac{1}{2}} = \sum_{s=0}^{[\log m]^*}\left\{\sum_{i=2^s+1}^{2^{s+1}} i^2 C_{ip}^2\right\}^{\frac{1}{2}} \leq$$

$$\leq \sum_{s=0}^{[\log m]} 2^{s+1}\left\{\sum_{i=2^s+1}^{2^{s+1}} C_{ip}^2\right\}^{\frac{1}{2}}.$$

Consequently

$$\sum_{m=2}^{\infty}\iint_R\left|\sigma_{m-1,n}(x,y)-\sigma_{mn}(x,y)\right|dx\,dy =$$

$$= O(1)\sum_{m=2}^{\infty}\frac{1}{(m-1)^2}\sum_{s=0}^{[\log m]}2^{s+1}\left\{\sum_{i=2^s+1}^{2^{s+1}}C_{ip}^2\right\}^{\frac{1}{2}}$$

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* [log m] is the integral part of log m with base 2.

$$= O(1) \sum_{s=0}^{\infty} 2^{s+1} \left\{ \sum_{i=2^s+1}^{2^{s+1}} c_{ip}^2 \right\}^{\frac{1}{2}} \sum_{[\log m] \geq s} \frac{1}{(m-1)^2}$$

$$= O(1) \sum_{s=0}^{\infty} \left\{ \sum_{i=2^s+1}^{2^{s+1}} c_{ip}^2 \right\}^{\frac{1}{2}}$$

(7.2.2) $$= O(1) \sum_{s=0}^{\infty} A_s^{(p)} < \infty .$$

The requirement (7.1.3) can be disposed off in a similar manner.

We shall indicate the expression

$$\sigma_{mn}(x,y) - \sigma_{m-1,n}(x,y) - \sigma_{m,n-1}(x,y) + \sigma_{m-1,n-1}(x,y)$$

by $u(m-1, n-1, m, n)$ and shall have

$$u(m-1, n-1, m, n) = \sum_{k=1}^{m} \sum_{l=1}^{n} \frac{kl}{m(m+1)\, n(n+1)} c_{kl} \phi_{kl}(x,y) .$$

Proceeding as before

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\iint_{R}\left|u\,(m-1,n-1,m,n)\right|\,dx\,dy =$$

$$= O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{m^2n^2}\left\{\sum_{k=2}^{m}\sum_{l=2}^{n}k^2l^2C_{kl}^2\right\}^{\frac{1}{2}}.$$

Denoting the k^{th} row of the sum

$$\sum_{k=2}^{m}\sum_{l=2}^{n}k^2l^2C_{kl}^2$$

by R_k we have by Jensen's inequality*

$$\left(\sum_{k=2}^{m}\sum_{l=2}^{n}k^2l^2C_{kl}^2\right)^{\frac{1}{2}} = \left(\sum_{k=2}^{m}R_k\right)^{\frac{1}{2}} < \sum_{k=2}^{m}\sqrt{R_k}\ .$$

Again

$$\sqrt{R_k} < k\left[2^2C_{k2}^2 + 2^{2\cdot 2}\left(C_{k3}^2 + C_{k4}^2\right) + \cdots\right.$$

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* Hardy, Littlewood and Pólya [23] Theorem 19, page 28.

$$+ 2^{2([\log n]+1)} \left(C^2_{k, 2^{[\log n]}+1} + \cdots + C^2_{k, 2^{[\log n]+1}} \right) \Bigg]^{\frac{1}{2}} =$$

$$= K \left\{ \sum_{r=0}^{[\log n]} 2^{2(r+1)} A^2_{kr} \right\}^{\frac{1}{2}} .$$

Applying the Jensen's inequality once more we get

$$\sqrt{R_k} < K \sum_{r=0}^{[\log n]} 2^{r+1} A_{kr} .$$

Therefore

$$\left(\sum_{k=2}^{m} \sum_{l=2}^{n} k^2 l^2 C^2_{kl} \right)^{\frac{1}{2}} < \sum_{k=2}^{m} \sum_{r=0}^{[\log n]} k \, 2^{r+1} A_{kr} .$$

Hence

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \iint_R |u(m-1, n-1, m, n)| \, dx \, dy =$$

$$= O(1) \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{m^2 n^2} \sum_{k=2}^{m} \sum_{r=0}^{[\log n]} k \, 2^{r+1} A_{kr}$$

$$= O(1) \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} m\, 2^{n+1} A_{mn} \sum_{k=m}^{\infty} \frac{1}{k^2} \sum_{[\log l] \geq n} \frac{1}{l^2}$$

$$(7.3.2) \qquad = O(1) \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} A_{mn} < \infty .$$

Thus from (7.3.1) and (7.3.2) our result follows by the application of the extension of the Levy's theorem for double series.