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Dec 14, 1974.

Dear Patel,

I inform you that your paper (with C. M. Patel) ''Absolute Cesàro Summability of Double Orthogonal Series'' contains no error in calculation and is therefore recommended for publication in Nanta Mathematica.

Yours sincerely,

sd/- Dr. K. M. Koh
The Editor,
Nanta Mathematica.

JOURNAL OF THE MAHARAJA SAYAJIRAO UNIVERSITY OF BARODA

Editor:

Office of the Journal,
M. S. University of Baroda,
Near Hansa Mehta Library,
BARODA-2 (India)

4th September 1974.

Dear Shri Patel,

I am happy to inform you that your paper entitled "ON CONNECTION BETWEEN CERTAIN SUMMABILITY METHODS AS APPLIED TO ORTHOGONAL SERIES" has been accepted for publication in the Science Number of the Journal of the M. S. University of Baroda, Vol. XXII and XXIII No.3 (1973-74).

Yours sincerely,

sd/-
(O. M. DHARMADHIKARI)
Editor,
Science Number.

Shri B. K. Patel,
Department of Mathematics.

МАТЕМАТИЧКИ ВЕСНИК
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R. K. Patel

ON STRONG EULER SUMMABILITY OF ORTHOGONAL SERIES

(Received August 17, 1972)

1. Let $\{\Phi_n(x)\}$ $(n=0, 1, 2, \ldots)$ be an orthogonal and normal function system in the interval $\langle a, b\rangle$. We consider the orthogonal series

$$\sum C_n \Phi_n(x), \tag{1.1}$$

with real coefficients sequence $\{C_n\}$.

The n-th Euler mean of the first order or the $(E, 1)$-mean of the sequence of partial sums $\{S_n(x)\}$ of the orthogonal series (1.1) is defined as

$$\tau_n(x)=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}S_k(x), \qquad n=0, 1, 2, \ldots$$

where

$$S_k(x)=\sum_{i=0}^{k} C_i \Phi_i(x).$$

Series (1.1) is said to be strongly summable $(E, 1)$ to the sum $S(x)$ if

$$\sum_{k=0}^{n}\binom{n}{k}(S_k(x)-S(x))^2=o(2^n) \text{ as } n\to\infty\,^*.$$

The strong summability $(C, 1)$ of orthogonal series, as well as that of Fourier series, has been investigated by several authors such as: A. Zygmund, S. Kaczmarz, S. Borgen, G. Alexits, K. Tandori, B. N. Prasad and U. N. Singh. A. Zygmund ([6] p. 356) has proved the following theorem.

Theorem A. If series (1.1) by condition $\sum C_n^2<\infty$ is summable $(c, 1)$ almost everywhere to a function $S(x)$, then it is strongly summable $(c, 1)$ to this function $S(x)$.

The strong summability $(R, \lambda_n, 1)$ of orthogonal series has been studied by G. Lorentz, J. Meder, C. Patel and A. Sapre. Meder [4] has proved the following theorem.

---

* If the sequence $\{f_n(x)/g_n(x)$ in $\langle a, b\rangle$ is bounded or convergent to zero for $n\to\infty$ almost everywhere, then we shall write $f_n(x)=O\{g_n(x)\}$ or $f_n(x)=o\{g_n(x)\}$ respectively.

Theorem B. If the orthogonal series (1.1) with coefficients satisfying condition $\sum C_n^2<\infty$ is summable $(R, \lambda_n, 1)$ almost everywhere to a function $S(x)$, then it is strongly summable $(R, \lambda_n, 1)$ almost everywhere to this function $S(x)$.

The strong Nörlund summability of (1.1) has been discussed by Meder. Meder [5] has proved the following:

Theorem C. If orthogonal series (1.1) is (N, p_n) summable to a function $S(x)$ almost everywhere with $\{p_n\}\in\overline{\overline{M}}^{\alpha *}$, $\alpha>\frac{1}{2}$, then it is strongly (N, p_n) summable to this function almost everywhere.

In this paper I propose to prove the analogous form for $(E, 1)$-summability. We prove the following theorem:

Theorem: *If the orthogonal series* (1.1), *with coefficients satisfying condition* $\sum C_n^2\sqrt{n}<\infty$, *is summable by the method* $(E, 1)$ *to a function* $s(x)$, *then it is strongly summable* $(E, 1)$ *almost everywhere to this function.*

2. For the proof of our theorem we need following lemmas:

Lemma 1: (refer Knopp [2] p. 136)

If $m=\left[\frac{n}{2}\right]^{**}$, then

$$\frac{\sqrt{n}\binom{n}{m}}{2^n}<20e \quad \text{for} \quad n=1, 2, 3, \ldots$$

Lemma 2***: Writing

$$W_{nk}=\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}-\frac{2k}{n+1},$$

we have

$$W_{nk}<0 \quad \text{for} \quad \left[\frac{n}{3}\right]+2\leqq k\leqq n.$$

Lemma 3***:

$$\left\{\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\right\}^2\leqq C_2\frac{k^2}{n^2}$$

---

* The sequence $\{p_n\}\in\overline{\overline{M}}^{\alpha}$ if $\{p_n\}$ is convex or concave and if

(i) $0<p_{n+1}<p_n$ or $0<p_n<p_{n+1}$ $(n=0, 1, 2, \ldots)$

(ii) $p_0+p_1+\cdots+p_n=P_n\uparrow\infty$

(iii) $\lim_{n\to\infty}\frac{n\,\Delta^2 p_{n-2}}{\Delta p_{n-1}}=2-\alpha$ where $\alpha\geqq 0$, $\Delta^2 p_{n-2}=\Delta p_{n-2}-\Delta p_{n-1}$

are satisfied.

** $\left[\frac{n}{2}\right]$ indicates, the integral part of $\frac{n}{2}$.

*** For the proof of lemmas, see Meder [3].

for $1 \leqq k \leqq \left[\frac{n}{3}\right]+1$, C_2 being an absolute constant and $n=1, 2, 3, \ldots$

3. *Proof of the Theorem*:

We have

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(S_k(x)-S(x))^2 \leqq$$

$$\leqq \frac{2}{2^n}\sum_{k=0}^{n}\binom{n}{k}(S_k(x)-\tau_k(x))^2+\frac{2}{2^n}\sum_{k=0}^{n}\binom{n}{k}(\tau_k(x)-S(x))^2 \leqq$$

$$\leqq \frac{4}{2^n}\sum_{k=0}^{n}\binom{n}{k}(S_k(x)-\sigma_k(x))^2+\frac{4}{2^n}\sum_{k=0}^{n}\binom{n}{k}(\sigma_k(x)-\tau_k(x))^2+$$

$$+\frac{2}{2^n}\sum_{k=0}^{n}\binom{n}{k}(\tau_k(x)-S(x))^2=$$

$$= S_1+S_2+S_3, \quad \text{say.} \tag{3.1}$$

From the hypothesis it is evident that

$$S_3 \to 0.$$

Coming now to S_1,

$$S_1 = O(1)\frac{\binom{n}{m}}{2^n}\sum_{k=0}^{n}(S_k(x)-\sigma_k(x))^2 = O(1)\frac{\sqrt{n}\binom{n}{m}}{2^n}\cdot\frac{1}{\sqrt{n}}\sum_{k=0}^{n}(S_k(x)-\sigma_k(x))^2$$

$$= O(1)\frac{1}{\sqrt{n}}\sum_{k=0}^{n}(S_k(x)-\sigma_k(x))^2, \text{ by Lemma 1.}$$

Now,

$$S_n(x)-\sigma_n(x)=\frac{1}{n+1}\sum_{k=1}^{n}kC_k\,\Phi_k(x).$$

Therefore by the orthonormality properties of $\{\Phi_n(x)\}$,

$$\int_a^b (S_n(x)-\sigma_n(x))^2\,dx = \frac{1}{(n+1)^2}\sum_{k=1}^{n}k^2C_k^2.$$

Consequently,

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\int_a^b (S_n(x)-\sigma_n(x))^2\,dx < \sum_{n=1}^{\infty}\frac{1}{n^{5/2}}\sum_{k=1}^{n}k^2C_n^2=$$

$$=\sum_{k=1}^{\infty}k^2C_k^2\sum_{n=k}^{\infty}\frac{1}{n^{5/2}}=O(1)\sum_{k=1}^{\infty}C_k^2\sqrt{k}<\infty.$$

Therefore by B. Levy's theorem (refer Alexits [1] p. 11)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (S_n(x) - \sigma_n(x))^2 < \infty.$$

Then, by Kronecker's lemma (refer Alexits [1] p. 72)

$$\sum_{k=1}^{n} (S_k(x) - \sigma_k(x))^2 = o(\sqrt{n}),$$

which proves that

$$S_1 \to 0.$$

Lastly let us consider S_2,

$$S_2 \leqq \frac{4}{2^n} \binom{n}{m} \sum_{k=0}^{n} (\sigma_k(x) - \tau_k(x))^2$$

$$= O(1) \frac{1}{\sqrt{n}} \sum_{k=0}^{n} (\sigma_k(x) - \tau_k(x))^2, \quad \text{by Lemma 1.}$$

Now, we have

$$\sigma_n(x) - \tau_n(x) = \sum_{k=0}^{n} C_k \Phi_k(x) \left[\frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} - \frac{k}{n+1} \right].$$

Whence

$$\int_a^b [\sigma_n(x) - \tau_n(x)]^2 \, dx =$$

$$= \sum_{k=1}^{n} C_k^2 \left\{ \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} \left[\frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} - \frac{2k}{n+1} \right] + \frac{k^2}{(n+1)^2} \right\} <$$

$$< \sum_{k=1}^{\left[\frac{n}{3}\right]+1} C_k^2 \left[\frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} \right]^2 +$$

$$\sum_{k=\left[\frac{n}{3}\right]+2}^{n} C_k^2 \left\{ \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} \left[\frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} - \frac{2k}{n+1} \right] \right\} + \sum_{k=1}^{n} \frac{k^2 C_k^2}{(n+1)^2}.$$

By lemma 2,

$$\sum_{k=\left[\frac{n}{3}\right]+2}^{n} C_k^2 \left\{ \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} \left[\frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} - \frac{2k}{n+1} \right] \right\} < 0.$$

Hence,

$$\int_a^b [\sigma_n(x) - \tau_n(x)]^2 \, dx < \sum_{k=1}^{\left[\frac{n}{3}\right]+1} C_k^2 \left[\frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} \right]^2 + \sum_{k=1}^{n} \frac{k^2 C_k^2}{(n+1)^2}.$$

Consequently, by Lemma 3,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_a^b [\sigma_n(x) - \tau_n(x)]^2 \, dx \leqq$$

$$\leqq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{\left[\frac{n}{3}\right]+1} C_k^2 \cdot C_2 \cdot \frac{k^2}{n^2} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{k^2 C_k^2}{(n+1)^2}$$

$$\leqq A \left[\sum_{k=1}^{\infty} k^2 C_k^2 \sum_{n=k}^{\infty} \frac{1}{n^{5/2}} + \sum_{k=1}^{\infty} k^2 C_k^2 \sum_{n=k}^{\infty} \frac{1}{n^{5/2}} \right]$$

where A is an absolute constant.

$$O(1) \sum_{k=1}^{\infty} C_k^2 \sqrt{k} < \infty .$$

Therefore by B. Levy's theorem,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\sigma_n(x) - \tau_n(x)\right)^2 \qquad \text{converges a.e. in } \langle a, b \rangle .$$

Then by Kronecker's lemma,

$$\sum_{k=1}^{n} \left(\sigma_k(x) - \tau_k(x)\right)^2 = o\left(\sqrt{n}\right) \qquad \text{a.e.}$$

which proves that $S_2 \to 0$ a.e.

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Thus the theorem is proved.

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ON VERY STRONG EULER SUMMABILITY OF ORTHOGONAL SERIES

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1. Introduction

Let ON $\{\phi_n(x)\}$ denote an orthonormal system defined in the interval $< a, b >$ and $\{C_n\} \varepsilon l^2$, that is,

$$(1.1) \quad \sum_{n=0}^{\infty} C_n^2 < \infty.$$

Further let

$$(1.2) \quad \sum_{n=0}^{\infty} C_n \phi_n(x)$$

denote orthogonal series being development of functions $f(x) \varepsilon L^2$ i.e. integrable with square in Lebesgue sense.

The n-th Euler mean of the first order or the (E, 1)-mean of the sequence of partial sums $\{S_n(x)\}$ of the orthogonal series (1.2) is defined as

$$\tau_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} S_k(x), \quad n = 0, 1, 2, \ldots \ldots$$

where

$$S_k(x) = \sum_{i=0}^{k} C_i \phi_i(x).$$

Series (1.2) is said to be very strong summable (E, 1) to the sum $S(x)$ if for every monotone increasing index sequence $\{\nu_n\}$ and for almost every x the relation

$$\sum_{k=0}^{n} \binom{n}{k} \left(S_{\nu_k}(x) - S(x) \right)^2 = 0 \left(2^n \right) \text{ as } n \to \infty \text{ holds.}$$

The very strong Cesáro summability of orthogonal series has been studied in great details by G. ALEXITS [2] and K. TANDORI [8]. Very strong Riesz summability of orthogonal series as well as very strong Nörlund summability of orthogonal series has been discussed by MEDER [6], [7].

In this paper, I propose to prove a theorem on very strong (E, 1)—summability of (1.2) which reads as follows:

Theorem : *Let $\{ C^{\star}_{\nu} \}$ be a numerical sequence of positive terms such that*

(1.3) $\sqrt{\nu}\, C^{\star}_{\nu} = \geqq \sqrt{\nu+1}\; C^{\star}_{\nu+1}$ $(\nu = 1, 2, 3, \ldots\ldots)$

and

(1.4) $\Sigma\, C^{\star 2}_{\nu} \sqrt{\nu} < \infty.$

Further let $\{ C_n \}$ be an arbitrary sequence of real numbers satisfying the relation

(1.5) $C_n = O\,(C^{\star}_n)$

Suppose that the orthogonal series $\Sigma\, C_n \phi_n(x)$ under these assumptions is (E, 1) summable to a function $f(x)$ almost everywhere in $< a, b >$, then it is very strongly summable (E, 1) to this function almost everywhere in $< a, b >$.

2. Lemmas :

For the proof of our theorem we need following lemmas:

Lemma 1 : (Refer KNOPP [3] p. 136).

If $m = \left[\frac{n}{2}\right]$ $\left(\text{the integral part of } \frac{n}{2}\right)$ then

$$\frac{\sqrt{n}\binom{n}{m}}{2^n} < 20\,e \qquad \text{for } n = 1, 2, 3, \ldots\ldots$$

Lemma 2 : (Refer MEDER [5] Lemma 1)

Writting $$W_{nk} = \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} - \frac{2k}{n+1}$$

We have

$$W_{nk} < 0 \quad \text{for } \left[\frac{n}{3}\right] + 2 \leqq k \leqq n.$$

Lemma 3 : (Refer MEDER [5] Lemma 1)

$$\left\{ \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} \right\}^2 \leqq C_1 \frac{k^2}{n^2}$$

for $1 \leqq k \leqq \left[\frac{n}{3}\right] + 1$, C_1 being an absolute constant and $n = 1, 2, 3, \ldots\ldots$

Lemma 4 : Under the condition (1.1) the relation

$S_{n_k}(x) - \tau_{n_k}(x) = 0_x(1)$ is valid almost everywhere for every index sequence

$\left\{ n_k \right\}$ with $\dfrac{n_{k+1}}{n_k} \geqq q > 1$.

Proof of Lemma 4 : We write

$$\sum_{k=1}^{\infty} \int_a^b \left[S_{n_k}(x) - \tau_{n_k}(x) \right]^2 dx \leqq$$

$$(2.1) \quad \leqq 2 \sum_{k=1}^{\infty} \int_a^b \left[S_{n_k}(x) - \sigma_{n_k}(x) \right]^2 dx +$$

$$+ 2 \sum_{k=1}^{\infty} \int_a^b \left[\sigma_{n_k}(x) - \tau_{n_k}(x) \right]^2 dx =$$

$$= I_1 + I_2 \text{, say.}$$

The convergence of I_1 under the assumed conditions follows from the theorem of A. N. KOLMOGOROFF [4].
For convergence of I_2
We have

$$\sigma_n(x) - \tau_n(x) = \sum_{\nu=0}^{n} C_\nu \phi_\nu(x) \left[\frac{1}{2^n} \sum_{i=0}^{\nu-1} \binom{n}{i} - \frac{\nu}{n+1} \right]$$

Whence

$$\int_a^b [\sigma_n(x) - \tau_n(x)]^2 dx =$$

$$= \sum_{\nu=0}^{n} C^2_\nu \left\{ \frac{1}{2^n} \sum_{i=0}^{\nu-1} \binom{n}{i} \left[\frac{1}{2^n} \sum_{i=0}^{\nu-1} \binom{n}{i} - \frac{2\nu}{n+1} \right] + \right.$$

$$\left. + \frac{\nu^2}{(n+1)^2} \right\} < \sum_{\nu=1}^{\left[\frac{n}{3}\right]+1} C^2_\nu \left[\frac{1}{2^n} \sum_{i=0}^{\nu-1} \binom{n}{i} \right]^2 +$$

$$+ \sum_{\nu=\left[\frac{n}{3}\right]+2}^{n} C^2_\nu \left\{ \frac{1}{2^n} \sum_{i=0}^{\nu-1} \binom{n}{i} W_{n\nu} \right\} +$$

$$+ \sum_{\nu=1}^{\infty} \frac{\nu^2 C^2_\nu}{(n+1)^2} = S_{21} + S_{22} + S_{23}, \quad \text{say.}$$

Now

$$(2.2) \quad S_{21} < \frac{C}{n^2} \sum_{\nu=1}^{n} \nu^2 C^2_{\nu}, \text{ by virtue of Lemma 3.}$$

Again

$$(2.3) \quad S_{22} = \sum_{\nu=\left[\frac{n}{3}\right]+2}^{n} C^2_{\nu} \left\{ \frac{1}{2^n} \sum_{i=0}^{\nu-1} \binom{n}{i} W_{n\nu} \right\} < 0,$$

by virtue of Lemma 2.

Whence from (2.2) and (2.3) we get

$$\int_a^b [\sigma_n(x) - \tau_n(x)]^2 \, dx < \frac{C}{n^2} \sum_{\nu=1}^{n} \nu^2 C^2_{\nu}.$$

Substituting n_k for n and summing we get

$$\sum_{k=1}^{\infty} \int_a^b \left[\sigma_{n_k}(x) - \tau_{n_k}(x)\right]^2 dx =$$

$$= O(1) \sum_{k=1}^{\infty} \frac{1}{n^2_k} \sum_{\nu=1}^{n_k} \nu^2 C^2_{\nu}$$

$$= O(1) \sum_{k-1}^{\infty} k^2 C^2_k \sum_{n_\nu \geqq k}^{\infty} \frac{1}{n^2_\nu}$$

$$= O(1) \frac{q^2}{q^2-1} \sum_{k=1}^{\infty} C^2_k < \infty.$$

Hence from (2.1)

$$\sum_{k=1}^{\infty} \int_a^b [S_{n_k}(x) - \tau_{n_k}(x)]^2 \, dx$$

converges almost everywhere from which our lemma follows by B. LEVY'S theorem (refer ALEXITS [1], p. 11).

3. Proof of the Theorem

Let $\{\nu_n\}$ be an arbitrary strictly increasing sequence of indices. We may suppose without loss of generality of theorem that $\nu_1 \geqq 1$.

Let $2^m \leqq \nu_k < 2^{m+1}$. Assume that $\mu_k = 2^{m+1}$ $(m = 0, 1, 2, \ldots)$.

Since from the assumption $\{C_n\} \varepsilon l^2$ and the series (1.2) is (E, 1) summable to a function $f(x)$ almost everywhere in $<a, b>$, so from Lemma 4 it follows that

$\lim_{m \to \infty} S_{2^m}(x) = f(x)$ is valid almost everywhere and subsequently

$$\lim_{k \to \infty} S_{\mu_k}(x) = f(x). \tag{3.1}$$

For every n we write

$$\frac{1}{2^n} \sum_{k=1}^{n} \binom{n}{k} \left(S_{\nu_k}(x) - f(x) \right)^2 \leqq$$

$$\leqq \frac{2}{2^n} \sum_{k=1}^{n} \binom{n}{k} \left(S_{\nu_k}(x) - S_{\mu_k}(x) \right)^2 + \tag{3.2}$$

$$+ \frac{2}{2^n} \sum_{k=1}^{n} \binom{n}{k} \left(S_{\mu_k}(x) - f(x) \right)^2 =$$

$$= S_{31} + S_{32}, \text{ say.}$$

In virtue of (3.1)

$$S_{32} \longrightarrow 0.$$

Also

$$S_{31} = \frac{2}{2^n} \sum_{k=1}^{n} \binom{n}{k} \left(S_{\nu_k}(x) - S_{\mu_k}(x) \right)^2$$

$$= O(1) \frac{\binom{n}{m}}{2^n} \sum_{k=1}^{n} \left(S_{\nu_k}(x) - S_{\mu_k}(x) \right)^2$$

$$= O(1) \frac{\sqrt{n}\binom{n}{m}}{2^n} \cdot \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(S_{\nu_k}(x) - S_{\mu_k}(x) \right)^2$$

$$= O(1) \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(S_{\nu_k}(x) - S_{\mu_k}(x) \right)^2,$$

by virtue of Lemma 1.

As per (1.3), (1.4) and (1.5)

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \int_a^b \left(S_{\nu_k}(x) - S_{\mu_k}(x) \right)^2 dx =$$

$$= O(1) \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left[\left(C^{\star}_{\nu_k + 1} \right)^2 + \dots + \left(C^{\star}_{\mu_k} \right)^2 \right]$$

$$= O(1) \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \frac{\left(\mu_k - \nu_k\right)\left(\nu_k + 1\right)\left(C^{\star}_{\nu_k + 1}\right)^2}{\nu_k + 1}$$

$$= O(1) \sum_{k=1}^{\infty} \frac{\left(\nu_k + 1\right)\left(C^{\star}_{\nu_k + 1}\right)^2}{\sqrt{k}}$$

$$= O(1) \sum_{k=1}^{\infty} \frac{k\, C^{\star 2}_k}{\sqrt{k}}$$

$$= O(1) \sum_{k=1}^{\infty} C^{\star 2}_k \sqrt{k} < \infty.$$

results by a simple calculation, from which we get by an application of B. LEVY'S theorem that the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left(S_{\nu_k}(x) - S_{\mu_k}(x) \right)^2$ converges almost everywhere. Then by KRONECKER'S lemma (refer ALEXITS [1] p. 72)

$$\sum_{k=1}^{n} \left(S_{\nu_k}(x) - S_{\mu_k}(x) \right)^2 = 0 (\sqrt{n})$$

which proves that

$$S_{31} \longrightarrow 0.$$

With that theorem is completely proved.

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