## CHAPTER I

## I N T R O D U C T I O N

1.1 In the last twenty years there has been remarkable activity in the field of convergence and summability of general orthogonal series and trigonometric series. We have paid much attention to orthogonal series and their generalisations, since, in our opinion, the theory of orthogonal series is less well developed than the theory of trigonometric series, so that in the former there are many interesting and important problems waiting for a solution. Mathematicians in India, as well abroad ( USSR, Hungary, U.S.A., Poland, Japan, England etc.), have worked in this direction; but the most significant results have been obtained in the USSR and in Hungary.

The theory of orthogonal expansions originated during the discussion of the problem of vibrating string almost two hundred years ago. During his investigations on the theory of conduction of heat, Fourier was led to consider the expansion of an arbitrary function f(x) in the form of an infinite trigonometric series. During the first half of the present century, some of the leading mathematicians like, Fejér, Hardy*, Hilbert, Hobson, Lebesgue, F. Riesz, M. Riesz, Weyl, Alexits, Kaczmarz, Steinhaus, Menchoff, Zygmund, Lorentz, Hoder and Tandori were (have been) occupied with the convergence and summability problems of orthogonal expansions. We would like to discuss some of the problems connected with the convergence and summability of orthogonal series to which we are naturally led from the researches of the previous workers. We shall start with a number of definitions and concepts relevant to the body work of our thesis.

1.2 Throughout the thesis we shall make use of either the Stieltjes-Lebesgue integral or the Lebesgue integral. The notion of orthogonality will be introduced by means of Stieltjes-Lebesgue integral. Let $\mu(x)$ be a positive,

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* Hardy [26].

bounded and monotone increasing function in the closed interval $[a, b]$. We call such a function a distribution function with moments* (in short m-distribution). A function $f(x)$ is called integrable in the sense of Stieltjes-Lebesgue or $L_\mu$-integrable, if it is $\mu$-measurable and

$$(1.2.1) \qquad \int_a^b |f(x)| \, d\mu(x) < \infty .$$

If $\mu(x)$ is absolutely continuous and $\mu'(x) = \rho(x)$, then for any $L_\mu$-integrable function $f(x)$ the relation

$$(1.2.2) \qquad \int_a^b f(x) \, d\mu(x) = \int_a^b f(x) \rho(x) \, dx$$

is valid. In this case we shall say that $f(x)$ is $L_{\rho(x)}$-integrable function and we shall call $\rho(x)$ a covering function or weight function. If, in particular, $\rho(x) \equiv 1$, then we shall say - in accordance with the usual terminology - that $f(x)$ is L-integrable.

A function $f(x)$ is called $L^2_\mu$ or $L^2_{\rho(x)}$-integrable, if it is $L_\mu$- or $L_{\rho(x)}$-integrable, respectively, and if, furthermore,

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* Freud [26] .

$$\int_a^b f^2(x)\, d\mu(x) < \infty \quad \text{or} \quad \int_a^b f^2(x)\, g(x)\, dx < \infty$$

holds, respectively. We shall talk about an $L^2$-integrable function if $g(x) \equiv 1$.

__Definition:__ Let $f, g \in L^2[a,b)$. The inner product of $f$ with $g$ is

$$(f,g) = \int_a^b f(x)\overline{g(x)}dx ,$$

(where $\bar{z}$ denotes the complex conjugate of $z$) and the norm of $f$ is $\|f\| = [(f,f)]^{\frac{1}{2}}$.

__ORTHOGONALITY:__ A finite or denumerably infinite system $\{\phi_n(x)\}$ of $L^2_\mu$-integrable functions is said to be orthogonal with respect to the distribution $d\mu(x)$ in the interval $[a , b]$ of the real variable $x$ provided that

$$(1.2.3) \quad (\phi_m, \phi_n) \equiv \int_a^b \phi_m(x)\, \phi_n(x)\, d\mu(x) = 0 \quad , \quad (m \neq n),$$

and none of the functions $\phi_n(x)$ vanishes almost everywhere (a.e.) in $[a,b]$ . If in addition to the condition (1.2.3), the system $\{\phi_n(x)\}$ also satisfies the condition

$$(1.2.4) \quad (\phi_n, \phi_n) \equiv \int_a^b \phi_n^2(x)\, d\mu(x) = \|\phi_n\|_2^2 = 1$$

for every $n$, then we say that $\{\phi_n\}$ is an orthonormal system (ONS) in $[a,b]$. The quantity

$$(1.2.5) \quad \|\psi_n(x)\| = \left\{ \int_a^b \psi_n^2(x)\, d\mu(x) \right\}^{\frac{1}{2}}$$

is known as the norm of the functions $\psi_n(x)$ in $[a,b]$. Every orthogonal system $\{\psi_n(x)\}$ can be converted into an orthonormal system by means of multiplying every one of its members by a suitably chosen constant factor. For, since none of the functions $\psi_n(x)$ can vanish a.e., the functions

$$\phi_n(x) = \frac{\psi_n(x)}{\left\{ \int_a^b \psi_n^2(x)\, d\mu(x) \right\}^{\frac{1}{2}}} \equiv \frac{\psi_n(x)}{\|\psi_n(x)\|}$$

exist and it is immediately evident that they constitute an orthonormal system with respect to $d\mu(x)$. If, in particular, $\mu(x) = x$, i.e. $\mu'(x) = \delta(x) = 1$, then $\{\phi_n(x)\}$ is simply an orthonormal system in the ordinary sense.

<u>ORTHOGONALIZATION</u>: A system consisting of $n+1$ functions $f_0, f_1, f_2, \ldots, f_n$ is called linearly dependent in $[a,b]$ if there exist $n+1$ constants $c_0, c_1, c_2, \ldots, c_n$, not all zero, for which

$$\sum_{k=0}^{n} c_k f_k(x) = 0$$

for $\mu$-almost $x \in [a,b]$. If no such constants exist, then the system is said to be linearly independent.* Every orthogonal system $\{\phi_n(x)\}$ is linearly independent+. Conversely, every linearly independent system $\{f_n(x)\}$ can be converted into an orthonormal system $\{\phi_n(x)\}$, such that the $\phi_n(x)$ are linear combinations of the functions $f_0(x)$, $f_1(x)$, ..., $f_n(x)$. The procedure of constructing an orthonormal system from a linearly independent system is known as Schmidt's** general procedure of orthogonalization.

ORTHOGONAL SERIES AND ORTHOGONAL EXPANSIONS: A series

$$(1.2.6) \qquad \sum_{n=0}^{\infty} a_n \phi_n(x)$$

constructed from an orthogonal system $\{\phi_n(x)\}$ and an arbitrary set of real numbers $a_0$, $a_1$, ... is called an orthogonal series. However, if the coefficients in the series (1.2.6) are not arbitrary but representable in the form of Fourier coefficients of a Stieltjes-Lebesgue integrable function $f(x)$, i.e.,

$$(1.2.7) \qquad c_n = \frac{1}{\|\phi_n\|^2} \int_a^b f \, \phi_n \, d\mu(x) = \frac{(f, \phi_n)}{(\phi_n, \phi_n)} \quad (n = 0, 1, 2, \ldots),$$

* Sansone [30], p.2.

+ Alexits [7], p.4.

** Schmidt [32].

then we shall say that

$$\sum_{n=0}^{\infty} c_n \phi_n(x)$$

is the <u>orthogonal expansion</u> of the function f(x) and we shall express this relation by the formula

$$(1.2.8) \qquad f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x) .$$

All these definitions are stated only for functions of $L^2_\mu$, since $(f, \phi_n)$ will not usually exist for other $f$; but they may be applied more widely if the $\phi_n$ are subject to further restrictions. If, for example, $\phi_n$ is bounded for each $n$, then any $f$ of $L_\mu$ has a Fourier series defined as in (1.2.7) and (1.2.8). The trigonometrical systems $\{1, \cos nx, \sin nx\}$ satisfy this condition.

The difference between orthogonal expansion and orthogonal series is characterized by the following minimum property given by Gram*:

Let $f(x) \in L^2_\mu[a,b]$ and $\{\psi_n(x)\}$ be an arbitrary orthonormal system. Among all the expressions of the form

$$(1.2.9) \qquad S_n(x) = \sum_{k=0}^{n} a_k \psi_k(x)$$

* Gram [28] (see also Szegö [106] p.29).

the integral

$$(1.2.10) \qquad I(S_n) = \int_a^b \left[f(x) - S_n(x)\right]^2 d\mu(x)$$

attains its minimum value when

$$S_n(x) = \sum_{k=0}^{n} c_k \phi_k(x) = s_n(x),$$

that is, when $S_n(x)$ is the nth partial sum of the expansion of $f(x)$.

An immediate consequence of Gram's theorem is the Bessel's inequality*:

$$(1.2.11) \qquad \sum_{n=0}^{\infty} c_n^2 \le \int_a^b f^2(x)\, d\mu(x).$$

Bessel's inequality implies that the expansion coefficients $c_n$ of an $L_\mu^2$-integrable function converge to zero as $n \to \infty$. This fact is important in the discussion of convergence of orthogonal series.

The fundamental theorem in the theory of orthogonal series is due to Riesz** and Fischer***, which may be stated

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* Tricomi [115], p.12.

** Riesz, F. [86].

*** Fischer [28].

as:

Let $\{\phi_n(x)\}$ denote an arbitrary orthonormal system and $\{c_n\}$ a sequence of real numbers. A necessary and sufficient condition that $\{c_n\}$ be the sequence of the expansion coefficients of an $L^2_\mu$-integrable function $f(x)$, is

$$\sum_{n=0}^{\infty} c_n^2 < \infty . \tag{1.2.12}$$

The partial sums

$$S_n(x) = \sum_{k=0}^{n} c_k \phi_k(x)$$

of the expansion of $f(x)$ then converge in the mean to the generating function $f(x)$.

For bounded systems $\left( |\phi_k(x)| \leq M < \infty , \; k = 0,1,2, \ldots \right)$ this theorem has been generalized by F. Riesz:
If $1 < p \leq 2$, $p^{-1} + q^{-1} = 1$ and condition

$$\sum_{k=0}^{\infty} |c_k|^p < \infty \tag{1.2.13}$$

is fulfilled, then the orthogonal series (1.2.6) is the orthogonal expansion of some function $f(x) \in L^q[a,b]$ $(q \geq 2)$. In case $p = 2$, condition (1.2.13) is equivalent* to the existence of such increasing sequence of positive numbers

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* Tomić [35].

$\{\nu_k\} \uparrow \infty$, that

$$(1.2.14) \qquad \sum_{k=0}^{\infty} \left( \frac{1}{\nu_k} - \frac{1}{\nu_{k+1}} \right) \int_a^b \left| \sum_{m=0}^{k} c_m \phi_m(x) \nu_m \right|^p dx < \infty .$$

Generalising the above theorem of F. Riesz, Tenia* has proved the following :

If $p \geq 1$ and condition (1.2.14) is fulfilled, then (1.2.6) is the orthogonal expansion of some function $f(x) \in L^p[a, b]$.

1.3 We would like to digress a little at this point from our main theme so as to define the various summability methods which will be used in the body-work of the thesis. All the methods of summability which we are going to consider belong to the class of sequence to sequence transformations.

CESÀRO SUMMABILITY :

Let $\alpha$ be any real number other than a negative integers, and let $A_n^\alpha = \binom{n+\alpha}{n}$, $(n > 0)$ $\left[ A_0^\alpha = 1 \right]$ denote the $n^{th}$ coefficient of the binomial series

$$\sum_{n=0}^{\infty} A_n^\alpha x^n = \frac{1}{(1-x)^{1+\alpha}} .$$

Let

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* Tenia [26] .

(1.3.1) $$\sum_{n=0}^{\infty} u_n$$

be a given infinite series with partial sums $S_n$. We write

$$\sigma_n^0 = S_n^0 = u_0 + u_1 + \dots + u_n , \quad S_n^\alpha = \sum_{v=0}^{n} A_{n-v}^{\alpha-1} S_v = \sum_{v=0}^{n} A_{n-v}^{\alpha} u_v ,$$

$$\tau_n^0 = T_n^0 = n u_n , \quad T_n^\alpha = \sum_{v=1}^{n} A_{n-v}^{\alpha-1} T_v^0 , \quad \tau_n^\alpha = \frac{T_n^\alpha}{A_n^\alpha} .$$

Then

(1.3.2) $$\sigma_n^\alpha = \frac{S_n^\alpha}{A_n^\alpha}$$

is termed as the $n^{th}$ Cesàro mean of order $\alpha$ of the sequence $\{S_n\}$ or simply $(C,\alpha)$-mean of $\sum u_n$. If $\sigma_n^\alpha \to S$ as $n \to \infty$, then the series $\sum u_n$ is said to be summable $(C,\alpha)$ to the sum $S$*.

If the sequence $\{\sigma_n^\alpha\}$ is of bounded variation, that is to say

(1.3.3) $$\sum |\Delta \sigma_n^\alpha| = \sum |\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty ,$$

then $\sum u_n$ is said to be absolutely summable $(C,\alpha)$ or simply summable $|C,\alpha|$**.

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* Cesàro [16] , Chapman [17] , Knopp [41,42] .

** Fekete [22] , Kogbetliantz [46] .

We have also

$$(1.3.4) \qquad \tau_n^{\alpha} = n\left(\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha}\right)^{\dagger}$$

and

$$(1.3.5) \qquad \tau_n^{\alpha} = -\alpha\left(\sigma_n^{\alpha} - \sigma_n^{\alpha-1}\right)^{\S}, \quad \alpha > 0 .$$

The infinite series $\sum u_n$ is summable $|C,\alpha|_k$ *, where $k \geq 1$, $\alpha > -1$, if the series

$$\sum n^{k-1} \left|\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha}\right|^k$$

is convergent.

Summability $|C,\alpha|_1$ is, of course, identical with summability $|C,\alpha|$.

In view of the identity (1.3.4) we may restate the definition of summability $|C,\alpha|_k$ in terms of the series

$$(1.3.6) \qquad \sum_1^{\infty} n^{-1} \left|\tau_n^{\alpha}\right|^k$$

the series $\sum u_n$ being summable $|C,\alpha|_k$ if the series (1.3.6) is convergent.

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† Kogbetliantz [45].

§ This is equivalent to an identity of Hardy [30].

* Flett [24].

The series $\sum u_n$ with partial sums $s_n$ is said to be strongly $(C,\alpha)$-summable with index $k$ to the sum $s$ if

$$(1.3.7) \qquad \frac{1}{A_n^{\alpha}} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_\nu - s|^{k} \longrightarrow 0 \quad \text{as } n \to \infty .$$

For $\alpha = 1$, this gives the definition of strong summability $(H,k)$*.

RIESZ SUMMABILITY:

Let $\{\lambda_n\}$ be a positive, strictly increasing numerical sequence with $\lambda_0 = 0$ and $\lambda_n \to \infty$ as $n \to \infty$. The relation

$$(1.3.8) \qquad \sigma_n^{\alpha}(\lambda, x) = \sum_{\nu=0}^{n} \left(1 - \frac{\lambda_\nu}{\lambda_{n+1}}\right)^{\alpha} u_\nu$$

defines the $n^{th}$ $(R, \lambda, \alpha)$-mean** $(\alpha > 0)$ of the series $\sum u_n$. In particular, for $\alpha = 1$,

$$\sigma_n^{1}(\lambda, x) = \sigma_n(\lambda, x) = \sum_{\nu=0}^{n} \left(1 - \frac{\lambda_\nu}{\lambda_{n+1}}\right) u_\nu$$

$$= \frac{1}{\lambda_{n+1}} \sum_{\nu=0}^{n} (\lambda_{\nu+1} - \lambda_\nu) s_\nu$$

defines the $n^{th}$ $(R, \lambda, 1)$-mean of the series $\sum u_n$.

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* Zygmund [126] II p.180, Bosy [10] p.2, Singh [93] p.3, Winn [121].

** Das [19], Peyerimhoff [76] p.24, Lorentz [51].

If

$$\lim_{n\to\infty} \sigma_n(\lambda, x) = S,$$

then the series $\sum u_n$ is said to be $(R, \lambda, 1)$-summable to the sum $S$.

Obviously, the Riesz-method of summation is a generalization of $(C,1)$-method, which is obtained by putting $\lambda_n = n$. In case $\lambda_n = \log(n+1)$, the Riesz-summability is known as Riesz logarithmic summability.

EULER SUMMABILITY:

Let $\sum u_n$ be an infinite series with the sequence of partial sums $\{s_n\}$. A sequence to sequence transformation given by the equation

$$(1.3.0) \quad \tau_n^{(q)} = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k , \quad n = 0, 1, 2, \ldots$$

defines the sequence $\{\tau_n^{(q)}\}$ of $(E,q)$ means $(q > 0)$ known as Euler-Knopp means*.

In place of $\tau_n^{(1)}$ we simply write $\tau_n$.

If

$$\lim_{n\to\infty} \tau_n^{(q)} = S,$$

* Hardy [31] p.180, Peyerimhoff [76] p.20.

then we say that the series $\sum u_n$ is (E,q)-summable to the sum S. If the sequence $\{\tau_n^{(q)}\}$ is of bounded variation, that is, if

$$\sum_{n=1}^{\infty} \left| \tau_n^{(q)} - \tau_{n-1}^{(q)} \right| < \infty,$$

then the series $\sum u_n$ is said to be absolutely (E,q)-summable or $|E,q|$-summable.

OR

The series $\sum u_n$ is said to be summable $|E,\alpha|$ $(0 < \alpha < 1)$ if

$$(1.3.10) \qquad t_n = \sum_{v=0}^{n} \binom{n}{v} \alpha^v (1-\alpha)^{n-v} S_v \;{}^{*},$$

where

$$S_v = u_0 + u_1 + \dots + u_v$$

and

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty.$$

It can be observed that in (1.3.10) if we let $\alpha = \dfrac{1}{1+q}$ ( $0 < \alpha < 1$ is equivalent to $q > 0$) we obtain (1.3.9).**

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* Knopp [42], Agnew [2].

** Powell and Shah [30] p.67.

The original Euler transformation [when $\alpha = \frac{1}{2}$ in (1.3.10)] was given by Euler in Institutiones calculi differentialis, page 281 in 1755. Much of the work on the Euler transform of order $\alpha (0 < \alpha < 1)$ was done by Knopp.*

The series $\sum u_n$ is said to be very strongly summable (E,1) to the sum $S$ if for every monotone increasing index sequence $\{v_n\}$ the relation

$$\sum_{k=0}^{n} \binom{n}{k} \left(S_{v_k} - S\right)^2 = o(2^n) \quad \text{as } n \to \infty$$

holds.

In particular, if $v_k = k$ $(k = 0, 1, 2, \ldots)$, we shall say that the series $\sum u_n$ is strongly summable (E,1) to the sum $S$.

CESÀRO SUMMABILITY OF DOUBLE SERIES:

Let

$$(1.3.11) \qquad \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} u_{kl}$$

be a given double infinite series and let us denote its mnth partial sum by $S_{mn}$, that is,

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* Knopp [43,44]. For examples on Euler transformation refer to Bromwich [15] p.92-6, 196-8.

$$S_{mn} = \sum_{k=0}^{m} \sum_{l=0}^{n} u_{kl} .$$

The series (1.3.11) is said to be (C,1,1)-summable to a sum S if

$$\lim_{m,n\to\infty} \sigma_{mn} = S ,$$

where

$$\sigma_{mn} = \sum_{k=1}^{m} \sum_{l=1}^{n} \left(1-\frac{k-1}{m}\right)\left(1-\frac{l-1}{n}\right) u_{kl} .$$

The series (1.3.11) is said to be absolutely summable (C,1,1) or summable |C,1,1|* if

$$\sum_{m=2}^{\infty} \left| \sigma_{m-1,n} - \sigma_{mn} \right| < \infty$$

for a fixed n ,

$$\sum_{n=2}^{\infty} \left| \sigma_{m,n-1} - \sigma_{mn} \right| < \infty$$

for a fixed n ,

and

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \left| \sigma_{mn} - \sigma_{m-1,n} - \sigma_{m,n-1} + \sigma_{m-1,n-1} \right| < \infty .$$

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* Chaman [98] , Sinha [114].

1.4 In this section we refer to some of the important and general results of real analysis which we shall have occasion to use many a time during the course of the proofs of our theorems.

Levi's theorem*: If $\{f_n(x)\}$ is a monotone increasing sequence of L-integrable functions and

$$\left| \int_a^b f_n(x)\,dx \right| \le C \quad , \quad (n = 0, 1, 2, \ldots),$$

then the limiting function

$$f(x) = \lim_{n\to\infty} f_n(x)$$

is also L-integrable and

$$(1.4.1) \qquad \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx .$$

If, in particular, $u_0(x)$, $u_1(x)$, ... are L-integrable functions such that

$$(1.4.2) \qquad \sum_{n=0}^{\infty} \int_a^b |u_n(x)|\,dx < \infty,$$

then the series $\sum_{n=0}^{\infty} u_n(x)$ is (absolutely) convergent a.e. in [a,b].

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* Alexits [7] p.11.

<u>An extension of Kronecker's theorem</u>*: If $\{\lambda_n\}$ is positive, monotone increasing and tending to infinity, then the condition

$$(1.4.3) \qquad \sum_{n=1}^{\infty} \frac{u_n}{\lambda_n} < \infty$$

implies

$$(1.4.4) \qquad \sum_{k=1}^{n} u_k = o(\lambda_n) .$$

## 1.5 <u>Convergence of orthogonal series</u>

Coming back now to the main theme of the thesis we would first of all consider the convergence of orthogonal series

$$(1.5.1) \qquad \sum_{n=1}^{\infty} a_n \phi_n(x)$$

where $\{a_n\}$ is an arbitrary sequence of real numbers. It is easily seen that the condition

$$(1.5.2) \qquad \sum_{n=1}^{\infty} |a_n| < \infty$$

implies the absolute convergence** of the series (1.5.1)

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* Alexits [7] p.72.

** Alexits [7] p.81.

a.e. in the interval of orthogonality. On the otherhand, it has been shown through the example of a series of Rademacher functions* that the condition

$$\sum_{n=1}^{\infty} a_n^2 < \infty \tag{1.5.3}$$

is essential for the convergence of the series (1.5.1) a.e. in the interval of orthogonality. It therefore follows that the sufficient condition for a.e. convergence will lie somewhere between the condition (1.5.2) and (1.5.3). Jerosch and Weyl+ were the first to formulate the condition

$$a_n = O\left(n^{-\frac{3}{4}-\epsilon}\right), \quad \epsilon > 0, \tag{1.5.4}$$

which would be sufficient for the convergence of the series (1.5.1) a.e. in the interval of orthogonality. Weyl** himself improved this result and showed that the condition

$$\sum_{n=1}^{\infty} a_n^2 \sqrt{n} < \infty \tag{1.5.5}$$

is sufficient for the convergence of (1.5.1). Hobson++ modified the condition (1.5.5) to the form

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* Zygmund [126] I p.212. ** Weyl [120]

\+ Jerosch and Weyl [37]. ++ Hobson [36]

$$(1.5.6) \qquad \sum_{n=1}^{\infty} a_n^2 \, n^{\epsilon} < \infty \, , \qquad \epsilon > 0 ,$$

and later on Plancherel* succeeded in replacing the condition by

$$(1.5.7) \qquad \sum_{n=1}^{\infty} a_n^2 \, (\log n)^3 < \infty \, .$$

The chain of ideas in this direction continued and Rademacher** (1922) and Menchoff+ (1923) independently obtained the sufficient condition in a form which has an 'air of finality'. They have shown that the series (1.5.1) is convergent a.e. in the interval of orthogonality if the coefficients $a_n$ satisfy the condition

$$(1.5.8) \qquad \sum_{n=1}^{\infty} a_n^2 \, (\log n)^2 < \infty \, .$$

Several generalisations of this theorem were given later, out of which we mention the earliest due to Kantorovitch++. He showed that

$$\int_a^b \max_n \left\{ \sum_{k=2}^{n} \frac{c_k \phi_k(x)}{\log k} \right\}^2 dx = O(1) \sum_{k=0}^{\infty} c_k^2 \, ,$$

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* Plancherel [72]     + Menchoff [62]

** Rademacher [85]     ++ Kantorovitch [40]

Other generalizations are due to Walfisz*, Salem**, Tandori*** and Gaposhkin+.

The theorem of Rademacher and Menshoff is the best of its kind is obvious from another fundamental theorem of Menshoff++. He showed that if $\{w(n)\}$ is an arbitrary positive monotone increasing sequence of numbers with $w(n) = o(\log n)$, then there exists an ONS $\{\psi_n(x)\}$ such that the series

$$\sum_{n=0}^{\infty} c_n \psi_n(x)$$

diverges everywhere even though the coefficients satisfy the condition

$$\sum_{n=1}^{\infty} c_n^2 \, \omega^2(n) < \infty .$$

Another result that need to be mentioned in this chain is that of Tandori+++ who showed that if the coefficients sequence $\{c_n\}$ be monotone decreasing then the condition (1.5.3) is also necessary for the convergence of (1.5.1).

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|---|---|---|---|---|---|
| * | Walfisz | [119] | + | Gaposhkin | [37] |
| ** | Salem | [93] | ++ | Menshoff | [62] |
| *** | Tandori | [110] | +++ | Tandori | [111] |

I. I. Privalov investigated the absolute convergence of orthogonal series. He proved that the absolute convergence a.e. of the orthogonal series (1.5.1) with respect to uniformly bounded functions implies the convergence of the series $\sum|a_n|$. But if we omit the condition of uniform boundedness of the functions of the system, it is impossible even to guarantee that the coefficients $a_n$ tend to zero. Absolute convergence of orthogonal series has also been discussed by Stechkin*, Alexits** and Bochkarev***.

For many special orthogonal series the condition for convergence has still better form. Kolmogoroff-Seliverstoff+ and Plessner++ have shown that for the trigonometric system the sequence $\{\log n\}$ is a sequence of Weyl factors for the convergence a.e. on $[0, 2\pi]$, i.e. that from

(1.5.9) $$\sum_{n=1}^{\infty} (a_n^2 + b_n^2) \log n < \infty$$

there follows the convergence a.e. on $[0,2\pi]$ of the

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* Stechkin [100,101].

** Alexits [5].

*** Bochkarev [13]

\+ Kolmogoroff-Seliverstoff [47]

++ Plessner [78].

series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right). \tag{1.5.10}$$

Further, it was proved by Plessner that the condition (1.5.9) is equivalent to

$$\int_0^{2\pi}\int_0^{2\pi} \frac{|f(x+t) - f(x-t)|^2}{t} \, dt \, dx < \infty$$

where $f(x)$ is the function whose Fourier series is (1.5.10). Hence, if the integral modulus of continuity* of the function $f \in L^2(0,2\pi)$ is such that**

$$\omega_2(\delta, f) = O\left\{\frac{1}{\left(\log\frac{1}{\delta}\right)^{\frac{1}{2}} \left(\log\log\frac{1}{\delta}\right)^{\frac{1}{2}+\epsilon}}\right\} \quad (\delta \to +0)$$

for some $\epsilon > 0$, then the series (1.5.10) converges a.e. on $[0, 2\pi]$.

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* We define

$$\omega_p(\delta, f) = \sup_{|h| \le \delta} \left\{\int_0^{2\pi} |f(x+h) - f(x)|^p \, dx\right\}^{\frac{1}{p}} \quad \text{for every } p \in [1, \infty]$$

Here if $p = \infty$, then $\omega_\infty(\delta, f) = \omega(\delta, f)$ and $L^\infty(0, 2\pi) = C(0, 2\pi)$.

** Ul'yanov [117].

The above mentioned theorem of Kolmogoroff-Seliverstoff and Plessner has been generalized by many authors (Alexits, Bari, Marcinkiewicz, Stechkin, Ul'yanov and others), but no considerable advance has been made. As regards the theory of trigonometric series there are the monographs of Zygmund* and Bari**. The excellent little book of Hardy and Rogosinski*** establishes many basic results in the theory of Fourier series+.

1.6 <u>Summability of orthogonal series:</u>

Besides the questions of the convergence of orthogonal series, D. E. Menchoff investigated questions of summability of these series by Cesàro processes and general Toeplitz processes, and also with the study of the effect of the rearrangement of orthogonal functions on the convergence and summability of the series of these functions.

Menchoff and Kaczmarz++ have shown that the sequence $w(n) = (\log\log n)^2$ is a sequence of Weyl factors for (C,1)-summability a.e. for orthogonal series

(1.6.1) $$\sum_{n=0}^{\infty} a_n \phi_n(x) .$$

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* Zygmund [126].

** Bari [9, 10].

*** Hardy and Rogosinski [34].

\+ See also Edwards [21].

\+\+ Kaczmarz-Steinhaus [39] p.190-191.

Moreover, it is an exact sequence of Weyl factors in the whole class of orthogonal systems.

Tandori* proved that if $\sqrt{n}\, \varepsilon_n \downarrow 0$, then for all orthogonal series (1.6.1) to be (C,1)-summable a.e., it is necessary and sufficient that

$$\sum_{n=3}^{\infty} a_n^2 (\log \log n)^2 < \infty .$$

New criteria for (C,1)-summability of orthogonal series have been found by Alexits**. He proved that if $q_n \downarrow 0$ and

$$\sum_{n=1}^{\infty} \frac{q_n}{\sqrt{n}} < \infty,$$

then the condition $|a_n| \leq q_n$ $(n = 1, 2, \ldots)$ implies that for any ONS $\{\phi_n(x)\}$ on $[0,1]$, the series (1.6.1) is (C,1)-summable a.e. on $[0,1]$.

It may be pointed out that neither of the theorems of Menchoff-Kaczmarz and Alexits contains the other. They are both valid for the Cesàro method $(C,\alpha)$ with $\alpha > 0$***.

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* Tandori [109].

** Alexits [4].

*** Alexits [7] p.125,133.

Analogue of the theorem of Menchoff-Kaczmarz for Riesz summability has been given by Zygmund*.

Meder** has studied the Euler and Nörlund*** summability of the series (1.6.1). He has established the following theorem connecting the Cesàro and Euler means of the Fourier orthogonal expansion

$$f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x) \tag{1.6.2}$$

Theorem A: If $\sigma_n(x)$ and $\tau_n(x)$ denote the Cesàro means and Euler means respectively, each of order unity of the series (1.6.2), then the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \left[ \sigma_n(x) - \tau_n(x) \right]^2 \tag{1.6.3}$$

converges a.e. in the interval of orthogonality.

In chapter II of the thesis we have established certain interconnections of the Riesz means of the series

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* Zygmund [124, 125].

** Meder [52, 53, 56].

*** For definitions refer to Nörlund [68], Vermes [118].

(1.6.2) with its Euler and Logarithmic means. Our theorems are as follows:

(i) *If $\{\lambda_n\}$ is a positive, strictly increasing numerical sequence with $\lambda_0 = 0$ such that*

$$\sum_{n=k}^{\infty} \frac{1}{n\lambda_{n+1}^{2}} = O\left(\frac{1}{\lambda_k^{2}}\right), \tag{1.6.4}$$

*then the series*

$$\sum_{n=1}^{\infty} \frac{1}{n}\left[\sigma_n(\lambda, x) - \tau_n(x)\right]^2$$

*converges a.e. in* $[a,b]$, $\sigma_n(\lambda, x)$ *denoting the Riesz mean of order unity of the series* (1.6.2).

(ii) *Let* $\{\lambda_n\}$ *be as defined in* (i) *satisfying* (1.6.4). *Then the series*

$$\sum_{n=2}^{\infty} \frac{1}{n\log n}\left[\sigma_n(\lambda, x) - L_n(x)\right]^2$$

converges a.e. in $[a,b]$, $l_n(x)$ denoting the logarithmic mean of order unity of the series (1.6.2).

1.7 Sunouchi* has discussed the convergence of

$$\sum_{n=1}^{\infty} \frac{|s_n(x) - \sigma_n(x)|^k}{n}, \quad k > 1 \tag{1.7.1}$$

under the restriction of boundedness of the functions $\phi_n(x)$.

In chapter III we discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{|s_n(x) - \tau_n(x)|^k}{n}, \quad k \geq 2, \tag{1.7.2}$$

and

$$\sum_{n=1}^{\infty} \frac{|s_n(x) - \sigma_n(\lambda, x)|^k}{n}, \quad k \geq 2. \tag{1.7.3}$$

The convergence of series (1.7.2) and (1.7.3) when $k = 2$ has been investigated by Meder** and Patel*** respectively.

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* Sunouchi [102].

** Meder [60].

*** Patel [73].

## 1.8 Euler summability of orthogonal series:

**Part I:**

The Euler summability of general orthogonal series

$$\sum_{n=0}^{\infty} c_n \phi_n(x) \tag{1.8.1}$$

with the coefficients $c_n$ satisfying the condition

$$\sum_{n=0}^{\infty} c_n^2 < \infty \tag{1.8.2}$$

has been discussed by Meder*. One of his interesting results is as follows:

The series

$$\sum_{n=1}^{\infty} \int_a^b n \left[ \tau_n(x) - \tau_{n-1}(x) \right]^2 dx \tag{1.8.3}$$

is convergent if

$$\sum_{n=1}^{\infty} c_n^2 \sqrt{n} < \infty . \tag{1.8.4}$$

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* Meder [53].

One of the results of chapter IV of our thesis includes the following generalization of above results of Meder :

<u>The series</u>

$$(1.8.5) \qquad \sum_{n=1}^{\infty} \int_a^b n \left[ \tau_n^{(q)}(x) - \tau_{n-1}^{(q)}(x) \right]^2 dx$$

<u>$(q > o)$ is convergent if</u>

$$(1.8.6) \qquad \sum_{n=1}^{\infty} c_n^2 \sqrt{n} < \infty .$$

<u>Part II:</u>

Absolute summability of Fourier trigonometric series by Cesàro, Nörlund and Riesz means has been engaging the attention of a large number of workers in this line. A systematic account of the available literature on absolute summability of a Fourier trigonometric series has been given by Prasad* in his presidential addresses delivered to the Indian Scientific Societies.

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* Prasad [82, 83].

In case of Fourier orthogonal expansion the earliest result on $|C,\alpha|$-summability are due to Tsuchikura[*] and Tandori[***]. Tandori[***] obtained the following criteria for the absolute (C,1)-summability of (1.3.1). He showed that the condition

$$(1.3.7) \qquad \sum_{m=0}^{\infty} \left( \sum_{k=2^m+1}^{2^{m+1}} c_k^2 \right)^{\frac{1}{2}} < \infty$$

is necessary and sufficient for $|C,1|$-summability of (1.3.1).

The necessity was later demonstrated in a very easy way by Billard[+]. Leindler[++], Grepacevskaja[+++] and Patel[**] extended Tandori's theorem to $|C,\alpha|$-summability. Considering generalised absolute Cesàro summability Szalay[†] has generalised further these theorems.

Absolute Riesz summability of orthogonal series was

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* Tsuchikura [116].

*** Tandori [112].

\+ Billard [12].

++ Leindler [49].

+++ Grepacevskaja [30].

** Patel [72].

† Szalay [106].

discussed by Alexits-Králik[*], Moricz[**] and Srivastava[***], whereas absolute Nörlund summability by Meder[+]. Absolute Euler summability of orthogonal series has been studied by Patel[++] and Bhatnagar[+++]. Bhatnagar[+++] has proved the following criteria for $|E,1|$-summability of (1.8.1):

If

(1.8.8) $$\sum_{n=1}^{\infty} c_n^2 n^p < \infty \quad , \quad 2 > p > \frac{1}{2} ,$$

then the series (1.8.1) is $|E,1|$-summable a.e. in $(a,b)$.

In chapter IV we generalise Bhatnagar's result to $|E,q|$ $(q > 0)$ summability. Our result may be stated as follows:

<u>If</u>

(1.8.8) $$\sum_{n=1}^{\infty} c_n^2 n^p < \infty \quad , \quad 2 > p > \frac{1}{2} ,$$

<u>then the series</u> (1.8.1) <u>is</u> $|E,q|$-<u>summable</u> $(q > 0)$ <u>a.e. in</u> $(a,b)$.

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* Alexits-Králik [8].

** Moricz [65].

*** Srivastava [99].

\+ Meder [57,58,60,61].

\++ Patel [71].

\+++ Bhatnagar [11].

Part III:

In this section we discuss the strong and very strong Euler summability of the orthogonal series (1.0.1).

The strong Cesàro summability of orthogonal series, as well as that of Fourier series, has been investigated by several authors such as : A. Zygmund, J. Meemure, S. Borgen, H. Selowasser, G. Alexits, K. Tandori*, G. Sunouchi, B. N. Prasad, U. N. Singh and T. D. Singh. Zygmund** has proved the following theorem:

Theorem A: If the series (1.0.1) satisfying (1.0.2) is summable (C,1) a.e. to a function f(x), then it is strongly summable (C,1) to this function f(x).

In chapter V we generalise this theorem of Zygmund transforming to the more general Euler-method of summation***. We prove :

If the orthogonal series (1.0.1) with coefficients satisfying the condition

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* Tandori [107] .

**Zygmund [126] .

*** Euler-method is more general than Cesaro method, Meder [69, Theorem 2] .

$$(1.3.9) \qquad \sum_{n=1}^{\infty} c_n^2 \sqrt{n} < \infty$$

is (C,1)-summable a.e. to S(x), then

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left(S_k(x) - S(x)\right)^2 = 0 ,$$

that is, the series (1.3.1) is strongly (C,1)-summable a.e. to S(x) with index 2.

In the same chapter we have extended our theorem to any index $p > 0$.

Very strong Cesàro summability of orthogonal series has been studied in great details by Alexits*, Sunouchi** and Tandori***. Tandori*** has proved the following theorems:

Theorem E: Let $\{c_v^*\} \in \ell^2$ be a positive number sequence with

$$\sqrt{v}\, c_v^* \geq \sqrt{v+1}\, c_{v+1}^* \qquad (v = 1, 2, 3, \ldots)$$

and $\{c_v\}$ be any sequence of real numbers with

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* Alexits [3,7].

** Sunouchi [104].

*** Tandori [109,110].

$$c_v = O\,(c_v^*).$$

If the orthogonal series (1.8.1) built with these coefficients $c_v$, is (C,1)-summable to the function $f(x)$ a.e. in $(a,b)$, then it is very strongly summable (C,1) a.e. to this function.

<u>Theorem C:</u> If

(1.8.10) $$\sum_{n=3}^{\infty} c_n^2 (\log\log n)^2 < \infty,$$

then the series (1.8.1) is very strongly summable (C,1) a.e. to a function $f(x) \in L^2(a,b)$.

In chapter VI we generalize the above theorems of Tandori transforming them to the more general Euler-method of summation. We are stating below two of the results proved by us:

(1) <u>Let $\{c_v^*\}$ be a numerical sequence of positive numbers such that</u>

(1.8.11) $$\sqrt{v}\, c_v^* \geq \sqrt{v+1}\, c_{v+1}^* \qquad (v=1,2,3,\ldots)$$

and

(1.8.12) $$\sum_{v=1}^{\infty} c_v^{*2} \sqrt{v} < \infty .$$

Further, let $\{c_v\}$ be an arbitrary sequence of real numbers satisfying the relation

(1.8.13) $$c_v = O\left(c_v^*\right) .$$

Suppose that the orthogonal series (1.8.1) under these assumptions is (C,1)-summable to a function $f(x)$ a.e. in $[a,b]$, then it is very strongly summable (C,1) to this function a.e. in $[a,b]$.

(ii) Under the condition

(1.8.14) $$\sum_{n=1}^{\infty} c_n^2 \sqrt{n} < \infty$$

the series (1.8.1) is very strongly summable (C,1) to a function $f(x)$ a.e.

## 1.9 Double orthogonal series:

Let $\{\phi_{ij}(x,y)\}$ $(i,j = 0, 1, 2, \ldots)$ be a double sequence of orthonormal functions in the rectangle $R\,[a \leq x \leq b,\ c \leq y \leq d]$ so that

$$(1.9.1) \qquad \iint_R \phi_{ij}(x,y)\,\phi_{kl}(x,y)\,dx\,dy = \begin{cases} 1 & \text{when } i = k,\ j = l \\ 0 & \text{otherwise}^{*} \end{cases}$$

The series

$$(1.9.2) \qquad \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}\,\phi_{ij}(x,y),$$

where $\{a_{ij}\}$ is an arbitrary double sequence of real numbers, is known as double orthogonal series. However, if $a_{ij}$ could be expressed in the form

$$(1.9.3) \qquad a_{ij} = \iint_R f(x,y)\,\phi_{ij}(x,y)\,dx\,dy$$

where $f(x,y) \in L^2(R)$, then (1.9.2) is known as the expansion** of $f(x,y)$ and is written as

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* Agnew [1].

** It may also be called the double Fourier series of the function $f(x,y)$.

$$(1.9.4) \qquad f(x,y) \sim \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}\, \phi_{ij}(x,y) .$$

Convergence and summability of a double Fourier series has been investigated by several authors such as Sunouchi[*], Timan[**], Scherey[***] and Sharma[+].

Agnew[++] was the first to investigate the convergence of double orthogonal series. He has generalised Rademacher-Menchoff convergence theorem of single orthogonal series to the case of double orthogonal series. He proved that if

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left[\log(i+1)\right]^2 \left[\log(j+1)\right]^2 a_{ij}^2 < \infty,$$

then the series (1.9.2) converges a.e. in R.

Subsequent papers in the line are due to Mitchell[+++], Pandzhikidze[≠], Gurasvili[#], Patel[†] and Sapre[‡].

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* Sunouchi [100].

** Timan [114].

*** Scherey [97].

\+ Sharma [95].

++ Agnew [1].

+++ Mitchell [53].

≠ Pandzhikidze [69].

\# Gurasvili [34].

† Patel [75].

‡ Sapre [91].

Absolute convergence of single orthogonal series has been studied by Bochkarev* and Patel** and those of double orthogonal series by Pandshokidse*** . Patel** has proved the following theorem :

Theorem A: If

$$\left(\sum_{k=n}^{\infty} c_k^2\right)^{\frac{1}{2}} = O\left(\frac{1}{n^\alpha}\right) \quad , \quad \alpha > \frac{1}{2} ,$$

then the series

$$(1.9.5) \qquad \sum_{n=0}^{\infty} c_n \phi_n(x)$$

will be absolutely convergent a.e. in (a,b).

For |C,1|-summability of (1.9.5) Tandori+ has proved the following theorem :

Theorem B: Under the condition

$$\sum_{m=0}^{\infty}\left(\sum_{k=2^m+1}^{2^{m+1}} c_k^2\right)^{\frac{1}{2}} < \infty$$

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* Bochkarev [19].

** Patel [74].

*** Pandshokidze [70].

\+ Alexits [7] p.194.

the orthogonal series (1.9.6) is absolutely (C,1)-summable a.e.

In the last chapter we generalize the above theorems of Patel and Tandori to double orthogonal series.