CHAPTER III

ON EULER AND RIESZ MEANS OF ORTHOGONAL SERIES

3.1 Let $\{\phi_n(x)\}$ be a system of normalized orthogonal functions in (a,b). Consider the orthogonal series

$$\sum_{n=0}^{\infty} c_n \phi_n(x) \tag{3.1.1}$$

such that

$$\sum_{n=0}^{\infty} c_n^2 < \infty . \tag{3.1.2}$$

By the Riesz-Fischer theorem, the orthogonal series (3.1.1) converges in the mean to a function $f(x) \in L^2(a,b)$.

We denote as usual the n^{th} partial sums, $(C,1)$-means, $(H,1)$-means and $(R, \lambda_n, 1)$-means of the orthogonal series (3.1.1) by $S_n(x)$, $\sigma_n(x)$, $\tau_n(x)$ and $\sigma_n(\lambda, x)$ respectively.

Sunouchi* has discussed the convergence of

$$\sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n(x)|^k}{n}, \quad k>1, \tag{3.1.3}$$

under the restriction of boundedness of the functions $\phi_n(x)$.

In this chapter we discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{|S_n(x) - \tau_n(x)|^k}{n}, \quad k \geq 2, \tag{3.1.4}$$

and

$$\sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n(\lambda, x)|^k}{n}, \quad k \geq 2. \tag{3.1.5}$$

The convergence of series (3.1.4) and (3.1.5)

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* Sunouchi [102].

when k = 2 has been investigated by Medor* and Patel** respectively.

__THEOREM 1:__ __If__

$$\left| \phi_n(x) \right| \leq K \quad (n = 0, 1, 2, \ldots),$$

__then__

$$\int_a^b \sum_{n=1}^{\infty} \frac{\left| s_n(x) - \tau_n(x) \right|^q}{n}\, dx \leq A \sum_{n=1}^{\infty} |c_n|^q\, n^{q-2}$$

__where__

$$q \geq 2.$$

__THEOREM 2:__ __If__

$$\left| \phi_n(x) \right| \leq K \quad (n = 0, 1, 2, \ldots),$$

__then__

$$\int_a^b \sum_{n=1}^{\infty} \frac{\left| s_n(x) - \sigma_n(\lambda, x) \right|^q}{n}\, dx \leq C \sum_{n=1}^{\infty} |c_n|^q\, n^{q-\frac{5}{2}}$$

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* Medor [58].

** Patel [73].

where $q \geq 2$ provided that $\{\lambda_n\}$ satisfies the condition

$$1 < r \leq \frac{\lambda_{n+1}}{\lambda_n} .$$

In all the above theorems A, B, C are positive constants.

3.2 For the proof of the theorems we need some preliminary lemmas.

Lemma 1

$$W_{nk} \equiv \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} - \frac{2k}{n+1} < 0$$

$$\text{for } \left[\frac{n}{3}\right]^{*} + 2 \leq k \leq n .$$

Lemma 2

$$\left\{ \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} \right\} \leq C_1 \frac{k}{n}$$

for $1 \leq k \leq \left[\frac{n}{3}\right] + 1$, C_1 being an absolute constant and $n = 1, 2, 3, \ldots$

Proof of Lemma 1: We shall consider the proof in two parts

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* $\left[\frac{n}{3}\right]$ indicates the integral part of $\frac{n}{3}$.

viz.,

(i) $\left[\frac{n}{3}\right] + 2 \leq k \leq \left[\frac{n}{2}\right]$

and

(ii) $\left[\frac{n}{2}\right] + 1 \leq k \leq n$.

Since $\left[\frac{n}{3}\right] \geq \frac{n-2}{3}$ we may write in case (i)

$$W_{nk} < \frac{1}{2^n} \sum_{i=0}^{\left[\frac{n}{2}\right]-1} \binom{n}{i} - \frac{2\left(\frac{n-2}{3}+2\right)}{n+1} \leq$$

$$\leq \frac{1}{2} - \frac{2n+8}{3(n+1)} < 0$$

and since

$$\left[\frac{n}{2}\right] \geq \frac{n-1}{2} ,$$

in case (ii) we have

$$W_{nk} < \frac{1}{2^n} \sum_{i=0}^{n-1} \binom{n}{i} - \frac{2\left(\frac{n-1}{2}+1\right)}{n+1} <$$

$$< 1 - \frac{n+1}{n+1} = 0 .$$

Thus, for $\left[\frac{n}{3}\right] + 2 \leq k \leq n$,

$$W_{nk} < 0 .$$

Proof of Lemma 2: Firstly we prove that

$$(2.2.1) \quad \frac{1}{2^n}\binom{n}{\left[\frac{n}{3}\right]} < 10\sqrt{\frac{3}{2}}\left(\frac{27}{32}\right)^{\left[\frac{n}{3}\right]} \qquad \text{for } n = 1,2,3,\ldots$$

When $k = \left[\frac{n}{3}\right]$, the number k equals one of the three numbers $\frac{n}{3}$, $\frac{n-1}{3}$, $\frac{n-2}{3}$ and we have

$$\frac{\binom{3k}{k}}{2^{3k}} > \frac{\binom{3k+1}{k}}{2^{3k+1}} > \frac{\binom{3k+2}{k}}{2^{3k+2}} .$$

Therefore if we prove that

$$\frac{\binom{3k}{k}}{2^{3k}} < 10\sqrt{\frac{3}{2}}\left(\frac{27}{32}\right)^{k} \qquad \text{for } k = 1,2,3,\ldots,$$

then the inequality (2.2.1) will also get proved.

Applying the inequality*

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* Knopp [44] p.236.

$$1 < \frac{n!}{n^n \, e^{-n} \sqrt{n}} < 10 \qquad \text{for } n = 1, 2, 3, \ldots,$$

we get

$$\frac{1}{2^{3k}} \binom{3k}{k} = \frac{1}{2^{3k}} \cdot \frac{(3k)!}{(2k)! \; k!} <$$

$$< \frac{10 \, (3k)^{3k} \, e^{-3k} \sqrt{3k}}{2^{3k} \, (2k)^{2k} \, e^{-2k} \sqrt{2k} \; k^k \, e^{-k} \sqrt{k}} <$$

$$< 10 \sqrt{\frac{3}{2k}} \left(\frac{27}{32}\right)^k \leq$$

$$\leq 10 \sqrt{\frac{3}{2}} \left(\frac{27}{32}\right)^k ,$$

proving thereby the inequality (2.2.1).

Returning to the estimation of

$$\left\{ \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} \right\}^2 \qquad \text{for } 1 \leq k \leq \left[\frac{n}{3}\right] + 1 ,$$

let us note that the sequence

$$\left\{ \frac{1}{2^n} \binom{n}{i} \right\} \qquad \text{increases for} \quad 0 \leq i \leq \left[\frac{n}{3}\right] .$$

Thus for $1 \le k \le \left[\frac{n}{3}\right] + 1$, we have

$$\left\{\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\right\}^2 \le \frac{1}{4^n}\sum_{i=0}^{k-1}\binom{n}{i}^2 \sum_{i=0}^{k-1} 1$$

$$\le \frac{k}{4^n}\sum_{i=0}^{k-1}\binom{n}{i}^2$$

$$\le \frac{k^2}{4^n}\binom{n}{k-1}^2$$

$$< \frac{k^2}{4^n}\binom{n}{\left[\frac{n}{3}\right]}^2 .$$

Applying inequality (2.2.1) we obtain

$$\left\{\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\right\}^2 < 150\, k^2 \left(\frac{27}{32}\right)^{2\left[\frac{n}{3}\right]} \le C_2 \frac{k^2}{n^2} .$$

This proves Lemma 2.

Lemma 3* (PALEY'S THEOREMS): Let $\{\phi_n(x)\}$ be an orthogonal normal system in (a,b) and $|\Phi_n(x)| \le M$ $(n = 1, 2, 3, \dots)$, $a < x < b$.

(1) If $1 < p \le 2$, $f(x) \in L^p$ and $c_1, c_2, \dots c_n, \dots$ be its Fourier coefficients with respect to the system $\{\phi_n(x)\}$,

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* Bary [9] p.225.

then

$$\left\{ \sum_{n=1}^{\infty} |c_n|^p n^{p-2} \right\}^{\frac{1}{p}}$$

is finite and

$$\left\{ \sum_{n=1}^{\infty} |c_n|^p n^{p-2} \right\}^{\frac{1}{p}} \leq A_p \left\{ \int_a^b |f|^p dx \right\}^{\frac{1}{p}},$$

where A_p depends only on p and M.

(8) If $q \geq 2$ and $c_1, c_2, \ldots, c_n, \ldots$ is a sequence of numbers for which

$$\sum_{n=1}^{\infty} |c_n|^q n^{q-2} < +\infty,$$

then a function $f(x) \in L^q(a,b)$ exists, for which the numbers c_n are Fourier coefficients with respect to the system $\{\phi_n(x)\}$ and

$$\left\{ \int_a^b |f|^q dx \right\}^{\frac{1}{q}} \leq B_q \left\{ \sum_{n=1}^{\infty} |c_n|^q n^{q-2} \right\}^{\frac{1}{q}},$$

where B_q depends only on q and M.

3.3 <u>Proof of Theorem 1</u>: We have

$$S_n(x) - T_n(x) =$$

$$= \sum_{k=0}^{n} c_k \phi_k(x) - \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} S_i(x)$$

$$= \sum_{k=0}^{n} c_k \phi_k(x) - \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{i} c_k \phi_k(x)$$

$$= \sum_{k=0}^{n} c_k \phi_k(x) - \frac{1}{2^n} \sum_{k=0}^{n} c_k \phi_k(x) \sum_{i=k}^{n} \binom{n}{i}$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} c_k \phi_k(x) \sum_{i=0}^{n} \binom{n}{i} - \frac{1}{2^n} \sum_{k=0}^{n} c_k \phi_k(x) \sum_{i=k}^{n} \binom{n}{i}$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} c_k \phi_k(x) \sum_{i=0}^{k-1} \binom{n}{i}$$

$$= \sum_{k=0}^{n} c_k \phi_k(x) R_k$$

where

$$R_k = \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} .$$

Using Lemma 3

$$\int_a^b |S_n(x) - \tau_n(x)|^q \, dx = \int_a^b \left| \sum_{k=0}^{n} c_k R_k \phi_k(x) \right|^q dx \leq$$

$$\leq A_1 \left\{ \sum_{k=1}^{n} |c_k|^q |R_k|^q k^{q-2} \right\}.$$

From Lemmas 1 and 2 we infer that

$$|R_k|^q = |(R_k)^2|^{\frac{q}{2}} \leq C \left(\frac{k^2}{n^2} \right)^{\frac{q}{2}} = C \frac{k^q}{n^q} .$$

Consequently

$$\int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - \tau_n(x)|^q}{n} dx \leq$$

$$\leq A_2 \sum_{n=1}^{\infty} \frac{1}{n^{1+q}} \sum_{k=1}^{n} |c_k|^q k^{2q-2}$$

$$\leq A_2 \sum_{k=1}^{\infty} |c_k|^q k^{2q-2} \sum_{n=k}^{\infty} \frac{1}{n^{1+q}}$$

$$\leq A \sum_{k=1}^{\infty} |c_k|^q \, k^{q-2} .$$

This completes the proof of theorem 1.

3.4. Proof of Theorem 2: We have

$$S_n(x) - \sigma_n(\lambda, x) =$$

$$= \sum_{k=0}^{n} c_k \phi_k(x) - \sum_{k=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right) c_k \phi_k(x)$$

$$= \frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} \lambda_k c_k \phi_k(x) .$$

Using lemma 3

$$\int_a^b |S_n(x) - \sigma_n(\lambda, x)|^q \, dx <$$

$$< \frac{C}{\lambda_{n+1}^q} \sum_{k=1}^{n} \lambda_k^q |c_k|^q \, k^{q-2} .$$

Consequently

$$\int_a^b \sum_{n=1}^{\infty} \frac{|S_n(x) - \sigma_n(\lambda, x)|^q}{n} \, dx <$$

$$< C \sum_{n=1}^{\infty} \frac{1}{n \lambda_{n+1}^q} \sum_{k=1}^{n} \lambda_k^q |c_k|^q \, k^{q-2}$$

$$< C \sum_{k=1}^{\infty} \lambda_k^q |c_k|^q k^{q-2} \sum_{n=k}^{\infty} \frac{1}{n \lambda_{n+1}^q} .$$

We now estimate the expression

$$\lambda_k^q \sum_{n=k}^{\infty} \frac{1}{n \lambda_{n+1}^q} .$$

By Cauchy's inequality

$$\left(\lambda_k^q \sum_{n=k}^{\infty} \frac{1}{n \lambda_{n+1}^q} \right)^2 \leq \sum_{n=k}^{\infty} \frac{1}{n^2} \sum_{n=k}^{\infty} \left(\frac{\lambda_k}{\lambda_{n+1}} \right)^{2q}$$

$$= O\left(\frac{1}{k}\right) \left\{ \left(\frac{\lambda_k}{\lambda_{k+1}}\right)^{2q} + \left(\frac{\lambda_k}{\lambda_{k+1}}\right)^{2q} \cdot \left(\frac{\lambda_{k+1}}{\lambda_{k+2}}\right)^{2q} + \cdots \right\}$$

$$= O\left(\frac{1}{k}\right) \left\{ \frac{1}{\varrho^{2q}} + \frac{1}{\varrho^{4q}} + \cdots \right\}$$

$$= O\left(\frac{1}{k}\right) \cdot O(1)$$

$$= O\left(\frac{1}{k}\right) .$$

Hence

$$\int_a^b \sum_{n=1}^{\infty} \frac{|s_n(x) - \sigma_n(\lambda, x)|^q}{n} dx < C \sum_{k=1}^{\infty} |c_k|^q k^{q-\frac{5}{2}} .$$

This completes the proof of theorem 2.