# CHAPTER IV

## SUMMABILITY OF ORTHOGONAL SERIES

## BY EULER METHOD

4.1 Let $\{\phi_n(x)\}$ $(n = 0, 1, 2, \ldots)$ be an orthonormal system of functions defined in the interval $(a,b)$ and $\{c_n\} \in \ell^2$, that is,

$$\sum_{n=0}^{\infty} c_n^2 < \infty . \tag{4.1.1}$$

Further, let

$$(4.1.2) \qquad \sum_{n=0}^{\infty} c_n \phi_n(x)$$

denote orthogonal series being development of functions $f(x) \in L^2(a,b)$, that is, integrable with the square in Lebesgue sense.

The $n^{th}$ Euler means of order $q > 0$ or $n^{th}$ $(E,q)$-mean of the sequence of partial sums $\{S_n(x)\}$ is given by

$$\tau_n^{(q)}(x) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} S_k(x) \, , \quad n = 0,1,2,\ldots, \; (q>0),$$

where

$$S_n(x) = \sum_{k=0}^{n} c_k \phi_k(x) .$$

In place of $\tau_n^{(1)}$ we simply write $\tau_n$ .

Series (4.1.2) is said to be summable by Euler means of order $q$ or more concisely $(E,q)$-summable to $S$ if

$$\lim_{n\to\infty} \tau_n^{(q)} = S .$$

Further, if

$$\sum_{n=1}^{\infty} \left| \tau_n^{(q)} - \tau_{n-1}^{(q)} \right| < \infty ;$$

then the series (4.1.2) is said to be absolutely (E,q)-summable or |E,q|-summable.

Absolute (C,1)-summability of (4.1.2) has been studied by Tandori* and Billard**, whereas absolute (C,α)-summability of (4.1.2) has been investigated by Leindler***, Moricz+ and Patel++. Generalised absolute Cesàro summability of (4.1.2) has been investigated by Szalay+++. Sapre† and Bhatnagar‡ have discussed absolute (N,1)-summability of the series (4.1.2). Summation of orthogonal series of $L^2$ by Euler's method was considered by Meder§, Zima§§ and Koljada††. Bhatnagar‡ has proved the following theorem:

<u>Theorem A:</u> If

$$\sum_{n=1}^{\infty} c_n^2 n^p < \infty , \quad 2 > p > \tfrac{1}{2} ,$$

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* Tandori [112].
** Billard [10].
*** Leindler [49].
\+ Moricz [66].
\++ Patel [71].
\+++ Szalay [105].
† Sapre [90].
‡ Bhatnagar [11].
§ Meder [53, 54].
§§ Zima [123].
†† Koljada [46].

then the series (4.1.2) is $|E,1|$-summable almost every-where in $(a,b)$.

Generalizing this theorem we prove

<u>THEOREM 1</u>: If

$$(4.1.3) \qquad \sum_{n=1}^{\infty} c_n^2 n^p < \infty \quad , \quad 2 > p > \tfrac{1}{2} \quad ,$$

<u>then the series</u> (4.1.2) <u>is</u> $|E,q|$<u>-summable</u> $(q > 0)$ <u>almost everywhere</u> <u>in</u> $(a,b)$.

Theorem 1 includes evidently the theorem A of Bhatnagar in special case $q = 1$.

Meder* has proved the following theorem:

<u>Theorem B</u>: The series

$$\sum_{n=1}^{\infty} \int_a^b n \left[ T_n(x) - T_{n-1}(x) \right]^2 dx$$

is convergent if

$$\sum_{n=1}^{\infty} c_n^2 \sqrt{n} < \infty .$$

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* Meder [53].

Generalising above result of Meder, we prove

**THEOREM 2: The series**

$$\sum_{n=1}^{\infty} \int_a^b n \left[ \tau_n^{(q)}(x) - \tau_{n-1}^{(q)}(x) \right]^2 dx$$

$(q > 0)$ **is convergent if**

$$(4.1.4) \qquad \sum_{n=1}^{\infty} c_n^2 \sqrt{n} < \infty .$$

Theorem 2 includes evidently the theorem D of Meder in special case $q = 1$.

4.2 The following lemmas will be required for the proofs of the theorems.

**Lemma 1:**

$$\sum_{i=k}^{n} \left[ \frac{1}{q^i} \binom{n-1}{i-1} - \frac{1}{q^{i+1}} \binom{n-1}{i} \right] = \frac{1}{q^k} \binom{n-1}{k-1} = \frac{1}{q^k} \frac{k}{n} \binom{n}{k} .$$

The proof is obvious.

**Lemma 2***: For any value of $q > 0$, the following evaluation

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* Also [103] .

is valid

$$(4.2.1) \qquad \max_{0\le k\le n} \binom{n}{k} q^k \le C_q \frac{(1+q)^n}{\sqrt{n}} \quad , \quad n=1,2,3,\ldots,$$

where constant $C_q$ does not depend on $n$.

4.3 <u>Proof of Theorem 1:</u> We have

$$T_n^{(q)}(x) - T_{n-1}^{(q)}(x) =$$

$$= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} S_k(x) - \frac{1}{(1+q)^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k} q^{n-1-k} S_k(x)$$

$$= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} S_k(x) - \frac{1}{(1+q)^{n-1}}\sum_{k=0}^{n}\binom{n-1}{k} q^{n-1-k} S_k(x)$$

$$= \frac{1}{(1+q)^n}\sum_{k=0}^{n} q^{n-1-k}\left[q\binom{n}{k} - (q+1)\binom{n-1}{k}\right] S_k(x) .$$

Applying the formula

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad , \qquad k = 0, 1, 2, \ldots, n,$$

we get

$$\tau_n^{(q)}(x) - \tau_{n-1}^{(q)}(x) =$$

$$= \frac{q^n}{(1+q)^n} \sum_{k=0}^{n} q^{-k-1} \left[ q \left\{ \binom{n-1}{k} + \binom{n-1}{k-1} \right\} - (q+1)\binom{n-1}{k} \right] S_k(x)$$

$$= \frac{q^n}{(1+q)^n} \sum_{k=0}^{n} \frac{1}{q^{k+1}} \left[ q\binom{n-1}{k-1} - \binom{n-1}{k} \right] \sum_{i=0}^{k} c_i \phi_i(x) .$$

On changing the order of summation and using lemma 1, we get

$$\tau_n^{(q)}(x) - \tau_{n-1}^{(q)}(x) =$$

$$= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} q^{n-k} c_k \phi_k(x) .$$

By using the orthonormality property of the system $\{\phi_n(x)\}$, we have

$$\int_a^b \left[ \tau_n^{(q)}(x) - \tau_{n-1}^{(q)}(x) \right]^2 dx =$$

$$= \frac{1}{(1+q)^{2n}} \cdot \frac{1}{n^2} \sum_{k=0}^{n} k^2 \binom{n}{k}^2 q^{2(n-k)} c_k^2 \leq$$

$$\leq \frac{1}{(1+q)^n} \left\{ \max_{0 \leq k \leq n} \binom{n}{k} q^{n-k} \right\} \left\{ \frac{1}{(1+q)^n} \cdot \frac{1}{n^2} \sum_{k=0}^{n} k^{2-p+p} \binom{n}{k} q^{n-k} c_k^2 \right\}$$

As $\binom{n}{k} = \binom{n}{n-k}$, using lemma 2

$$\max_{0 \leq k \leq n} \binom{n}{k} q^{n-k} = \max_{0 \leq k \leq n} \binom{n}{k} q^k \leq C_q \frac{(1+q)^n}{\sqrt{n}} .$$

Therefore

$$\int_a^b \left[ \tau_n^{(q)}(x) - \tau_{n-1}^{(q)}(x) \right]^2 dx \leq$$

$$\leq C_q \left[ \frac{1}{\sqrt{n}} \cdot \frac{1}{n^2} \cdot n^{2-p} \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} k^p c_k^2 \right]$$

Using Schwarz's inequality, we get

$$\int_a^b \left| \tau_n^{(q)}(x) - \tau_{n-1}^{(q)}(x) \right| dx \leq$$

$$\leq \sqrt{(b-a)} \left\{ \int_a^b \left[ \tau_n^{(q)}(x) - \tau_{n-1}^{(q)}(x) \right]^2 dx \right\}^{\frac{1}{2}}$$

$$= O(1) \left[ \frac{1}{n^{p+\frac{1}{2}}} \cdot \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} c_k^2 k^p \right]^{\frac{1}{2}} .$$

Applying Cauchy's inequality, we get

$$\sum_{n=1}^{\infty} \int_a^b \left| \tau_n^{(q)}(x) - \tau_{n-1}^{(q)}(x) \right| dx =$$

$$= O(1) \left[ \left\{ \sum_{n=1}^{\infty} \frac{1}{n^{p+\frac{1}{2}}} \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\infty} \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} c_k^2 k^p \right\}^{\frac{1}{2}} \right]$$

(4.3.1) $$= O(1) \left\{ \sum_{k=1}^{\infty} c_k^2 k^p \sum_{n=k}^{\infty} \binom{n}{k} q^{n-k} \frac{1}{(1+q)^n} \right\}^{\frac{1}{2}}, \text{ as } p > \tfrac{1}{2} .$$

Let us now show that the internal sum in (4.3.1) is limited to a constant not depending on k. The quantity

$$\sum_{n=k}^{\infty} \binom{n}{k} q^{n-k} \frac{1}{(1+q)^n}$$ is a mean of the order k which

can be found out by the circle method $\left(\gamma, \frac{1}{1+q}\right)$* from the sequence $s_n = 1 + q$, $n = 0, 1, 2, \ldots$ The circle method $(\gamma, a)$, $0 < a < 1$ is regular+ and consequently, the sum in question tends to $1 + q$ as $k \to \infty$, i.e. it is limited. Consequently from (4.3.1), using the condition (4.1.3), theorem 1 follows.

4.4 Proof of Theorem 2: We have

$$\int_a^b \left[ \tau_n^{(q)}(x) - \tau_{n-1}^{(q)}(x) \right]^2 dx =$$

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* The sequence $\{s_n\}$ is summed to $s$ by the circle method $(\gamma, a)$ $(0 < a < 1)$ if

$$\lim_{k \to \infty} a^{k+1} \sum_{n \geq k} \binom{n}{k} (1-a)^{n-k} s_n = s .$$

In 1916 G. H. Hardy and J. E. Littlewood [33] introduced the Taylor method (known then as the circle method of order a) for $0 < a < 1$.
See also Hardy [31] p. 210 and 279.

\+ Hardy [31], p.279.

$$= \frac{1}{(1+q)^{2n}} \cdot \frac{1}{n^2} \sum_{k=0}^{n} k^2 \binom{n}{k}^2 q^{2(n-k)} c_k^2 \leq$$

$$\leq \frac{1}{(1+q)^{2n}} \cdot \frac{1}{\sqrt{n}} \sum_{k=0}^{n} \binom{n}{k}^2 q^{2(n-k)} c_k^2 \sqrt{k} .$$

Hence

$$\sum_{n=1}^{\infty} \int_a^b n \left[ \tau_n^{(q)}(x) - \tau_{n-1}^{(q)}(x) \right]^2 dx \leq$$

$$\leq \sum_{n=1}^{\infty} \left[ n \cdot \frac{1}{(1+q)^n} \left\{ \max_{0 \leq k \leq n} \binom{n}{k} q^{n-k} \right\} \left\{ \frac{1}{\sqrt{n}} \cdot \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} c_k^2 \sqrt{k} \right\} \right]$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} c_k^2 \sqrt{k} \text{ , by Lemma 2}$$

$$\leq \sum_{k=1}^{\infty} c_k^2 \sqrt{k} \sum_{n=k}^{\infty} \binom{n}{k} q^{n-k} \frac{1}{(1+q)^n} < \infty$$

as in the proof of Theorem 1 and by the assumed condition (4.1.4).

Thus theorem 2 is completely proved.