CHAPTER V

ON STRONG EULER SUMMABILITY OF ORTHOGONAL SERIES

5.1 Let $\{\phi_n(x)\}$ $(n = 0, 1, 2, \ldots)$ be an orthogonal and normal function system in the interval (a,b). We consider the orthogonal series

$$(5.1.1) \qquad \sum_{n=0}^{\infty} c_n \phi_n(x)$$

with real coefficients sequence $\{c_k\}$. We denote the n^{th} partial sum of (5.1.1) by $S_n(x)$.

Let $T = (\alpha_{nk})$ $(n,k = 0,1,2,\ldots)$ be an infinite matrix of positive elements satisfying the condition

$$(6.1.2) \qquad \lim_{n\to\infty} \sum_{k=0}^{\infty} \alpha_{nk} = 1 .$$

We say that the series (6.1.1) is T-summable to $S(x)$ at a point $x \in (a,b)$ if

$$(6.1.3) \qquad t_n(x) = \sum_{k=0}^{\infty} \alpha_{nk} S_k(x)$$

exists for all n (except perhaps finitely many of them) and

$$\lim_{n\to\infty} t_n(x) = S(x) .$$

The series (6.1.1) is called strongly T-summable with an index $p > 0$ to the sum $S(x)$ if the relation

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \alpha_{nk} \left| S_k(x) - S(x) \right|^p = 0$$

holds.

In (6.1.3) if we put

$$\alpha_{nk} = \frac{\binom{n}{k}}{2^n} \text{ for } k \leq n$$

$= 0 \qquad \text{for } k > n ,$

then the T-means are the classical (C,1)-means.

The strong Cesàro summability of orthogonal series, as well as that of Fourier series, has been studied in great details by several authors such as : A. Zygmund, S. Kaczmarz, S. Borgen, G. Alexits, K. Tandori, G. Sunouchi, B. N. Prasad and U. N. Singh. Strong Riesz summability as well as strong Nörlund summability of orthogonal series has been discussed by Meder*, whereas strong T-summability as applied to orthogonal series has been investigated by Moricz**. Zygmund*** has proved the following theorem :

<u>Theorem A</u>: If the series (5.1.1) satisfying

$$(5.1.4) \qquad \sum_{n=0}^{\infty} c_n^2 < \infty$$

is summable (C,1) a.e. to a function S(x), then it is strongly summable (C,1) to this function S(x).

In this chapter we generalise this theorem of Zygmund transferring to the more general Euler-method of summation.

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* Meder [55, 56]

** Moricz [66]

*** Zygmund [106]

We prove :

THEOREM 1: *If the orthogonal series (5.1.1) with coefficients satisfying the condition*

$$(5.1.5) \qquad \sum_{n=1}^{\infty} c_n^2 \sqrt{n} < \infty$$

is (C,1)-summable a.e. to $s(x)$, *then*

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left(s_k(x) - s(x)\right)^2 = 0 ,$$

that is, the series (5.1.1) is strongly (E,1)-summable a.e. to $s(x)$ *with index 2.*

We intend to generalise further our theorem 1, for any index $p > 0$, to the following form.

THEOREM 2: *If the orthogonal series (5.1.1) under the condition (5.1.5) is (C,1)-summable a.e. to* $s(x)$, *then*

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left| s_k(x) - s(x)\right|^p = 0 \quad \text{for } 0 < p \leq 2 .$$

THEOREM 3: Let $\{\phi_n(x)\}$ be an orthonormal system in (a,b) and

$$|\phi_n(x)| \leq M \quad (n = 1, 2, 3, \ldots),\ a < x < b.$$

Further, let the coefficients of (6.1.1) satisfy the condition

$$(6.1.6) \qquad \sum_{n=1}^{\infty} |c_n|^q \, n^{q-\frac{3}{2}} < \infty, \quad q \geq 2.$$

Suppose that the series (6.1.1) under these assumptions is $(E,1)$-summable to $s(x)$ a.e. in (a,b), then

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} |S_k(x) - S(x)|^q = 0$$

$(q \geq 2)$ a.e. in (a,b).

6.2 For the proof of the theorems we need some preliminary lemmas.

Lemma 1*: If

$$m = \left[\frac{n}{2}\right]$$

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* Knopp [44] p.136.

then

$$\frac{\sqrt{n}\binom{n}{m}}{2^n} < 20e \quad \text{for } n = 1, 2, 3, \ldots$$

<u>Lemma 2:</u>

$$W_{nk} \equiv \frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i} - \frac{2k}{n+1} < 0$$

$$\text{for} \left[\frac{n}{3}\right] + 2 \leq k \leq n .$$

<u>Lemma 3 :</u>

$$\left\{\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\right\} \leq C\,\frac{k}{n} \qquad \text{for } 1 \leq k \leq \left[\frac{n}{3}\right] + 1 ,$$

C being an absolute constant and $n = 1,2,3,\ldots$

Lemmas 2 and 3 are proved in chapter III.

<u>Lemma 4 :</u> If a series $\sum u_n$ with partial sums S_n is strongly E-summable to f with index q, then it is strongly E-summable to f with an index p, $0 < p < q$.

Proof of lemma 4 : Let r and s be two numbers greater than 1 satisfying $r^{-1} + s^{-1} = 1$. Then by Hölder's inequality we get

$$\sum_{k=0}^{\infty} \alpha_{nk} |S_k - f|^p = \sum_{k=0}^{\infty} \alpha_{nk}^{\frac{1}{r}} |S_k - f|^p \alpha_{nk}^{\frac{1}{s}} \leq$$

$$\leq \left(\sum_{k=0}^{\infty} \alpha_{nk} |S_k - f|^{pr} \right)^{\frac{1}{r}} \left(\sum_{k=0}^{\infty} \alpha_{nk} \right)^{\frac{1}{s}} .$$

Now set pr = q and s will also change accordingly by them

$$\sum_{k=0}^{\infty} \alpha_{nk} |S_k - f|^p \leq \left(\sum_{k=0}^{\infty} \alpha_{nk} |S_k - f|^q \right)^{\frac{1}{r}} \left(\sum_{k=0}^{\infty} \alpha_{nk} \right)^{\frac{1}{s}} = o(1)$$

as $n \to \infty$ by hypothesis and (5.1.2), which completes the proof of Lemma 4.

5.3 Proof of Theorem 1 : We have

$$\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left(S_k(x) - S(x) \right)^2 \leq$$

$$\leq \frac{2}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left(S_k(x) - \tau_k(x) \right)^2 +$$

$$+ \frac{2}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left(\tau_k(x) - s(x) \right)^2 \leq$$

$$\leq \frac{4}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left(s_k(x) - \sigma_k(x) \right)^2 +$$

$$+ \frac{4}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left(\sigma_k(x) - \tau_k(x) \right)^2 +$$

$$+ \frac{2}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left(\tau_k(x) - s(x) \right)^2 =$$

(5.3.1) $\qquad = S_1 + S_2 + S_3$, say.

From the hypothesis it is evident that

$$S_3 \longrightarrow 0 .$$

Coming now to S_1

$$S_1 = O(1) \frac{\binom{n}{m}}{2^n} \sum_{k=0}^{n} \left(s_k(x) - \sigma_k(x) \right)^2$$

$$= O(1) \frac{\sqrt{n} \binom{n}{m}}{2^n} \cdot \frac{1}{\sqrt{n}} \sum_{k=0}^{n} \left(s_k(x) - \sigma_k(x) \right)^2$$

$$= O(1) \frac{1}{\sqrt{n}} \sum_{k=0}^{n} \left(s_k(x) - \sigma_k(x) \right)^2, \text{ by lemma 1.}$$

Now,

$$S_n(x) - \sigma_n(x) =$$

$$= \sum_{k=0}^{n} c_k \phi_k(x) - \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) c_k \phi_k(x)$$

$$= \frac{1}{n+1} \sum_{k=1}^{n} k\, c_k \phi_k(x) \quad .$$

Therefore by the orthonormality property

$$\int_a^b \left(S_n(x) - \sigma_n(x)\right)^2 dx = \frac{1}{(n+1)^2} \sum_{k=1}^{n} k^2 c_k^2 \quad .$$

Consequently

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_a^b \left(S_n(x) - \sigma_n(x)\right)^2 dx <$$

$$< \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \sum_{k=1}^{n} k^2 c_k^2 =$$

$$= \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^{5/2}}$$

$$= O(1) \sum_{k=1}^{\infty} c_k^2 \sqrt{k} < \infty \quad .$$

Therefore by B. Levy's theorem

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(S_n(x) - \sigma_n(x)\right)^2 < \infty$$

Then, by Kronecker's lemma

$$\sum_{k=1}^{n} \left(S_k(x) - \sigma_k(x)\right)^2 = o\left(\sqrt{n}\right)$$

which proves that

$$S_1 \longrightarrow 0 ,$$

Lastly let us consider S_2.

$$S_2 \leq \frac{4}{2^n}\binom{n}{m} \sum_{k=0}^{n} \left(\sigma_k(x) - \tau_k(x)\right)^2$$

$$= O(1)\frac{1}{\sqrt{n}} \sum_{k=0}^{n} \left(\sigma_k(x) - \tau_k(x)\right)^2 , \text{ by Lemma 1.}$$

Now, we have

$$\sigma_n(x) - \tau_n(x) =$$

$$= \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) c_k \phi_k(x) - \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} S_i(x)$$

$$= \sum_{k=0}^{n} c_k \phi_k(x) - \frac{1}{n+1} \sum_{k=0}^{n} k c_k \phi_k(x) - \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{i} c_k \phi_k(x)$$

$$= \frac{1}{2^n}\sum_{k=0}^{n} c_k\phi_k(x)\sum_{i=0}^{n}\binom{n}{i} - \frac{1}{n+1}\sum_{k=0}^{n} k c_k\phi_k(x) - \frac{1}{2^n}\sum_{k=0}^{n} c_k\phi_k(x)\sum_{i=k}^{n}\binom{n}{i}$$

$$= \frac{1}{2^n}\sum_{k=0}^{n} c_k\phi_k(x)\sum_{i=0}^{k-1}\binom{n}{i} - \frac{1}{n+1}\sum_{k=0}^{n} k c_k\phi_k(x)$$

$$= \sum_{k=0}^{n} c_k\phi_k(x)\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i} - \frac{k}{n+1}\right],$$

atunci

$$\int_a^b \left[\sigma_n(x) - \tau_n(x)\right]^2 dx =$$

$$= \sum_{k=1}^{n} c_k^2\left\{\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i} - \frac{2k}{n+1}\right] + \frac{k^2}{(n+1)^2}\right\} <$$

$$< \sum_{k=1}^{\left[\frac{n}{3}\right]+1} c_k^2\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\right]^2 +$$

$$+ \sum_{k=\left[\frac{n}{3}\right]+2}^{n} c_k^2\left\{\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i}\left[\frac{1}{2^n}\sum_{i=0}^{k-1}\binom{n}{i} - \frac{2k}{n+1}\right]\right\} +$$

$$+\sum_{k=1}^{n} \frac{k^2 c_k^2}{(n+1)^2} .$$

By Lemma 2

$$\sum_{k=\left[\frac{n}{3}\right]+2}^{n} c_k^2 \left\{ \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} \left[\frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} - \frac{2k}{n+1} \right] \right\} < 0 .$$

Hence

$$\int_a^b \left[\sigma_n(x) - \tau_n(x) \right]^2 dx <$$

$$< \sum_{k=1}^{\left[\frac{n}{3}\right]+1} c_k^2 \left[\frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i} \right]^2 + \sum_{k=1}^{n} \frac{k^2 c_k^2}{(n+1)^2} .$$

Consequently, by Lemma 3

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_a^b \left[\sigma_n(x) - \tau_n(x) \right]^2 dx \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{\left[\frac{n}{3}\right]+1} c_k^2 \cdot C_0 \cdot \frac{k^2}{n^2} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{k^2 c_k^2}{(n+1)^2}$$

$$\leq A \left[\sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^{5/2}} + \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{\infty} \frac{1}{n^{5/2}} \right] .$$

where A is an absolute constant,

$$= O(1) \sum_{k=1}^{\infty} c_k^2 \sqrt{k} < \infty .$$

Therefore by P. Levy's theorem

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\sigma_n(x) - \tau_n(x)\right)^2 < \infty .$$

Then, by Kronecker's lemma

$$\sum_{k=1}^{n} \left(\sigma_k(x) - \tau_k(x)\right)^2 = o\left(\sqrt{n}\right)$$

which proves that

$$C_2 \longrightarrow 0 .$$

This completes the proof of Theorem 1.

5.4 <u>Proof of Theorem 2</u> : For $p = 2$ the proof has been given by us in Theorem 1 and for $0 < p < 2$ the proof follows from Lemma 4.

5.5 <u>Proof of Theorem 3</u> : We have

$$\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left| S_k(x) - S(x) \right|^q \leq$$

$$\leq \frac{2^q}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left| S_k(x) - \tau_k(x) \right|^q +$$

$$+ \frac{2^q}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left| \tau_k(x) - S(x) \right|^q =$$

$= S_1 + S_2$, say.

By virtue of hypothesis

$$S_2 \longrightarrow 0 .$$

For S_1

$$S_1 = O(1) \frac{\binom{n}{m}}{2^n} \sum_{k=0}^{n} \left| S_k(x) - \tau_k(x) \right|^q$$

$$= O(1) \frac{1}{\sqrt{n}} \sum_{k=0}^{n} \left| S_k(x) - \tau_k(x) \right|^q , \text{ by Lemma 1 .}$$

Now,

$$S_n(x) - \tau_n(x) =$$

$$= \sum_{k=0}^{n} c_k \phi_k(x) - \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} S_i(x)$$

$$= \sum_{k=0}^{n} c_k \phi_k(x) - \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{i} c_k \phi_k(x)$$

$$= \sum_{k=0}^{n} c_k \phi_k(x) - \frac{1}{2^n} \sum_{k=0}^{n} c_k \phi_k(x) \sum_{i=k}^{n} \binom{n}{i}$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} c_k \phi_k(x) \sum_{i=0}^{n} \binom{n}{i} - \frac{1}{2^n} \sum_{k=0}^{n} c_k \phi_k(x) \sum_{i=k}^{n} \binom{n}{i}$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} c_k \phi_k(x) \sum_{i=0}^{k-1} \binom{n}{i}$$

$$= \sum_{k=0}^{n} c_k \phi_k(x) R_k$$

Where

$$R_k = \frac{1}{2^n} \sum_{i=0}^{k-1} \binom{n}{i}$$

Using Paley's theorem*, we get

$$\int_a^b |S_n(x) - T_n(x)|^q \, dx = \int_a^b \Big| \sum_{k=0}^{n} c_k \phi_k(x) R_k \Big|^q dx \le$$

$$\le A_1 \left\{ \sum_{k=1}^{n} |c_k|^q \, |R_k|^q \, k^{q-2} \right\}$$

* See Chapter III - Lemma 3. Also refer to Bary [5] p.225.

Now, from Lemmas 2 and 3,

$$|R_k|^q = \left|(R_k)^2\right|^{q/2} \leq C_0\left(\frac{k^2}{n^2}\right)^{q/2} = C_0\,\frac{k^q}{n^q} .$$

Consequently

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_a^b |S_n(x) - T_n(x)|^q \, dx \leq$$

$$\leq A_2 \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}+q}} \sum_{k=1}^{n} |c_k|^q \, k^{2q-2}$$

$$\leq A_2 \sum_{k=1}^{\infty} |c_k|^q \, k^{2q-2} \sum_{n=k}^{\infty} \frac{1}{n^{\frac{1}{2}+q}}$$

$$\leq A \sum_{k=1}^{\infty} |c_k|^q \, k^{q-\frac{3}{2}} < \infty .$$

Therefore by B. Levi's theorem

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} |S_n(x) - T_n(x)|^q < \infty .$$

Then, by Kronecker's lemma

$$\sum_{K=1}^{n} \left| S_K(x) - T_K(x) \right|^q = o(\sqrt{n})$$

which proves that

$$S_1 \longrightarrow 0 .$$

This completes the proof of Theorem 3.