CHAPTER VI

ON VERY STRONG RIESZ SUMMABILITY OF ORTHOGONAL SERIES

6.1 Let $\{\phi_n(x)\}$ $(n = 0, 1, 2, \ldots)$ be an orthogonal and normal function system defined in the interval (a,b) and $\{c_n\} \in \ell^2$, that is,

$$(6.1.1) \quad \sum_{n=0}^{\infty} c_n^2 < \infty .$$

Further, let

$$(G.2.2) \qquad \sum_{n=0}^{\infty} c_n \phi_n(x)$$

denote orthogonal series being development of functions $f(x) \in L^2(a,b)$, i.e. integrable with the square in Lebesgue sense.

The n^{th} Euler-mean of the first order or the $(E,1)$-mean of the sequence of partial sums $\{S_n(x)\}$ of the orthogonal series (G.2.2) is defined as

$$\tau_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} S_k(x) \quad , n = 0, 1, 2, \dots ,$$

where

$$S_k(x) = \sum_{i=0}^{k} c_i \phi_i(x) \quad .$$

Series (G.1.2) is said to be very strongly summable $(E,1)$ to the sum $S(x)$ a.e. if for every monotone increasing index sequence $\{v_n\}$ and for almost every x the relation

$$\sum_{k=0}^{n} \binom{n}{k} \left(S_{v_k}(x) - S(x) \right)^2 = o\left(2^n\right) \quad \text{as } n \to \infty$$

holds.

In particular, if $v_k = k$ ($k = 0, 1, 2, \ldots$), we shall say that series (6.1.2) is strongly summable (C,1) to the sum S(x).

The strong summability (C,1) of orthogonal series, as well as that of Fourier series, has been investigated by several authors such as : A. Zygmund*, S. Kaczmarz**, S. Borgen***, Z. Zalcwasser+, G. Alexits++, B. N. Prasad+++, U. N. Singh† and D. D. Singh‡. Very strong Cesàro summability of orthogonal series has been studied in great details by G. Alexits≠, K. Tandori# and Leindler°°. Very strong Riesz summability of orthogonal series as well as very strong Nörlund summability of orthogonal series has been discussed by Meder§. K. Tandori# has proved the following theorems :

<u>Theorem A.1</u> Let $\{c_\nu^*\} \in \ell^2$ be a positive number sequence with

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*	Zygmund	[125].	†	Singh	[77].
**	Kaczmarz	[39].	‡	Singh	[90].
***	Borgen	[14].	≠	Alexits	[3].
+	Zalcwasser	[122].	#	Tandori	[100, 110].
++	Alexits	[3].	§	Meder	[56, 58].
+++	Prasad	[61].	°°	Leindler	[50].

$$\sqrt{v}\, C_v^* \geq \sqrt{v+1}\, C_{v+1}^* \quad (v=1,2,3,\ldots)$$

and $\{c_v\}$ be any sequence of real numbers with

$$c_v = O\left(c_v^*\right).$$

If the orthogonal series (G.1.2) built with these coefficients c_v, is (C,1)-summable to the function $f(x)$ almost everywhere in $[a,b]$, then it is very strongly summable (C,1) a. e. to this function.

Theorem D.: Under the condition (G.1.1), for the very strong (C,1)-summability almost everywhere of the orthogonal series (G.1.2) it is necessary and sufficient that the sequence $\{S_{v_{2^n}}(x)\}$ of the partial sums should converge almost everywhere for any arbitrary increasing index sequence $v_1 < v_2 < \ldots < v_k < \ldots$

Theorem E.: If

$$\sum_{n=3}^{\infty} c_n^2 (\log\log n)^2 < \infty,$$

then the series (G.1.2) is very strong summable (C,1) almost everywhere to a function $f(x) \in L^2(a,b)$.

In this chapter we generalize the above theorems of Tandori transforming them to the more general Euler-method of summation*.

THEOREM 1.1 Let $\{c^*_\nu\}$ be a numerical sequence of positive terms such that

$$(6.1.3) \qquad \sqrt{\nu}\, c^*_\nu \geq \sqrt{\nu+1}\, c^*_{\nu+1} \quad (\nu = 1, 2, 3, \ldots)$$

and

$$(6.1.4) \qquad \sum_{\nu=1}^{\infty} c^{*2}_\nu \sqrt{\nu} < \infty .$$

Further, let $\{c_\nu\}$ be an arbitrary sequence of real numbers satisfying the relation

$$(6.1.5) \qquad c_\nu = O\left(c^*_\nu\right) .$$

Suppose that the orthogonal series (6.1.3) under these assumptions is (E,1)-summable to a function f(x)

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* Euler-method is more general than Cesàro-method, Fodor [53, Theorem [illegible]].

almost everywhere in $[a,b]$, then it is very strongly summable (C,1) to this function almost everywhere in $[a,b]$.

For the following theorems we assume

$$(6.1.6) \qquad \sum_{t=v_i+1}^{v_{i+1}} c_t^2 = O\left(c_i^2\right) \qquad \text{where } v_1 < v_2 < \dots$$

__THEOREM [illegible].__ __Under the conditions__ (6.1.6) __and__

$$(6.1.7) \qquad \sum_{n=1}^{\infty} c_n^2 \sqrt{n} < \infty,$$

__the series__

$$\sum_{n=1}^{\infty} \int_a^b (n+1) \left[\tau_{n+1}^{(v)}(x) - \tau_n^{(v)}(x) \right]^2 dx$$

__converges almost everywhere in__ (a,b), __where__

$$\tau_n^{(v)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} S_{v_k}(x),$$

__where__ $\{v_n\}$ __denotes the sequence defined above.__

THEOREM 3 : Under the conditions (6.1.6) and (6.1.7), the convergence almost everywhere of the sequence $\{S_{v_{2^n}}(x)\}$ is necessary and sufficient for the convergence almost everywhere of $\{\tau_n^{(v)}(x)\}$.

THEOREM 4 : Under the conditions (6.1.6) and (6.1.7) for the convergence almost everywhere of the sequence $\{\tau_n^{(v)}(x)\}$ to a certain function $f(x)$ it is necessary and sufficient that the series (6.1.2) be very strongly summable (C,1) almost everywhere to this function.

THEOREM 5 : Under the conditions (6.1.6) and (6.1.7), for the very strong (C,1)-summability almost everywhere of the orthogonal series (6.1.2) it is necessary and sufficient that the sequence $\{S_{v_{2^n}}(x)\}$ of the partial sums should converge almost everywhere for any arbitrary increasing index sequence $v_1 < v_2 < \cdots$

THEOREM 6 : Under the conditions (6.1.6) and (6.1.7), the series (6.1.2) is very strongly summable (C,1) to a function $f(x)$ almost everywhere.

6.2 For the proof of our theorems we need some preliminary lemmas.

Lemma 1*: If

$$m = \left[\frac{n}{2}\right]$$

then

$$\frac{\sqrt{n}\binom{n}{m}}{2^n} < 20\, e \qquad \text{for } n = 1, 2, 3, \ldots$$

Lemma 2:

$$W_{nK} \equiv \frac{1}{2^n}\sum_{i=0}^{K-1}\binom{n}{i} - \frac{2K}{n+1} < 0 \quad \text{for } \left[\frac{n}{3}\right] + 2 \le K \le n.$$

Lemma 3:

$$\left\{\frac{1}{2^n}\sum_{i=0}^{K-1}\binom{n}{i}\right\} \le C_1\frac{K}{n} \quad \text{for } 1 \le K \le \left[\frac{n}{3}\right] + 1\,,$$

C_1 being an absolute constant and $n = 1, 2, 3, \ldots$

Lemma 2 and 3 are proved in chapter III.

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* Knopp [44] p.136.

<u>Lemma 4</u>.[*] If the series

$$\sum_{n=1}^{\infty} (\log n)^2 \sum_{k=m_n+1}^{m_{n+1}} c_k^2$$

is convergent, then the sequence $\{S_{m_n}(x)\}$ of the partial sums of the orthogonal series (6.1.2) is convergent almost everywhere.

<u>Lemma 5</u>.[**] If

$$\sum_{n=1}^{\infty} c_n^2 \sqrt{n} < \infty$$

and $\{n_k\}$ is an arbitrary increasing sequence of indices such that

$$1 < q \leq \frac{n_{k+1}}{n_k} \leq r \qquad \text{for } k = 0, 1, 2, \ldots,$$

where r and q are constants, then the orthogonal series (6.1.2) is (C,1)-summable almost everywhere in (a,b) if and only if the sequence $\{S_{n_k}(x)\}$ is convergent almost everywhere in (a,b).

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* Alexits [7] p.93.
** Sharma [23 , Theorem 2] .

<u>Lemma 6</u> : If the coefficients of the orthogonal series (6.1.2) satisfy the condition (6.1.7), then the series (6.1.2) is (E,1)-summable almost everywhere in (a,b).

This lemma follows from lemmas 4 and 5 above or by regularity of Euler summability.

<u>Lemma 7*</u> : Under the condition (6.1.1) the relation

$$S_{n_k}(x) - T_{n_k}(x) = o_x(1)$$

is valid almost everywhere for every index sequence $\{n_k\}$ with

$$\frac{n_{k+1}}{n_k} \geq q > 1 .$$

<u>Lemma 8**</u> : If the real numbers $a_0, a_1, \ldots, a_n$ and the orthonormal system $\{\psi_n(x)\}$ are arbitrarily given, there exists an L^2-integrable function $\delta_n(x) \geq 0$ with the following properties :

$$\max_{\nu \leq n} \left| \sum_{k=0}^{\nu} a_k \psi_k(x) \right| \leq \delta_n(x) ,$$

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* [illegible] [58, Theorem 1].

** Alexits [7], p.79.

$$\int_a^b \delta_n^2(x)\,dx = O(\log^2 n) \sum_{k=0}^{n} a_k^2 .$$

6.3 <u>Proof of Theorem 1.1</u> Let v_n be an arbitrary strictly increasing sequence of indices. We may suppose without loss of generality of theorem that $v_1 \geq 1$. Let $2^m \leq v_k < 2^{m+1}$. Assume that $\mu_k = 2^{m+1}$ $(n = 0,1,2,\ldots)$. Since from the assumption $\{c_n\} \in l^2$ and the series (6.1.3) is (E,1)-summable to a function $f(x)$ almost everywhere in $[a,b]$, so from Lemma 7 it follows that

$$\lim_{m\to\infty} S_{2^m}(x) = f(x)$$

is valid almost everywhere and subsequently

$$\lim_{k\to\infty} S_{\mu_k}(x) = f(x). \tag{6.3.1}$$

For every n we write

$$\frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k}\left(S_{v_k}(x) - f(x)\right)^2 \leq$$

$$\leq \frac{2}{2^n} \sum_{k=1}^{n} \binom{n}{k} \left(S_{\nu_k}(x) - S_{\mu_k}(x)\right)^2 +$$

$$+ \frac{2}{2^n} \sum_{k=1}^{n} \binom{n}{k} \left(S_{\mu_k}(x) - f(x)\right)^2 =$$

$$= S_{31} + S_{32} \text{ , say.}$$

In virtue of (6.2.1)

$$S_{32} \longrightarrow 0 \text{ .}$$

Also

$$S_{31} = \frac{2}{2^n} \sum_{k=1}^{n} \binom{n}{k} \left(S_{\nu_k}(x) - S_{\mu_k}(x)\right)^2$$

$$= O(1) \frac{\binom{n}{m}}{2^n} \sum_{k=1}^{n} \left(S_{\nu_k}(x) - S_{\mu_k}(x)\right)^2$$

$$= O(1) \frac{\sqrt{n}\binom{n}{m}}{2^n} \cdot \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(S_{\nu_k}(x) - S_{\mu_k}(x)\right)^2 .$$

$$= O(1)\,\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\left(S_{\nu_k}(x)-S_{\mu_k}(x)\right)^2,$$

by virtue of lemma 1.

As per (6.1.3), (6.1.4) and (6.1.5)

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\int_a^b\left(S_{\nu_k}(x)-S_{\mu_k}(x)\right)^2 dx =$$

$$= O(1)\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\left[\left(c^*_{\nu_k+1}\right)^2+\cdots+\left(c^*_{\mu_k}\right)^2\right]$$

$$= O(1)\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\;\frac{(\mu_k-\nu_k)(\nu_k+1)\left(c^*_{\nu_k+1}\right)^2}{(\nu_k+1)}$$

$$= O(1)\sum_{k=1}^{\infty}\frac{(\nu_k+1)\left(c^*_{\nu_k+1}\right)^2}{\sqrt{k}}$$

$$= O(1)\sum_{k=1}^{\infty}\frac{k\,c_k^{*2}}{\sqrt{k}}$$

$$= O(1)\sum_{k=1}^{\infty}c_k^{*2}\sqrt{k} < \infty$$

results by a simple calculation from which we get by an application of B. Levy's theorem that the series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left(S_{\nu_k}(x) - S_{\mu_k}(x) \right)^2$$

converges almost everywhere. Then by Kronecker's lemma

$$\sum_{k=1}^{n} \left(S_{\nu_k}(x) - S_{\mu_k}(x) \right)^2 = o\left(\sqrt{n}\right)$$

which proves that

$$J_{31} \longrightarrow 0 .$$

With that theorem 1 is completely proved.

6.4 <u>Proof of Theorem 2</u> : We have

$$T_{n+1}^{(\nu)}(x) - T_{n}^{(\nu)}(x) =$$

$$= \frac{1}{2^{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k} S_{\nu_k}(x) - \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} S_{\nu_k}(x)$$

$$= \frac{1}{2^{n+1}} \left[S_{v_{n+1}}(x) + \sum_{k=0}^{n} \left[\binom{n+1}{k} - 2\binom{n}{k} \right] S_{v_k}(x) \right]$$

$$= \frac{1}{2^{n+1}} \left[S_{v_{n+1}}(x) + \sum_{k=1}^{n} \left[\binom{n}{k-1} - \binom{n}{k} \right] S_{v_k}(x) \right],$$

as $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

$$= \frac{1}{2^{n+1}} \left[S_{v_{n+1}}(x) + \sum_{k=1}^{n} C_{nk} S_{v_k}(x) \right],$$

where $C_{nk} = \binom{n}{k-1} - \binom{n}{k}$

$$= \frac{1}{2^{n+1}} \left[\sum_{k=1}^{n+1} C_{nk} \sum_{t=1}^{v_k} c_t \phi_t(x) \right]$$

$$= \frac{1}{2^{n+1}} \left[\sum_{k=1}^{n+1} C_{nk} \sum_{i=1}^{k} \sum_{t=v_i+1}^{v_{i+1}} c_t \phi_t(x) \right].$$

$$= \frac{1}{2^{n+1}} \sum_{i=1}^{n+1} \left(\sum_{t=v_i+1}^{v_{i+1}} c_t \phi_t(x) \right) \sum_{k=i}^{n+1} c_{nk}$$

$$= \frac{1}{(n+1)\, 2^{n+1}} \sum_{i=1}^{n+1} \binom{n+1}{i} i \sum_{t=v_i+1}^{v_{i+1}} c_t \phi_t(x) ,$$

$$\text{as} \sum_{k=i}^{n+1} c_{nk} = \frac{i}{n+1} \binom{n+1}{i} .$$

Thus,

$$\tau_{n+1}^{(v)}(x) - \tau_n^{(v)}(x) =$$

$$= \frac{1}{(n+1)\, 2^{n+1}} \sum_{i=1}^{n+1} \binom{n+1}{i} i \sum_{t=v_i+1}^{v_{i+1}} c_t \phi_t(x) .$$

Therefore with $m_1 = \left[\frac{n+1}{2}\right]$, by virtue of Lemma 1

$$\int_a^b (n+1)\left[\tau_{n+1}^{(v)}(x) - \tau_n^{(v)}(x)\right]^2 dx =$$

$$= \frac{1}{(n+1)\,(2n+1)^2} \sum_{i=0}^{n+1} \binom{n+1}{i}^2 i^2 \sum_{t=v_i+1}^{v_{i+1}} c_t^2$$

$$= O(1)\,\frac{1}{(2n+1)^2} \sum_{i=0}^{n+1} \binom{n+1}{i}^2 i \sum_{t=v_i+1}^{v_{i+1}} c_t^2$$

$$= O(1)\,\frac{\sqrt{(n+1)}\binom{n+1}{m_1}}{2^{n+1}} \cdot \frac{1}{2^{n+1}} \sum_{i=0}^{n+1} \binom{n+1}{i}\sqrt{i} \sum_{t=v_i+1}^{v_{i+1}} c_t^2$$

$$= O(1)\,\frac{1}{2^{n+1}} \sum_{i=0}^{n+1} \binom{n+1}{i}\sqrt{i} \sum_{t=v_i+1}^{v_{i+1}} c_t^2$$

$$= O(1)\,\frac{1}{2^{n+1}} \sum_{i=0}^{n+1} \binom{n+1}{i}\sqrt{i}\; c_i^2$$

Hence

$$\sum_{n=1}^{\infty} \int_a^b (n+1) \left[T_{n+1}^{(v)}(x) - T_n^{(v)}(x) \right]^2 dx =$$

$$= O(1) \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \sum_{i=0}^{n+1} \binom{n+1}{i} \sqrt{i}\, c_i^2 \; .$$

However as is known for numerical series, the convergence of $\sum_{n=0}^{\infty} u_n$ implies the convergence of the series

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{i=0}^{n} \binom{n}{i} u_i$$

to the same sum.

Therefore if we assume the convergence of the series

$$\sum_{K=1}^{\infty} c_K^2 \sqrt{K} < \infty ,$$

then

$$\sum_{n=1}^{\infty} \int_a^b (n+1) \left[T_{n+1}^{(v)}(x) - T_n^{(v)}(x) \right]^2 dx =$$

$$= O(1) \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \sum_{i=0}^{n+1} \binom{n+1}{i} \sqrt{i}\, c_i^2$$

$$= O(1) \sum_{k=1}^{\infty} c_k^2 \sqrt{k} < \infty .$$

Thus theorem 2 is completely proved.

6.5. Proof of Theorem 3.1

Necessity : Assume that the sequence $\{\tau_n^{(v)}(x)\}$ and therefore the sequence $\{\tau_{2^n}^{(v)}(x)\}$ also, are convergent almost everywhere.

In order to show the necessity it suffices to prove that

$$(6.5.1) \qquad \lim_{n\to\infty} \left[S_{v_{2^n}}(x) - \tau_{2^n}^{(v)}(x) \right] = 0$$

For this purpose, we first notice that, applying to the expression $\tau_{2^n}^{(v)}(x)$ the Abel transformation, we may write

$$S_{\nu_{2^n}}(x) - \tau_{2^n}^{(\nu)}(x) =$$

$$= -\frac{1}{2^{2^n}} \sum_{k=0}^{2^n-1} \sum_{\nu=0}^{k} \binom{2^n}{\nu} \sum_{\nu=\nu_k+1}^{\nu_{k+1}} c_\nu \phi_\nu(x) .$$

Hence by Lemma 2 and 3, we have,

$$\sum_{n=0}^{\infty} \int_a^b \left[S_{\nu_{2^n}}(x) - \tau_{2^n}^{(\nu)}(x) \right]^2 dx =$$

$$= O(1) \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \left[\frac{1}{2^{2^n}} \sum_{\nu=0}^{k} \binom{2^n}{\nu} \right]^2 \sum_{\nu=\nu_k+1}^{\nu_{k+1}} c_\nu^2$$

$$= O(1) \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \frac{k^2}{(2^n)^2} \sum_{\nu=\nu_k+1}^{\nu_{k+1}} c_\nu^2$$

$$= O(1) \sum_{k=0}^{\infty} k^2 \sum_{\nu=\nu_k+1}^{\nu_{k+1}} c_\nu^2 \sum_{2^n \geq k} \frac{1}{(2^n)^2}$$

$$= O(1) \sum_{k=0}^{\infty} c_k^2 < \infty .$$

Thus by P. Levy's theorem

$$\sum_{n=0}^{\infty} \left[S_{\nu_{2^n}}(x) - \tau_{2^n}^{(\nu)}(x) \right]^2$$

is convergent almost everywhere. Hence there follows formula (C.8.1), which proves, by virtue of convergence almost everywhere of the sequence $\left\{ \tau_{2^n}^{(\nu)}(x) \right\}$ that the sequence $\left\{ S_{\nu_{2^n}}(x) \right\}$ is convergent almost everywhere.

<u>Sufficiency</u>: It is easily seen from the proof of necessity that the convergence almost everywhere of $\left\{ S_{\nu_{2^n}}(x) \right\}$ implies that of the sequence $\left\{ \tau_{2^n}^{(\nu)}(x) \right\}$.

The only thing remaining to be proved is the validity of the relation

$$\tau_k^{(\nu)}(x) - \tau_{2^n}^{(\nu)}(x) \longrightarrow 0 \quad (n \longrightarrow \infty)$$

for all k between 2^n and 2^{n+1} and for almost every x.

Applying the Cauchy's inequality we find that

$$\left[\tau_k^{(v)}(x)-\tau_{2^n}^{(v)}(x)\right]^2=$$

$$=\left[\sum_{i=2^n+1}^{k}\left(\tau_i^{(v)}(x)-\tau_{i-1}^{(v)}(x)\right)\right]^2\leq$$

$$\leq\sum_{i=2^n+1}^{2^{n+1}} i\left[\tau_i^{(v)}(x)-\tau_{i-1}^{(v)}(x)\right]^2\sum_{i=2^n+1}^{2^{n+1}}\frac{1}{i}\quad.$$

By theorem 3 the first sum on the right side of the last inequality tends to zero almost everywhere and the second one is bounded. Thus we have proved our assertion concerning the sufficiency.

6.6. Proof of Theorem 4 : The sufficiency follows from the relation

$$\left|\tau_n^{(v)}(x)-f(x)\right|\leq\sqrt{\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(S_{v_k}(x)-f(x)\right)^2}$$

To prove the necessity we assume that

$$\mu_k = \nu_{2^m} \quad \text{for} \quad 2^m \leq k < 2^{m+1} \qquad (m = 1, 2, 3, \ldots)$$

Next let us suppose the sequence $\left\{\tau_n^{(\nu)}(x)\right\}$ to be convergent almost everywhere to the function $f(x)$. Hence and from theorem 3 we conclude that the sequence $\left\{S_{\nu_{2^n}}(x)\right\}$ is convergent almost everywhere to the function $f(x)$ and subsequently

$$\lim_{k\to\infty} S_{\mu_k}(x) = f(x) . \tag{6.6.1}$$

Let us observe that

$$\frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k}\left(S_{\nu_k}(x) - f(x)\right)^2 \leq$$

$$\leq \frac{2}{2^n}\sum_{k=1}^{n}\binom{n}{k}\left(S_{\nu_k}(x) - S_{\mu_k}(x)\right)^2 +$$

$$+ \frac{2}{2^n}\sum_{k=1}^{n}\binom{n}{k}\left(S_{\mu_k}(x) - f(x)\right)^2 =$$

$$= S_{61} + S_{62} \text{, say.}$$

In view of (6.6.1)

$$S_{62} \longrightarrow 0 .$$

As in the proof of theorem 1

$$S_{61} = O(1) \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(S_{\nu_k}(x) - S_{\mu_k}(x) \right)^2 .$$

Now

$$\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}} \int_a^b \left(S_{\nu_k}(x) - S_{\mu_k}(x) \right)^2 dx =$$

$$= \sum_{m=0}^{\infty} \sum_{k=2^m+1}^{2^{m+1}} \frac{1}{\sqrt{k}} \left[C^2_{\nu_{2^m+1}} + \cdots + C^2_{\nu_k} \right] \leq$$

$$\leq \sum_{m=0}^{\infty} \left[C^2_{\nu_{2^m+1}} + \cdots + C^2_{\nu_{2^{m+1}}} \right] \sum_{l=1}^{2^m} \frac{1}{\sqrt{l}}$$

$$\leq \sum_{k=2}^{\infty} C_k^2 \sqrt{k} < \infty$$

results by a simple calculation from which we get by an application of P. Levy's theorem that the series

$$\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}} \left(S_{\nu_k}(x) - S_{\mu_k}(x)\right)^2$$

is convergent almost everywhere.

Then by Kronecker's lemma

$$\sum_{k=2}^{n} \left(S_{\nu_k}(x) - S_{\mu_k}(x)\right)^2 = o\left(\sqrt{n}\right)$$

which proves that

$$S_{n1} \longrightarrow 0 .$$

With that we have proved theorem 4.

6.7 <u>Proof of Theorem 5 :</u>

Theorem 5 establishes immediately from theorem 3 and theorem 4.

6.8 <u>Proof of Theorem 6 :</u> Let

$v_1 < v_2 < v_3 \ldots$ be an arbitrary increasing index sequence.

According to theorem 5 we have to show that $\{S_{\nu_{2n}}(x)\}$ is convergent almost everywhere. We first of all notice that by reason of lemmas 5 and 6, the sequence $\{S_{2^m}(x)\}$ converges almost everywhere and then intend to prove that

$$S_{\nu_{2n}}(x) - S_{2^m}(x)$$

converges to zero almost everywhere for $2^m < \nu_{2n} < 2^{m+1}$ as $m \to \infty$. For this purpose let us denote by k and l the lower and upper bound of the set of integers n fulfilling the condition $2^m < \nu_{2n} < 2^{m+1}$, respectively.

Now let $F_r(x)$ be defined by

$$F_0(x) = \frac{S_{\nu_{2k}}(x) - S_{2^m}(x)}{C_0}$$

$$F_r(x) = \frac{S_{\nu_{2k+r}}(x) - S_{\nu_{2k+r-1}}(x)}{C_r}$$

with

$$C_0 = \sqrt{\sum_{i=2^m+1}^{\nu_{2k}} c_i^2}, \qquad C_r = \sqrt{\sum_{i=\nu_{2k+r-1}+1}^{\nu_{2k+r}} c_i^2}$$

$$(r = 1, 2, \ldots, l-k+1)$$

(or respectively, with $C_r = 1$ if on the right hand side every c_r vanishes).

It is obvious that the system $\{F_r(x)\}$ $(r = 1, 2, \ldots, l - k + 1)$ is orthonormal in (a,b). Then by Lemma 8 there exists a function $0 \leq \delta_m(x) \subset L^2(a,b)$ such that

$$1^\circ \quad \max_{0 \leq t \leq l-k+1} \left| \sum_{r=0}^{t} C_r F_r(x) \right| \leq \delta_m(x)$$

$$2^\circ \quad \int_a^b \delta_m^2(x)\, dx = O\left(\log^2(l-k+2)\right) \sum_{r=0}^{l-k+1} C_r^2 .$$

Since $l - k + 1 \leq n$, by (6.1.7) and the definition of we obtain the following estimate :

$$\sum_{m=1}^{\infty} \int_a^b \delta_m^2(x)\, dx =$$

$$= O(1) \sum_{m=1}^{\infty} \log^2(m+1) \sum_{i=2^m+1}^{2^{m+1}} c_i^2$$

$$= O(1) \sum_{i=3}^{\infty} c_i^2 (\log \log i)^2$$

$$= O(1) \sum_{i=1}^{\infty} c_i^2 \sqrt{i} < \infty .$$

The series

$$\sum_{m=1}^{\infty} \delta_m^2(x)$$

is then by the B. Levy's theorem convergent almost everywhere. Hence it follows that $\delta_m(x) \longrightarrow 0$ almost everywhere. Because of

$$\max_{2^m < \nu_{2^n} < 2^{m+1}} \left| S_{\nu_{2^n}}(x) - S_{2^m}(x) \right| = \max_{0 \le t \le l-k+1} \left| \sum_{r=0}^{t} C_r F_r(x) \right| \le \delta_m(x) ,$$

the sequences $\{ S_{2^m}(x) \}$ and $\{ S_{\nu_{2^n}}(x) \}$ with $2^m < \nu_{2^n} < 2^{m+1}$ converge simultaneously almost everywhere. The sequence $\{ S_{\nu_{2^n}}(x) \}$ is then convergent almost everywhere to the function $f(x)$ and from theorem 3 it follows that

$$T_n^{(\nu)}(x) \longrightarrow f(x)$$

almost everywhere, which finally by virtue of theorem 4 completes the proof of theorem C.