ON THE PARTIAL SUMS OF A LACUNARY FOURIER SERIES

By

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1. Let $f(x) \varepsilon L(-\pi, \pi)$ and be periodic with period 2π.

Let

$$(1.1) \qquad \sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x)$$

be the Fourier Series of $f(x)$ with an infinity of gaps (n_k, n_{k+1}) such that $(n_{k+1} - n_k) \to \infty$, as $k \to \infty$.

We shall denote, as usual, by S_n the partial sums of the n^{th} order of (1.1).

The following theorem is known, ([1], p. 256-257, Note 3).

Theorem A: If $f(x)$ is bounded and if (1.1) is the Fourier series of $f(x)$ satisfying Hadamard lacunarity condition *viz.*

$$(1.2) \qquad n_{k+1}/n_k > \lambda > 1,$$

then,

$$(1.3) \qquad S_{n_k} = O(1), \text{ as } k \to \infty.$$

The purpose of this note is to examine the behaviour of the partial sums S_{n_k} when the sequence $\{n_k\}$ satisfies a lacunarity condition appreciably weaker than the condition (1.2) of Hadamard. In fact, we shall consider the sequence $\{n_k\}$ given by

$$(1.4) \qquad n_k = [a^{k^\alpha}], \; a > 1, \text{ and } 1/2 < \alpha \leq 1, \; [a^{k^\alpha}] \text{ being the}$$

greatest integer not greater than a^{k^α}. It is easily seen that the sequence $\{n_k\}$ of (1.4), with $\alpha = 1$, satisfies the Hadamard lacunarity condition (1.2) for all sufficiently large k. On the other hand, when $1/2 < \alpha < 1$, we have

$$n_{k+1}/n_k \to 1, \text{ as } k \to \infty,$$

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$= O(k^{1-\alpha})$, by using the well known result ([2], p. 413, 13.31) that for every positive integer n,

$$(2.4) \qquad \frac{1}{n\pi} \int_0^\pi \frac{\sin^2 \frac{1}{2} nt}{\sin^2 \frac{1}{2} t}\, dt = 1,$$

and observing that n_{k+1}/n_k is bounded.

If $\alpha = 1$, the result (1.3) follows from our result (2.1).

The result (2.1) can be sharpened for a value of x for which the expression $f(x+0)+f(x-0)/2$ is finite. This is done in the following theorem. Let $s = f(x+0)+f(x-0)/2$.

Theorem 2 : Let $n_k = [a^{k^\alpha}]$, where $a > 1$, and $1/2 < \alpha \leqslant 1$. If for such a sequence $\{n_k\}$, the series (1.1) is the Fourier series of a function $f(x)$, then

$$(2.5) \qquad S_{n_k} - s = 0(k^{1-\alpha}), \text{ as } k \to \infty,$$

for a value of x for which the expression $f(x+0)+f(x-0)/2$ is finite.
Proof : By virtue of the lacunarity of the Fourier Series, we have,

$$S_{n_k} - s = \frac{1}{2\pi(n_{k+1} - n_k)} \int_0^\pi \varphi(t) \frac{\sin^2 n_{k+1}\frac{1}{2}t - \sin^2 n_k \frac{1}{2}t}{\sin^2 \frac{1}{2}t}\, dt,$$

where $\varphi(t) = f(x+t) + f(x-t) - 2s/2$.
Hence,

$$|S_{n_k} - s| \leqslant A_1 k^{1-\alpha} \left| \frac{1}{n_{k+1}} \int_0^\pi \varphi(t) \frac{\sin^2 n_{k+1}\frac{1}{2}t}{\sin^2\frac{1}{2}t}\, dt \right|$$

$$+ A_1 k^{1-\alpha} \left| \frac{1}{n_k} \int_0^\pi \varphi(t) \frac{\sin^2 n_k \frac{1}{2}t}{\sin^2 \frac{1}{2}t}\, dt \right|$$

$$= A_1 k^{1-\alpha} I_1 + A_1 k^{1-\alpha} I_2.$$

We shall show that $I_1 = 0(1)$, $I_2 = 0(1)$.
Let us consider I_2. Let $|\varphi(t)| < \varepsilon$, for $0 \leqslant t \leqslant \delta$. It is possible to choose such a $\delta > 0$, since $\varphi(t) \to$ as $t \to 0$.

$$I_2 \leqslant \frac{1}{n_k} \int_0^\delta |\varphi(t)| \frac{\sin^2 n_k \frac{1}{2}t}{\sin^2\frac{1}{2}t}\, dt + \frac{1}{n_k}\int_\delta^\pi |\varphi(t)| \frac{\sin^2 n_k \frac{1}{2}t}{\sin^2 \frac{1}{2}t}\, dt$$

$$= I_3 + I_4.$$

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Reprinted from the Indian Journal of Mathematics,
Vol. 11, No. 3, September 1969, Pages 111-114.

Indian Journal of Mathematics
Vol. 11, No. 3, *September*, 1969

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(*Received October* 10, 1967)

1. Suppose $f(x) \in L\ [-\pi, \pi]$, and periodic with period 2π.

Let

(1.1) $$\Sigma(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) = \Sigma A_{n_k}$$

be the Fourier series of $f(x)$ with an infinity of gaps (n_k, n_{k+1}) such that $n_{k+1} - n_k \to \infty$.

We shall be concerned in this note with the series

(1.2) $$\sum\left(\frac{s_{n_k} - s}{n_k}\right),$$

where $s_{n_k} = \sum_{p=1}^{k} A_{n_p}$ and s is an appropriate number independent of n_k.

Let

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s\}.$$

2. We prove the following theorems.

THEOREM 1. *If $f(x)$ is bounded and if*

(2.1) $$\sum\left(\frac{1}{n_{k+1} - n_k}\right)$$ *is convergent, then* (1.2) *is absolutely convergent*

THEOREM 2. *If* (i) $\frac{n_{k+1}}{n_k} \to 1$, *as* $k \to \infty$,

(ii) $$\omega\left(\frac{\pi}{n_{k+1} - n_k}\right) \log\left(1 - \frac{n_k}{n_{k+1}}\right) = O(1),$$

(iii) $\Sigma \frac{1}{n_k}$ *is convergent,*

then (1.2) *is absolutely convergent, where* $\omega(\delta)$ *is the modulus of continuity of* $f(x)$.

hence the convergence of $\sum\left|\frac{s_{n_k}-s}{n_k}\right|$ follows from the convergence of $\sum\left(\frac{1}{n_{k+1}-n_k}\right)$.

Proof of Theorem 2.

By a method similar to the one used by Tomić[1] and under the conditions of the theorem, we have

$$|s_{n_k}-s|=O(1),$$

$$\therefore \qquad \left|\frac{s_{n_k}-s}{n_k}\right|=O\left(\frac{1}{n_k}\right).$$

Hence the convergence of $\sum\left|\frac{s_{n_k}-s}{n_k}\right|$ follows from the convergence of

$$\sum\frac{1}{n_k}.$$

Proof of Theorem 3.

$$\frac{s_{n_k}-s}{n_k}=\frac{A_{n_1}+A_{n_2}+\ldots+A_{n_k}-s}{n_k}.$$

Now, under the conditions of the theorem, we have[2]

$$a_{n_k}=O\left(\frac{1}{n_k}\right)$$

$$b_{n_k}=O\left(\frac{1}{n_k}\right),$$

and hence

$$A_{n_k}=O\left(\frac{1}{n_k}\right).$$

Thus

$$\left|\frac{s_{n_k}-s}{n_k}\right|\leqslant A\frac{(1/n_1+1/n_2+\ldots+1/n_k)+|s|}{n_k},$$

where A is an absolute constant. Therefore

$$\left|\frac{s_{n_k}-s}{n_k}\right|=O\left(\frac{\log n_k}{n_k}\right).$$

1. Tomić (2).
2. Noble (1).

INDIAN JOURNAL OF MATHEMATICS

(Founded by the late Prof. B. N. Prasad)

PUBLISHED BY

THE ALLAHABAD MATHEMATICAL SOCIETY

LAKSHMI NIWAS, GEORGE TOWN, ALLAHABAD-2 (INDIA)

Editor : Prof. U. N. Singh

Founded and first published in December 1958, the *Indian Journal of Mathematics* is devoted to the publication of original research papers in different branches of Pure and Applied Mathematics and Theoretical Physics. It publishes papers after careful refereeing with the assistance of an Editorial Board consisting of eminent Indian and foreign mathematicians. Having published important research papers by a large number of mathematicians from different parts of the world in its ten volumes, the Journal has now entered its 11th year of existence. With effect from Vol. 11, No. 1, January 1969 issue, the Journal is being published in three numbers each year (January, May and September) instead of two.

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