CHAPTER I

INTRODUCTION

The present thesis is devoted to the study of certain problems relating to convergence, absolute convergence, absolute summability (c,1) and estimates of the partial sums of a lacunary Fourier series.

Let $\{n_k\}$ be a sequence of strictly increasing positive integers.

A lacunary Fourier series corresponding to a 2π-periodic and Lebesgue integrable function f is the trigonometric series

$$\text{(L)} \qquad \sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x)$$

with an infinity of gaps $(n_k\ ,\ n_{k+1})$ such that $(n_{k+1} - n_k) \longrightarrow \infty$ as $k \longrightarrow \infty$, and

$$a_{n_k} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos n_k t \, dt,$$

$$b_{n_k} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin n_k t \, dt.$$

The numbers a_{n_k} and b_{n_k} are called the Fourier coefficients of the function f.

The series

$$(L_1) \qquad \sum_{k=1}^{\infty} (b_{n_k} \cos n_k x - a_{n_k} \sin n_k x)$$

is called the conjugate series of the series (L).

The function

$$(1) \qquad \bar{f}(x) = -\frac{1}{2\pi} \int_0^{\pi} \frac{\psi(t)}{\tan(t/2)}\, dt$$

$$= \lim_{\epsilon \to 0} \left(-\frac{1}{2\pi} \int_{\epsilon}^{\pi} \frac{\psi(t)}{\tan(t/2)}\, dt \right)$$

where

$$\psi(t) = f(x+t) - f(x-t),$$

is called the conjugate function of the function f.

The behaviour of the Fourier series (L) is intimately connected with the behaviour of the sequence $\{n_k\}$.

If the sequence $\{n_k\}$ satisfies the condition

$$(2) \qquad \frac{n_{k+1}}{n_k} > \lambda > 1 \ , \quad \text{for all } k,$$

then it is said to satisfy Hadamard lacunarity condition.

Considerable amount of work has been done with regard to the convergence problems of lacunary Fourier series satisfying Hadamard lacunarity condition, but much remains to be done in respect of the behaviour of the series under less stringent conditions. In the present thesis, we have studied the behaviour of the series under conditions which are less stringent than (2).

This chapter is of introductory character and seeks to give a brief survey of the problems dealt with in the thesis.

Let $[t]$ denote, as usual, the greatest integer not greater than t.

Consider the sequence $\{n_k\}$ defined by

$$(3) \qquad n_k = [a^{k^r}] , \text{ where } a > 1 , 0 < r \leq 1.$$

It can be easily verified that

$$\frac{n_{k+1}}{n_k} \longrightarrow 1 , \text{ as } k \longrightarrow \infty \quad \text{for } o < r < 1,$$

and for $r = 1$, the sequence $\{n_k\}$ in (3) satisfies the condition (2) for all sufficiently large k. Thus a sequence $\{n_k\}$ of the type described in (3) is less restrictive than a Hadamard sequence.

We also consider certain other conditions pertaining to the behaviour of the sequence $\{n_k\}$, which are appreciably weaker than Hadamard's condition (2), and which relate to the behaviour of

$$\frac{N_k}{\log n_k} \quad , \quad \text{as} \quad k \longrightarrow \infty ,$$

where

$$N_k = \min \{ n_k - n_{k-1} , n_{k+1} - n_k \} \quad ,$$

and

$$\frac{n_{k+1} - n_k}{n_k^{\beta} \log n_k} \quad , \quad \text{as} \quad k \longrightarrow \infty ; \quad o < \beta < 1.$$

In order to state precisely the results proved in the present thesis, it is necessary to give some definitions and notations.

Let $\omega(h,f)$ denote the modulus of continuity of a function f in an interval $[a,b]$, i.e.

$$\omega (h,f) = \sup_{|x_2-x_1| \leq h} | f(x_1) - f(x_2) |$$

for $h > o$ and x_1 , $x_2 \in [a,b]$.

The modulus of continuity of f at a point x_o will be defined by

$$\omega (x_o , h, f) = \sup_{|t| \leq h} |f(x_o + t) - f(x_o)| .$$

A function is said to satisfy a Lipschitz condition of order α, $o < \alpha < 1$, in a set E of real numbers, if,

$$\omega(h) = O(h^{\alpha})$$

uniformly for x in E, as $h \to o$ through unrestricted real values.

The fact that f satisfies a Lipschitz condition of order α is expressed in symbols as $f \in \text{Lip}\,\alpha$.

We also write, for $h > o$,

$$\ell_1(h) = \log(e + h^{-1}),$$

$$\ell_2(h) = \log\log(e^{e} + h^{-1}), \quad \ldots . \text{ etc.,}$$

and denote by $S_m(x)$, the partial sums of order m of the series (L). Thus

$$S_{n_k}(x) = \sum_{p=1}^{K}(a_{n_p}\cos n_p x + b_{n_p}\sin n_p x).$$

We shall denote by I a subinterval $|x - x_o| \lesseqgtr \delta$ of $[-\pi, \pi]$.

One of the problems investigated by us in chapter II of the present thesis relates to the estimates of the partial sums S_{n_k} of the series (L). We may recall in this connection the following well known result[1].

---

1) Bary ([2], p. 256-257, note 3)

(1.1) If the sequence $\{n_k\}$ satisfies the Hadamard lacunarity condition (2) and if f is bounded, then

$$(4) \qquad S_{n_k} = O(1), \text{ as } k \longrightarrow \infty .$$

It is natural to ask as to how the partial sums S_{n_k} behave, in general, when the sequence $\{n_k\}$ is of the type (3). One of the first theorems proved by us in chapter II gives an estimate which appears to be very naturally related to the estimate (4). Our theorem is as follows:

(1.2) Let $\frac{1}{2} < r \leq 1$, and let the sequence $\{n_k\}$ be as in (3). If the function f is bounded, then

$$(5) \qquad S_{n_k} = O(k^{1-r}), \text{ as } k \longrightarrow \infty .$$

If $r = 1$, our sequence $\{n_k\}$ satisfies Hadamard's condition (2), as has been pointed out earlier, and (5) gives the same estimate as (4).

It is of some interest to observe that the estimate (5) can be appreciably sharpened in respect of the behaviour of the partial sums S_{n_k} at a point x where the function f is either continuous or has a discontinuity of the first kind. We have proved a theorem in chapter II which states :

(1.3) If the conditions of the theorem (1.2) are satisfied and if, at a point x, $f(x+o)$ and $f(x-o)$ are finite, then

$$S_{n_k}(x) - s(x) = o(k^{1-r}), \text{ as } k \longrightarrow \infty, \tag{6}$$

where $s(x) = f(x+o) + f(x-o)/2$.

In particular,

$$S_{n_k}(x) - f(x) = o(k^{1-r})$$

at a point of continuity of f.

It may be pointed out in this connection that our results regarding the behaviour of the partial sums S_{n_k} of the Fourier series with

$$n_k = [a^{k^r}] \ , \ a > o \ , \text{ and } \ \tfrac{1}{2} < r < 1 \ ,$$

are sharper than the general conclusions deducible from the known results pertaining to the behaviour of partial sums S_n of a Fourier series, viz. $S_n = O(\log n)$ if f is bounded and $S_n = o(\log n)$ when $f(x)$ is continuous at x. Our assertion becomes clear if we observe that according to the general results, applied to the lacunary Fourier series one can atmost conclude that

$S_{n_k} = O(k^r)$, (or in case of continuity, $S_{n_k} = o(k^r)$),
where as our results give the estimate $S_{n_k} = O(k^{1-r})$,
(or in case of continuity $S_{n_k} = o(k^{1-r})$).

The results (5) and (6) have been further improved by us by proving in chapter II the following two theorems (1.4) and (1.5). In these theorems we have been able to replace k^{1-r} by logk in the estimates (5) and (6) of the partial sums. Our theorems are:

(1.4) Let $o < r_1 < r \leqq 1$. Let the sequence be as in (3). If the function f is bounded, then

$$S_{n_k} = O(\log k), \text{ as } k \longrightarrow \infty .$$

(1.5) If the conditions of the theorem (1.4) are satisfied and if, at a point x, $f(x + o)$ and $f(x - o)$ are finite, then

$$S_{n_k}(x) - s(x) = o(\log k), \text{ as } k \longrightarrow \infty.$$

In particular

$$S_{n_k}(x) - f(x) = o(\log k)$$

at a point x of continuity of f.

In a paper published in the year 1954 M. E. Noble[1] studied a lacunarity condition which

---

1) Noble [11]

enabled him to deduce results of general character concerning the behaviour of the Fourier coefficients and the absolute convergence of the lacunary Fourier series (L) under the assumption that the corresponding function f has certain property e.g. being of bounded variation or belonging to Lip α, in a small subinterval of the interval of periodicity. Noble's lacunarity condition makes it possible to relax restrictions on the behaviour of f.

One of the theorems proved by him is the following:

(1.6) If

$$\lim \frac{N_k}{\log n_k} = \infty, \quad \text{as} \quad k \longrightarrow \infty,$$

and if the function f satisfies the Lipschitz condition of order α, $0 < \alpha < 1$ in a subinterval I of $[-\pi, \pi]$, then

(8)
$$a_{n_k} = O(1/n_k^{\alpha}),$$
$$b_{n_k} = O(1/n_k^{\alpha}).$$

Kennedy[1)] has shown that the conclusion (8) holds under the weaker lacunarity condition that $n_{k+1} - n_k \longrightarrow \infty$. The same author, in a subsequent

---

1) Kennedy ([6] ; [7])

paper, has replaced the subinterval by a set E of positive measure, to draw the conclusion (8), but in doing so, the author has replaced the condition $n_{k+1} - n_k \longrightarrow \infty$ by a stronger condition. The following theorem is due to Kennedy.

(1.7) If the sequence $\{n_k\}$ satisfies the condition (2) of Hadamard and if

$$f \in \text{Lip}\, \alpha(E)\ ,\ 0 < \alpha < 1,$$

where E is a set of positive measure, then (8) holds.

Tomić[1], while retaining the condition (2) of Hadamard, replaced the set of positive measure by a single point and studied the behaviour of the Fourier coefficients. In fact, he proved the following theorem.

(1.8) If the sequence $\{n_k\}$ satisfies the condition (2) of Hadamard and if

(9) $$\omega(x_0\ ;\ h) = O(h^{\alpha})\ ,\ 0 < \alpha \leq 1,$$

then

(10) $$a_{n_k} = O(1/n_k^{\beta})\ ,$$

$$b_{n_k} = O(1/n_k^{\beta})\ ;$$

where $\beta = \dfrac{\alpha}{2+\alpha}$.

---

1) Tomić [19]

In chapter III, we have discussed the estimates of the Fourier coefficients of the series (L), and have used these estimates in our study of the problem of absolute convergence and almost everywhere convergence of the series (L).

We have studied the behaviour of the Fourier coefficients by replacing the Hadamard condition (2) by (3), while retaining the condition (9), and have proved the following theorem.

(1.9) Let $o < r_1 < r \leq 1$. Let the sequence $\{n_k\}$ be as in (3). If the function f satisfies the condition (9), then

$$(11) \qquad a_{n_k} = O\left(\frac{k^{2(1-r)}}{n_k^{\beta}}\right),$$

$$b_{n_k} = O\left(\frac{k^{2(1-r)}}{n_k^{\beta}}\right);$$

where $\beta = \dfrac{\alpha}{2+\alpha}$.

If $r = 1$, the theorem (1.8), due to Tomić, and our theorem (1.9) give the same result.

It follows as a corollary that under the conditions of theorem (1.9), the series (L) is absolutely convergent.

Turning to the problem of absolute convergence of the lacunary series, it may be recalled that Zygmund[1)] proved in the year 1928 the following theorem relating to the absolute convergence of a Fourier series.

(1.10) If $f(x)$ is of bounded variation and

$$\omega(x\,;\,h) \leq A \log^{-2-\eta}\left(\frac{1}{h}\right), (\eta > 0, 0 \leq x < 2\pi), \tag{12}$$

then the Fourier series of $f(x)$ converges absolutely.

This theorem does not require any lacunarity condition. Salem[2)] has proved that this theorem is best possible in the sense that η cannot be replaced by zero. While studying the problem of absolute convergence of a lacunary Fourier series Szidon[3)] proved the following theorem.

(1.11) If the sequence $\{n_k\}$ satisfies Hadamard lacunarity condition (2) and if $f(x)$ is bounded, then the series (L) converges absolutely.

Now the following problem can be posed.

Can we replace η by zero in (12) by imposing some lacunarity condition ? Of course, in view of Szidon's theorem (1.11), we have to look for a weaker lacunarity condition than Hadamard's condition (2). In this chapter III, we study this problem and show that for a

---

1) Zygmund [21]
2) Salem [13] .
3) Szidon ([15], [16])

sequence $\{n_k\}$ of the type defined in (3), the conclusion of theorem (1.10) will hold when $\eta = o$, and even more is true. In fact, we prove the following theorem.

(1.12) Let $o < r_1 < r \leq 1$, and let the sequence $\{n_k\}$ be as in (3).

If

$$\omega(x_0, h) = O(\log \tfrac{1}{h})^{-p}, \quad 1 < p \leq 2, \tag{13}$$

then the series (L) is absolutely convergent provided that $\frac{1}{p} < r \leq 1$.

We have also discussed in chapter III the question of almost everywhere convergence of the series (L) mainly because, certain aspects of this question have connections with the chain of ideas developed in this chapter. We have not studied in the present thesis the problem of almost everywhere convergence systematically and in great detail, and one of the theorems proved by us in this respect runs as follows:

(1.13) Let the sequence $\{n_k\}$ be as in theorem (1.12). If p is such that $\frac{1}{2} < p \leq 1$ and the condition (13) is satisfied for this p, then the Fourier series (L) converges almost everywhere provided that $\frac{1}{2p} < r \leq 1$.

Chapter IV of this thesis is devoted to the study of certain questions concerning the absolute convergence of the series (L). In order to explain the significance of the results established in this chapter it is desirable to recall here briefly the developments that have taken place during recent years in respect of the problem of absolute convergence of a Fourier series.

S. Bernstein was one of the first mathematicians who investigated the problem of absolute convergence of a Fourier series from the point of view of relating it to the continuity properties of the generating function in the whole interval periodicity. Bernstein proved the following theorem which is the starting point of a chain of theorems that were subsequently proved.

The following theorem is due to Bernstein[1].

(1.14) If $f \in \mathrm{Lip}\,\alpha$ in $[-\pi, \pi]$, $\frac{1}{2} < \alpha < 1$,

then the Fourier series of $f(x)$ converges absolutely. For $\alpha = \frac{1}{2}$, the series may not converge.

This important theorem of Bernstein gave rise to other theorems of the kind and was improved upon in various ways by authors like Zygmund and others. Later L. Neder[2] proved a theorem which is as follows.

---

1) Bernstein [3]
2) L. Neder [10]

(1.15) If

(14) $$|f(x+h)-f(x)| \leq \frac{Ah^{\alpha}}{\ell_1(h)\,\ell_2(h)\ldots\,\ell_n^{1+\epsilon}(h)}\ , \quad \epsilon > o\ ,$$

$h > o$, in $[-\pi\ ,\ \pi]$, the Fourier series of f converges absolutely for $\alpha = \frac{1}{2}$.

The ideas of Bernstein were applied to the question of absolute convergence of a lacunary Fourier series by M. E. Noble[1] who proved the following theorem.

(1.16) If

(15) $$\lim \frac{N_k}{\log n_k} = \infty\ ,$$

and if $f(x) \in \mathrm{Lip}\ \alpha$, $\frac{1}{2} < \alpha < 1$, in some subinterval I, then the series (L) converges absolutely.

Without the lacunarity condition (15) and with I = $[-\pi\ ,\ \pi]$, Noble's theorem (1.16) reduces to Bernstein theorem (1.14).

We consider a slightly weaker lacunarity condition than Noble's condition (15) and prove the following theorem.

(1.17) If

(16) $$\underline{\lim}\ \frac{N_k}{\log n_k} = B\ ,\quad B \geq \frac{73(3+\epsilon)}{\delta}\ ,$$

---

1) M. E. Noble [11]

and if the condition (14) is satisfied in some subinterval $I = \{x; |x - x_0| \leq \delta, \delta > o\}$ of $[-\pi, \pi]$, then the series (L) converges absolutely for $\alpha = \frac{1}{2}$.

Without the lacunarity condition (16) and with $I = [-\pi, \pi]$, our theorem reduces to the theorem (1.15) of L. Neder.

In this chapter we also prove the following two theorems.

(1.18) If the lacunarity condition (16) is satisfied and if

$$|f(x+h) - f(x)| \leq \frac{Ah^{\alpha}}{[\ell_1(h)\,\ell_2(h)\ldots\ldots\,\ell_m^{1+\epsilon}(h)]^{\frac{2\alpha+1}{2}}} \quad \text{in } I,$$

$o < \alpha < 1$, then

$$(17) \qquad \sum_{k=1}^{\infty}(|a_{n_k}|^t + |b_{n_k}|^t) < \infty \quad ,$$

for $t = \dfrac{2}{2\alpha + 1}$.

(1.19) Under the conditions of theorem (1.17)

$$(18) \qquad \sum_{k=1}^{\infty} n_k^{t-\frac{1}{2}}(|a_{n_k}| + |b_{n_k}|) < \infty ,$$

for $t = \alpha$.

If we omit the lacunarity condition (16) and take the interval $I = [-\pi, \pi]$, then theorems (1.18) and (1.19) are reduced to theorems proved by A.C.Zannen[1].

In order to explain certain other results proved in this chapter we introduce a definition due to Kennedy.

Definition: A subset E of the interval $[-\pi, \pi]$ is said to have positive spread if there is a number $d > o$ such that, for every integer $P > 1$, E contains P points $x_1, x_2, \ldots\ldots, x_P$; satisfying

$$|x_p - x_q| > d\,P^{-1}, \quad (p \neq q).$$

Kennedy[2] discussed the absolute convergence of the series (L) by replacing the subinterval I by a set $E \subset [-\pi, \pi]$ of positive spread ; but in Kennedy's theorem , Noble's lacunarity condition (15) has been replaced by a stronger lacunarity condition. In fact, Kennedy's theorem is as follows:

(1.20) Let

(19)
$$\lim_{k \to \infty} \frac{n_{k+1} - n_k}{n_k^{\beta} \log n_k} = \infty, \quad (o < \beta < 1).$$

Let $f(x) \in \mathrm{Lip}\,\alpha$, $o < \alpha < 1$ in E, a subset of $[-\pi, \pi]$ of positive spread, Then

$$a_{n_k} = O(n_k^{-\alpha\beta}),$$

$$b_{n_k} = O(n_k^{-\alpha\beta});$$

---

1) A. C. Zannen [20]

2) Kennedy [7]

and the series (L) is absolutely convergent if

(20) $$\alpha > \frac{1}{2}(\beta^{-1} - 1).$$

Kennedy has remarked in the same paper that it is an open question as to whether the conclusion of the theorem (1.20) breaks down when $\alpha = \frac{1}{2}(\beta^{-1}-1)$.

The question raised by Kennedy has not yet been answered. However, we have investigated conditions bearing upon the function f under which the series (L) converges absolutely when $\alpha = \frac{1}{2}(\beta^{-1} - 1)$.

Our results is as follows:

(1.21) If the lacunarity condition (19) holds and if the condition (14) holds in E, as $h \to o$, through unrestricted real values, then the series (L) is absolutely convergent for $\alpha = \frac{1}{2}(\beta^{-1} - 1)$.

We also prove the following theorems.

(1.22) Under the conditions of theorem (1.20), (17) holds for

$$t > \frac{1 - \beta}{\alpha\beta + (\frac{1-\beta}{2})} .$$

(1.23) If the lacunarity condition (19) holds and if

$$|f(x+h) - f(x)| = O\left(\frac{h^{\alpha}}{(l_1(h)\, l_2(h) \cdots l_m^{1+\epsilon}(h))^{\frac{2\alpha\beta+(1-\beta)}{2(1-\beta)}}}\right)$$

$\varepsilon > o$, $h > o$, in E, as $h \to o$ through unrestricted real values, then (17) holds for

$$t = \frac{1-\beta}{\alpha\beta + \left(\frac{1-\beta}{2}\right)} .$$

(1.24) Under the conditions of theorem (1.20), (18) holds for

$$t < \frac{1}{2}\beta + \alpha\beta .$$

(1.25) Under the conditions of theorem (1.21), (18) holds for

$$t = \frac{1}{2}\beta + \alpha\beta$$

Masako Sato[1)] discussed the absolute convergence of the series (L) when the function f satisfies some continuity condition at a point, instead of in a small subinterval or in a set of positive spread, and proved the following theorems.

(1.26) Let $o < \alpha < 1$, and $o < \beta < \min\left(1-\alpha , \frac{2-\alpha}{3}\right)$.

If

(21) $$k^{2/2-\alpha-2\beta} < n_k < e^{2k/2+\alpha+\beta}$$

(22) $$| n_{k\pm 1} - n_k | > 4e\, k\, n_k^{\beta} ,$$

---

1) Masako Sato ([8] , [9])

(23) $$\frac{1}{h^{\beta}} \int_0^{h^{\beta}} | f(t) - f(t \pm h)| dt = O(h^{\alpha}),$$

(24) $$\frac{1}{\Gamma} \int_0^{\Gamma} |f(t) - f(t \pm h)| dt = O(1), \text{ unif.in } \Gamma > h^{\beta} ,$$

then

(25) $$a_{n_k} = O(1/n_k^{\alpha}) ,$$

$$b_{n_k} = O(1/n_k^{\alpha}) .$$

(1.27) Let $\frac{1}{2} < a < \alpha < 1$, $o < \beta < (2-\alpha)/3$,

and

$$\frac{\beta}{2} < \alpha - a \leq (2-\alpha-\beta)/4.$$

If

(26) $$k^{1/2\alpha-2a-\beta} < n_k < e^{2k/2+\alpha+\beta} ,$$

(22) is satisfied,

(27) $$\frac{1}{h^{\beta}} \int_0^{h^{\beta}} |f(t) - f(t \pm h)|^2 dt = O(h^{2\alpha}) \quad \text{as } h \to o ,$$

(28) $$\frac{1}{\Gamma} \int_0^{\Gamma} |f(t) - f(t \pm h)|^2 dt = O(1) \text{ unif.in } \Gamma > h^{\beta} ,$$

then, the series (L) converges absolutely.

We also discuss in chapter IV, the absolute convergence of the series (L) only under the conditions of the theorem (1.26). We are also able to cover a greater range of values of α viz. $\frac{1}{2} \leq \alpha < 1$.

We prove the following theorem.

(1.28) Let $\frac{1}{2} \leq \alpha < 1$ and $o < \beta < \min\left(1 - \alpha, \frac{2-\alpha}{3}\right)$.

If the conditions (21), (22),(23) and (24) are satisfied, then the series (L) is absolutely convergent.

We also prove the following theorems.

(1.29) Let $o < \alpha < 1$. Under the hypothesis of theorem (1.28), the conclusion (17) holds $t \geq 1/2\alpha$.

(1.30) Let $o < \alpha < 1$. Under the hypothesis of theorem (1.28), the conclusion (18) is valid for $t \leq \alpha$.

The range $\frac{1}{2} \leq \alpha < 1$ in theorem (1.28) can be extended in the discussion of almost everywhere convergence of the series (L). In this connection, we prove the following theorem.

(1.31) Under the hypothesis of theorem (1.28), the series (L) is almost everywhere convergent for $\frac{1}{4} \leq \alpha < 1$.

In chapter V of this thesis we discuss convergence of the series (L) and (L_1) ; and the absolute convergence of a series associated with the series (L).

The convergence problems of the series (L) and (L_1) have been studied under the condition that $n_{k+1} - n_k \longrightarrow \infty$, imposing some conditions on the behaviour of the function f in a subinterval of $[-\pi , \pi]$.

The theorems that we have proved in this respect are as follows:

(1.32) If $f(x) \in L^2(I)$ then the series (L) and its conjugate series (L_1) are almost everywhere convergent.

(1.33) If f(x) is of bounded variation in some subinterval I, then the series (L) is convergent to $f(x + o) + f(x - o)/2$ at any point where this expression has a meaning and the conjugate series is convergent to $\bar{f}(x)$ whenever it exists, and when x is a point of the Lebesgue set.

Further in this chapter we discuss the absolute convergence of the series

(29)
$$\sum_{k=1}^{\infty} \left(\frac{s_{n_k} - s}{n_k} \right) ,$$

and prove the following theorems.

(1.34) If f(x) is bounded [illegible] and if

$$\sum_{k=1}^{\infty} \left(\frac{1}{n_{k+1} - n_k} \right)$$

is convergent, then the series (29) is absolutely convergent.

(1.35) If

(i) $$\frac{n_{k+1}}{n_k} \to 1 , \quad \text{as} \quad k \to \infty ,$$

(ii) $$\omega \left(\frac{\pi}{n_{k+1} - n_k} \right) \log\left(1 - \frac{n_k}{n_{k+1}} \right) = O(1),$$

(iii) $$\sum_{k=1}^{\infty} \frac{1}{n_k} \quad \text{is convergent,}$$

then the series (29) is absolutely convergent.

(1.36) If $f(x)$ is of bounded variation in some subinterval I , and if

$$\sum_{k=1}^{\infty} \left(\frac{\log n_k}{n_k} \right)$$

is convergent, then the series (29) is absolutely convergent.

(1.37) If $f(x) \in \mathrm{Lip}\, \alpha$, $0 < \alpha < 1$, in some subinterval I , and if

$$\sum_{k=1}^{\infty} \left(\frac{1}{n_k^{\alpha}} \right)$$

is convergent, then the series (29) is absolutely convergent.

It can be seen from the proofs of theorems (1.36) and (1.37) that theorems analogous to these theorems hold for the conjugate series $(\bar{L}_1)$.

Finally, in chapter VI, which is the last chapter of the thesis, we discuss the problem of absolute summability (c , 1) of the series (L) and its conjugate series $(\bar{L}_1)$.

The following theorems have been proved.

(1.38) If

$$\underline{\lim} \; \frac{n_{k+1} - n_k}{\log k} = \beta \quad , \quad \beta > o ,$$

and if f(x) is of bounded variation in some subinterval I, then the series (L) and $(\bar{L}_1)$ are everywhere absolutely summable (c , 1).

(1.39) If

(i) $n_{k+1} - n_k \longrightarrow \infty$,

(ii) $\sum_{k=1}^{\infty} \left(\frac{k}{n_k^2} \right) < \infty$;

and (iii) f(x) is of bounded variation in some subinterval I, then the series (L) and $(\bar{L}_1)$ are everywhere absolutely summable (c , 1).