CHAPTER II

ESTIMATES OF THE PARTIAL SUMS OF A LACUNARY FOURIER SERIES

1. The notations and definitions will be the same as given in the introduction. In particular S_{n_k} will denote the partial sum of the series (L).

It is known[1] that when the sequence $\{n_k\}$ satisfies the Hadamard lacunarity condition namely

$$\frac{n_{k+1}}{n_k} > \lambda > 1 , \tag{1}$$

and if the function f is bounded, then

$$S_{n_k} = O(1) , \text{ as } k \longrightarrow \infty . \tag{2}$$

Our object in the present chapter is to obtain estimates for the partial sums S_{n_k} when the sequence $\{n_k\}$ is given by

$$n_k = [a^{k^r}] , \ a > 1 , \ o < r \leq 1 . \tag{3}$$

$[a^{k^r}]$ being the greatest integer not greater than a^{k^r}.

As has been remarked in the first chapter, for $o < r < 1$ such a sequence $\{n_k\}$ is less restrictive than a Hadamard sequence and it is a Hadamard sequence

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1) Bary [2]

when $r = 1$. This may be seen from the following considerations.

The sequence $\{n_k\}$ in (3), is such that

$$\frac{n_{k+1}}{n_k} \longrightarrow 1 \ , \ \text{as } k \longrightarrow \infty \ ,$$

when $o < r < 1$.

When $r = 1$, it satisfies the Hadamard lacunarity condition (1) for all sufficiently large k.

Firstly, consider the behaviour of

$$P_k = \frac{a^{(k+1)^r}}{a^{k^r}} \ , \ a > 1 \ , \ o < r < 1.$$

We have

$$\log P_k = \{(k+1)^r - k^r\} \log a \ , \text{ and hence}$$

$$\frac{\log P_k}{\log a} = k^r \left\{ (1 + \frac{r}{k} + \frac{r(r-1)}{\underline{|2}} \cdot \frac{1}{k^2} + \ldots\ldots) - 1 \right\}$$

$$= \frac{k^r . r}{k} \left(1 + \frac{r-1}{\underline{|2} \cdot k} + \ldots\ldots \right)$$

$$= \frac{r}{k^{1-r}} \left(1 + \frac{r-1}{\underline{|2} \cdot k} + \ldots\ldots \right) , \ 1 - r > o \ ,$$

$$= \left(\frac{r}{k^{1-r}}\right) . O(1) \ ,$$

hence $\log P_k \longrightarrow o$ as $k \longrightarrow \infty$ and hence $P_k \longrightarrow 1$ as $k \longrightarrow \infty$.

Now, consider the sequence $\{n_k\}$ as defined in (3),

$$n_k = [a^{k^r}] = a^{k^r} - \delta_1 , \quad o \leq \delta_1 < 1 ,$$

and

$$n_{k+1} = [a^{(k+1)^r}] = a^{(k+1)^r} - \delta_2 , \quad o \leq \delta_2 < 1.$$

Therefore

$$\frac{n_{k+1}}{n_k} = \frac{a^{(k+1)^r} - \delta_2}{a^{k^r} - \delta_1}$$

$$= \frac{\dfrac{a^{(k+1)^r}}{a^{k^r}} - \dfrac{\delta_2}{a^{k^r}}}{1 - \dfrac{\delta_1}{a^{k^r}}}$$

Using the fact that $a^{k^r} \longrightarrow \infty$ as $k \longrightarrow \infty$ and

$\dfrac{a^{(k+1)^r}}{a^{k^r}} \longrightarrow 1$ as $k \longrightarrow \infty$, we get $\dfrac{n_{k+1}}{n_k} \longrightarrow 1$ as $k \longrightarrow \infty$.

When $r = 1$ in (3), we have,

$$n_k = [a^k] , \quad a > 1 ,$$

and hence,

$$\frac{n_{k+1}}{n_k} = \frac{a^{k+1} - \delta_2}{a^k - \delta_1} = \frac{a - (\delta_2/a^k)}{1 - (\delta_1/a^k)} \longrightarrow a , \text{ as } k \longrightarrow \infty .$$

Hence there exists $\lambda > 1$, such that

$$\frac{n_{k+1}}{n_k} > \lambda > 1 ,$$

for all sufficiently large k.

Now, we prove the following two theorems.

THEOREM 1. Let

$$n_k = [a^{k^r}] \quad , a > 1 , \text{ and } \quad \tfrac{1}{2} < r \leq 1.$$

If the function f is bounded, then

$$(4) \qquad S_{n_k} = O(k^{1-r}) , \text{ as } k \longrightarrow \infty .$$

THEOREM 2. Let

$$n_k = [a^{k^r}], \ a > 1 , \text{ and } \tfrac{1}{2} < r \leq 1.$$

If at a point x , f(x + o) and f(x - o) are finite and s = f(x + o) + f(x - o)/2 , then for this x

$$(5) \qquad S_{n_k} - s = o(k^{1-r}) , \text{ as } k \longrightarrow \infty .$$

PROOF OF THEOREM 1.

By virtue of the lacunarity of the Fourier series, we have,

$$(6) \qquad S_{n_k} = \frac{1}{2\pi(n_{k+1} - n_k)} \int_0^{\pi} \varphi(t) \frac{\sin^2 n_{k+1}\frac{1}{2}t - \sin^2 n_k \frac{1}{2}t}{\sin^2 \frac{1}{2}t} dt,$$

where $\varphi(t) = f(x + t) + f(x - t)$.

Now, when $n_k = [a^{k^r}]$, $a > 1$, and $o < r_1 < r < 1$, it can be observed that

$$(7) \qquad \frac{1}{n_{k+1} - n_k} < \frac{A\, k^{1-r}}{n_k},$$

for all sufficiently large k , where A is a positive constant independent of k.

It follows from (6) and (7) that

$$|S_{n_k}| \leq \frac{A_1 k^{1-r}}{n_k} \left| \int_0^{\pi} \varphi(t) \frac{\sin^2 n_{k+1} \frac{1}{2} t}{\sin^2 \frac{1}{2} t} \, dt \right|$$

$$+ \frac{A_1 k^{1-r}}{n_k} \left| \int_0^{\pi} \varphi(t) \frac{\sin^2 n_k \frac{1}{2} t}{\sin^2 \frac{1}{2} t} \, dt \right| ,$$

A_1 being a constant which may be different at different occurrences.

The hypothesis implies that $|\varphi(t)| \leq M$, and consequently,

$$|S_{n_k}| \leq A_1 k^{1-r} \cdot \frac{n_{k+1}}{n_k} \cdot \frac{1}{n_{k+1}} \int_0^{\pi} \frac{\sin^2 n_{k+1} \frac{1}{2} t}{\sin^2 \frac{1}{2} t} \, dt$$

$$+ A_1 k^{1-r} \cdot \frac{1}{n_k} \int_0^{\pi} \frac{\sin^2 n_k \frac{1}{2} t}{\sin^2 \frac{1}{2} t} \, dt$$

$$= O(k^{1-r}) + O(k^{1-r})$$

$$= O(k^{1-r}) ,$$

by using the well known[1] result that for every positive integer p,

$$(8) \qquad \frac{1}{p\pi} \int_0^{\pi} \frac{\sin^2 \frac{1}{2}pt}{\sin^2 \frac{1}{2}t}\, dt = 1 ,$$

and observing that $\frac{n_{k+1}}{n_k}$ is bounded.

This proves theorem 1.

If r = 1 , the result (4) gives the same estimate as (2).

The result (4) of this theorem is sharpened, for a value of x for which the expression $f(x + o) + f(x - o)/2$ is finite, in theorem 2.

PROOF OF THEOREM 2.

It follows from (6) that

$$(9) \quad S_{n_k} - s = \frac{1}{2\pi(n_{k+1}-n_k)} \int_0^{\pi} \varphi(t) \frac{\sin^2 n_{k+1}\frac{1}{2}t - \sin^2 n_k \frac{1}{2}t}{\sin^2 \frac{1}{2}t}\, dt,$$

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1) Titchmarsh ([17] , p.413,13·31)

where $\varphi(t) = f(x + t) + f(x - t) - 2s/2.$

Hence,

$$|S_{n_k} - s| \leq A_1 k^{1-r} \left| \frac{1}{n_{k+1}} \int_0^{\pi} \varphi(t) \frac{\sin^2 n_{k+1}\frac{1}{2}t}{\sin^2 \frac{1}{2}t} dt \right|$$

$$+ A_1 k^{1-r} \left| \frac{1}{n_k} \int_0^{\pi} \varphi(t) \frac{\sin^2 n_k \frac{1}{2}t}{\sin^2 \frac{1}{2}t} dt \right|$$

(10) $$= A_1 k^{1-r} I_1 + A_1 k^{1-r} I_2 .$$

Now, we shall show that $I_1 = o(1), I_2 = o(1)$.

Let us consider I_2. Let $|\varphi(t)| < \epsilon$, for $o \leq t \leq \delta$. It is possible to choose such a $\delta > o$, since $\varphi(t) \longrightarrow o$ as $t \longrightarrow o$.

$$I_2 \leq \frac{1}{n_k} \int_0^{\delta} |\varphi(t)| \frac{\sin^2 n_k \frac{1}{2}t}{\sin^2 \frac{1}{2}t} dt$$

$$+ \frac{1}{n_k} \int_{\delta}^{\pi} |\varphi(t)| \frac{\sin^2 n_k \frac{1}{2}t}{\sin^2 \frac{1}{2}t} dt$$

$$= I_3 + I_4 \text{, say.}$$

Now , $I_3 < \epsilon \, \frac{1}{n_k} \int_{0}^{\delta} \frac{\sin^2 n_k \frac{1}{2} t}{\sin^2 \frac{1}{2} t} \, dt$

$< A_1 \, \epsilon$.

Hence, $I_3 = o(1)$.

$$I_4 < A_1 \, \frac{1}{n_k} \int_{\delta}^{\pi} \frac{|\varphi(t)|}{t^2} \, dt \, .$$

Having fixed δ, it is clear that $I_4 \longrightarrow o$ as $k \longrightarrow \infty$.

Hence, $I_2 = o(1)$.

Similarly, we can prove that $I_1 = o(1)$. Therefore, from (10), we have

$$|S_{n_k} - s| = o(k^{1-r}) \, ,$$

which proves the theorem.

As a particular case

$$|S_{n_k} - f(x)| = o(k^{1-r})$$

at a point of continuity of the function.

It will not be out of place to make a remark pertaining to the connection of this result with a known general result relating to the behaviour of the partial sum S_n of the Fourier series of a bounded function. It is well known that for a bounded f, $S_n = O(\log n)$ and if in addition f is also continuous at a point x then $S_n = o(\log n)$. More than this cannot be asserted in general. If we apply these results to a lacunary Fourier series with $\{n_k\}$ as in (3), our conclusion will be that $S_{n_k} = O(k^r)$ when f is supposed to be bounded and $S_{n_k} = o(k^r)$ in the case of continuity, whereas our theorem gives the better estimate that $S_{n_k} = O(k^{1-r})$ and $S_{n_k} = o(k^{1-r})$ respectively.

2. The results (4) and (5) of theorems 1 and 2 respectively have been improved in the following theorems 3 and 4. In these theorems, we have been able to replace k^{1-r} by $\log k$ in the estimates of the partial sums.

THEOREM 3. Let

$$n_k = [a^{k^r}], \quad a > 1, \text{ and } 0 < r_1 < r < 1.$$

If the function f is bounded, then

$$S_{n_k} = O(\log k), \text{ as } k \to \infty. \tag{11}$$

THEOREM 4. Let

$$n_k = [a^{k^r}], \quad a > 1, \text{ and } 0 < r_1 < r < 1.$$

If at a point x , $f(x + o)$ and $f(x - o)$ are finite and $s = f(x + o) + f(x - o)/2$, then for that x

$$S_{n_k} - s = o(\log k), \text{ as } k \to \infty .$$

For proving these theorems, we need a result which we are stating in the form of a lemma.

LEMMA :

$$(11) \quad \frac{1}{n_{k+1} - n_k} \int_0^{\frac{\pi}{n_{k+1} - n_k}} \frac{|\sin^2 n_{k+1} \frac{1}{2} t - \sin^2 n_k \frac{1}{2} t|}{\sin^2 \frac{1}{2} t} \, dt$$

$$\leq A \frac{n_{k+1} + n_k}{n_k} \log \frac{n_{k+1} + n_k}{n_{k+1} - n_k} ,$$

where A is an absolute constant. A will be used as an absolute constant which may be different at different occurrences.

PROOF OF THE LEMMA:

Let

$$n_{k+1} = m , \quad n_k = p .$$

Let

$$I = \frac{1}{m - p} \int_0^{\frac{\pi}{m-p}} \frac{|\sin^2 \frac{1}{2} mt - \sin^2 \frac{1}{2} pt|}{\sin^2 \frac{1}{2} t} \, dt .$$

(12) $$= \frac{2}{m-p} \int_0^{\frac{\pi}{2(m-p)}} \frac{|\sin^2 mt - \sin^2 pt|}{\sin^2 t}\, dt .$$

Put $mt = u$, $\frac{p}{m} = r$.

Then

$$I = \frac{2}{m-p} \int_0^{\frac{\pi m}{2(m-p)}} \frac{|\sin^2 u - \sin^2 ru|}{\sin^2(u/m)} \cdot \frac{du}{m}$$

$$\leq \frac{Am}{m-p} \int_0^{\frac{\pi m}{2(m-p)}} \frac{|\sin^2 u - \sin^2 ru|}{u^2}\, du$$

$$= \frac{Am}{m-p} \int_0^{\frac{\pi m}{2(m-p)}} \frac{|\sin(1-r)u \cdot \sin(1+r)u|}{u^2}\, du .$$

Now, since $\frac{\sin \alpha u}{u} \leq \frac{2\alpha}{1+\alpha u}$, we have

$$I \leq \frac{Am}{m-p} \int_0^{\infty} \frac{(1-r^2)du}{\{1+u(1+r)\}\{1+u(1-r)\}}$$

$$= \frac{Am(1-\frac{p^2}{m^2})}{m-p} \int_0^{\infty} \frac{du}{\{1+u(1+r)\}\{1+u(1-r)\}}$$

$$= A\left(\frac{m+p}{m}\right) \int_0^{\infty} \frac{du}{\{1+u(1+r)\}\ \{1+(1-r)u\}}$$

$$= A\left(\frac{m+p}{m}\right) \cdot \frac{2}{r}\log\left(\frac{1+r}{1-r}\right)$$

$$= A\left(\frac{m+p}{m}\right). \ \frac{m}{p}\log\left(\frac{1+\frac{p}{m}}{1-\frac{p}{m}}\right)$$

$$= A\left(\frac{m+p}{p}\right) \ \log\left(\frac{m+p}{m-p}\right)$$

$$= A\left(\frac{n_{k+1}+n_k}{n_k}\right)\log\left(\frac{n_{k+1}+n_k}{n_{k+1}-n_k}\right).$$

This proves the lemma.

We are now in a position to prove theorem 3.

PROOF OF THEOREM 3:

Using (6), we get,

$$(13)\quad S_{n_k} = \frac{1}{2\pi(n_{k+1}-n_k)}\left\{\int_0^{\frac{\pi}{n_{k+1}-n_k}} + \int_{\frac{\pi}{n_{k+1}-n_k}}^{\delta} + \int_{\delta}^{\pi}\right\}$$

$$\varphi(t)\ \frac{\sin^2 n_{k+1}\frac{1}{2}t - \sin^2 n_k\frac{1}{2}t}{\sin^2\frac{1}{2}t}\,dt$$

$= I_1 + I_2 + I_3$, say.

It results from the hypothesis that $|\varphi(t)| \leq M$.

Hence

$$|I_1| \leq \frac{A}{n_{k+1}-n_k} \int_0^{\frac{\pi}{n_{k+1}-n_k}} \frac{|\sin^2 n_{k+1}\frac{1}{2}t - \sin^2 n_k\frac{1}{2}t|}{\sin^2\frac{1}{2}t}\, dt$$

$$\leq A\left(\frac{n_{k+1}+n_k}{n_k}\right) \log\left(\frac{n_{k+1}+n_k}{n_{k+1}-n_k}\right),$$

by the lemma,

$$= A \log\left(\frac{n_{k+1}+n_k}{n_{k+1}-n_k}\right), \text{ as } \frac{n_{k+1}}{n_k} = O(1).$$

Now, using (7), we get,

(14) $|I_1| = O(\log k)$.

Again,

$$|I_2| \leq \frac{A}{n_{k+1}-n_k} \int_{\frac{\pi}{n_{k+1}-n_k}}^{\delta} \frac{|\sin^2 n_{k+1}\frac{1}{2}t - \sin^2 n_k\frac{1}{2}t|}{\sin^2\frac{1}{2}t}\, dt$$

$$\leq \frac{A}{n_{k+1}-n_k} \int_{\frac{\pi}{n_{k+1}-n_k}}^{\delta} \frac{1}{t^2}\, dt$$

$$= \frac{A}{n_{k+1} - n_k} \left\{ \frac{n_{k+1} - n_k}{\pi} - \frac{1}{\delta} \right\}$$

$$(15) \qquad = O(1)$$

Also

$$| I_3 | \leq \frac{A}{n_{k+1} - n_k} \int_{\delta}^{\pi} \frac{| \sin^2 n_{k+1} \frac{1}{2} t - \sin^2 n_k \frac{1}{2} t |}{\sin^2 \frac{1}{2} t} \, dt$$

$$\leq \frac{A}{(n_{k+1} - n_k) \, \delta^2}$$

$$(16) \qquad = O(1) .$$

Putting (14), (15) and (16) together and using (13), we get,

$$S_{n_k} = O(\log k).$$

This completes the proof of theorem 3.

PROOF OF THEOREM 4:

From (9), we have,

$$S_{n_k} - s = I_1 + I_2 + I_3 ,$$

where I_1, I_2 and I_3 are integrals corresponding to their counterparts in (13).

$$| I_1 | \leq A \frac{\omega(\delta)}{n_{k+1} - n_k} \int_0^{\frac{\pi}{n_{k+1}-n_k}} \frac{|\sin^2 n_{k+1}\frac{1}{2}t - \sin^2 n_k \frac{1}{2}t|}{\sin^2 \frac{1}{2}t} \, dt$$

$$= A\, \omega(\delta) \log \left(\frac{n_{k+1} + n_k}{n_{k+1} - n_k} \right) \text{, by the lemma,}$$

and using the method followed in the proof of theorem 3 we conclude that $I_1 = o(\log k)$.

By the method used in the proof of theorem 3 and by choosing δ first and then making $n_{k+1} - n_k$ sufficiently large, we get $I_2 = o(1)$, $I_3 = o(1)$.

Hence, we finally have,

$$S_{n_k} - s = o(\log k).$$