CHAPTER IV

ON THE ABSOLUTE CONVERGENCE OF A LACUNARY FOURIER SERIES

1. In a paper published in the year 1954 Noble[1)] studied a lacunarity condition which enabled him to deduce results of general character concerning the behaviour of the Fourier coefficients and the absolute convergence of the lacunary Fourier series (L) under the assumption that the corresponding function f has certain property e.g. being of bounded variation or belonging to Lip α , in a small subinterval of the interval of periodicity. Noble's lacunarity condition makes it possible to relax restrictions on the behaviour of f. This approach of Noble gives rise to a question which was, in fact, posed by Noble himself. Suppose that the Fourier series of a function f converges absolutely as a consequence of a certain property P possessed by f in the whole interval $[-\pi , \pi]$. Is it possible to ensure the absolute convergence of the Fourier series of f under the assumption that f satisfies the property P in a subinterval (which may be arbitrarily small) instead of the whole interval $[-\pi , \pi]$, by imposing some lacunarity conditions ? One may further ask as to

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1) Noble [11]

what will be the weakest lacunarity condition which can ensure the absolute convergence of the Fourier series (L).

We have investigated in this chapter a lacunarity condition which is slightly weaker than Noble's condition and which enables us to prove theorems of general character.

Let $I = \{x \; ; \; |x - x_0| \leq \delta \; , \; \delta > 0\}$ denote a subinterval of $[-\pi , \pi]$.

For $h > 0$, let

(1) $$\ell_1(h) = \log(e + h^{-1}) \; ,$$

$$\ell_2(h) = \log\log(e^e + h^{-1}) \; , \; \ldots\ldots \text{ etc.,}$$

Let

(2) $$N_k = \min\,(n_k - n_{k-1} , n_{k+1} - n_k).$$

The following theorem is due to Noble[1].

<u>THEOREM A:</u> If

(3) $$\lim \frac{N_k}{\log n_k} = \infty \; ,$$

and if $f(x) \in \mathrm{Lip}\,\alpha$, where $\frac{1}{2} < \alpha < 1$, in some subinterval I, then the series (L) is absolutely convergent.

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1) Noble [11]

It may be noted that if we omit the lacunarity condition (3) and take the interval $I = [-\pi, \pi]$, then this theorem reduces to a theorem due to Bernstein[1].

In the following, we discuss a condition on f under which the series (L) converges absolutely even when $f \in Lip\ \frac{1}{2}$. Our lacunarity condition is weaker than Noble's condition (3). In fact, we prove the following theorem.

THEOREM 8: If

$$(4) \qquad \underline{\lim} \frac{N_k}{\log n_k} = B \quad , \qquad B \geq \frac{73\ (3 + \epsilon)}{\delta} ,$$

and if,

$$\omega(h) = \omega(h, f) \leq \frac{Ah^{\alpha}}{\ell_1(h)\, \ell_2(h) \ldots \ell_m^{1+\epsilon}(h)} \quad , \epsilon > 0 ,$$

in a closed subinterval I, where $h > 0$, then the series (L) converges absolutely for $\alpha = \frac{1}{2}$.

It may be noted that if we omit the lacunarity condition (4) and take the interval $I = [-\pi, \pi]$ then the theorem 8 is reduced to a theorem due to L.Neder[2].

For proving theorem 8, we require the following lemmas:

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1) Bernstein [3]
2) L. Neder [10]

LEMMA 1 : Let $o < \delta < \pi$ and let m be a positive integer. There exists a trigonometric polynomial

(5) $$T_m(x) = 1 + \sum_{j=1}^{m} t_j \cos jx$$

such that

(6) $$|T_m(x)| < \tfrac{1}{2} A_1 \delta^{-1} , \text{ for all } x ,$$

(7) $$|T_m(x)| < A_2' m^2 \delta^{-1} \exp(-2A_3\delta m), \quad (\delta \leq |x| \leq 2\pi-\delta)$$

where A_1 , A_2' and A_3 are positive absolute constants.

Further if it is supposed that

(8) $$\delta > \frac{3 \log m + 1}{A_3 m} ,$$

then (7) gives the simpler inequality

(9) $$|T_m(x)| < A_2 \exp(-A_3\delta m), (\delta \leq |x| \leq 2\pi-\delta),$$

where A_2 is an absolute constant.

This lemma is due to Noble[1), but it has been stated here in a form which was given to it by Kennedy[2). The constant A_3 can be chosen, as was done by Bary[3), to be 1/8e .

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1) Noble [11]
2) Kennedy [7]
3) Bary ([2] , p.270)

LEMMA 2: If (L) is a Fourier series of $f(x)$ and if $f(x) \in L^2(I)$, then $f(x) \in L^2 [-\pi , \pi]$.

This is a particular case of a very general theorem due to Paley - Wiener[1], theorem XLII'.

PROOF OF THEOREM 8:

Without loss of generality we choose $x_o = o$. We shall prove this theorem for $m = 2$.

Let $n_k > \pi/\delta$. Choose a sequence M_k such that

$$(10) \qquad \frac{18(3 + \epsilon)}{\delta} \log n_k \leq M_k \leq \frac{36(3 + \epsilon)}{\delta} \log n_k .$$

If k is large enough

$$(11) \qquad M_k \leq \frac{1}{2} N_p , \text{ when } n_k \leq n_p .$$

Let

$$(12) \qquad g_k(x) = f(x + \frac{\pi}{2n_k}) - f(x - \frac{\pi}{2n_k})$$

so that $g_k(x)$ has its Fourier series

$$(13) \qquad \sum_{p=1}^{\infty} 2 \sin \frac{n_p \pi}{2n_k} (b_{n_p} \cos n_p x - a_{n_p} \sin n_p x) .$$

Consequently by the choice of M_k , if k is is large enough, then $g_k(x)\, T_{M_k}(x)$ has Fourier coefficients

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1) Paley - Wiener [12]

α_{n_p} , β_{n_p} ; where,

(14)
$$\alpha_{n_p} = 2 \sin\left(\frac{n_p\pi}{2n_k}\right) b_{n_p} ,$$

$$\beta_{n_p} = -2\sin\left(\frac{n_p\pi}{2n_k}\right) a_{n_p} ,$$

for $n_p \geq n_k$.

Now, by the hypothesis f(x) is bounded in I, hence $f(x) \in L^2(I)$ and hence by lemma 2, $f(x) \in L^2[-\pi , \pi]$ and consequently by Bessel's in equality

(15)
$$\sum_{n_k}^{2n_k}(a_{n_p}^2 + b_{n_p}^2)\sin^2 \frac{n_p\pi}{2n_k} = \frac{1}{4} \sum_{n_k}^{2n_k}(\alpha_{n_p}^2 + \beta_{n_p}^2)$$

$$\leq \frac{1}{4\pi} \int_{-\pi}^{\pi} g_k^2(x)\ T_{M_k}^2(x)dx$$

$$= O(\delta^{-2} \int_{|x| \leq \frac{\delta}{2}} |f(x+ \frac{\pi}{2n_k}) - f(x- \frac{\pi}{2n_k})|^2 dx)$$

$$+ O\left(e^{-2A_3'(\delta/2)M_k} \int_{-\pi}^{\pi} |f(x)|^2 dx\right), \text{ by lemma 1,}$$

$$= O\left(\frac{1}{n_k^{2\alpha}\left\{\ell_1\left(\frac{\pi}{2n_k}\right) \quad \ell_2^{1+\epsilon}\left(\frac{\pi}{2n_k}\right)\right\}^2}\right)$$

$$+ O\left(e^{-\frac{\delta}{8e}M_k} \int_{-\pi}^{\pi} |f(x)|^2 dx\right) .$$

Now, using (10), it can be observed that

$$\frac{\delta}{8e} M_k > 2\log n_k ,$$

and hence, we get,

$$e^{-\frac{\delta}{8e}M_k} \int_{-\pi}^{\pi} |f(x)|^2 dx = O(1/n_k^2).$$

Therefore, we get

$$(16) \quad \sum_{n_k}^{2n_k} (a_{n_p}^2 + b_{n_p}^2) = O\left(\frac{1}{n_k^{2\alpha}\left\{\ell_1\left(\frac{\pi}{2n_k}\right) \quad \ell_2^{1+\epsilon}\left(\frac{\pi}{2n_k}\right)\right\}^2}\right).$$

Using Cauchy's inequality, we get,

$$(17)\quad \sum_{2^k}^{2^{k+1}} (|a_{n_p}|+|b_{n_p}|) = O\left(\frac{2^{(\frac{1}{2}-\alpha)k}}{\ell_1(\frac{\pi}{2^{k+1}})\ \ell_2^{1+\epsilon}(\frac{\pi}{2^{k+1}})} \right)$$

$$= O\left(\frac{1}{\ell_1(\frac{\pi}{2^{k+1}})\ \ell_2^{1+\epsilon}(\frac{\pi}{2^{k+1}})} \right), \text{ when } \alpha = \frac{1}{2}.$$

Now, for $k \geq 3$, we have,

$$\ell_1(\frac{\pi}{2^{k+1}}) = \log(e + \frac{2^{k+1}}{\pi})$$

$$> \log 2^{k-1}$$

$$(18)\quad = (k-1)\log 2\ ;$$

$$\ell_2(\frac{\pi}{2^{k+1}}) = \log\log(e^e + \frac{2^{k+1}}{\pi})$$

$$> \log\log 2^{k-1}$$

$$(19)\quad > \frac{1}{2}\log(k-1).$$

Therefore

$$(20)\quad \sum_{2^k}^{2^{k+1}} (|a_{n_p}| + |b_{n_p}|) = O\left(\frac{1}{(k-1)\log^{1+\epsilon}(k-1)} \right).$$

Hence, absolute convergence of the series (L) follows from the convergence of the series

$$\frac{1}{(k-1)\log^{1+\epsilon}(k-1)} .$$

This completes the proof of the theorem.

The following theorems are due to Noble[1].

THEOREM B:

If the lacunarity condition (3) is satisfied and if $f(x) \in \text{Lip}\,\alpha$, $o < \alpha < 1$, in some subinterval I, and $t > 2/2\alpha+1$, then,

(21) $$\sum_{k=1}^{\infty} (|a_{n_k}|^t + |b_{n_k}|^t) < \infty .$$

THEOREM C:

If the lacunarity condition (3) is satisfied and if $f(x) \in \text{Lip}\,\alpha$, $o < \alpha < 1$, in some subinterval I, and $t < \alpha$, then,

(22) $$\sum_{k=1}^{\infty} n_k^{t-\frac{1}{2}} (|a_{n_k}| + |b_{n_k}|) < \infty .$$

It may be noted that if we omit the lacunarity condition (3) and take the interval $I = [-\pi , \pi]$, then theorem B reduces to a theorem due to O. Szasz[2]

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1) Noble [11]
2) O. Szasz [14]

and theorem C reduces to a theorem due to Hardy[1].

In the following, we discuss the conditions on f under which (21) holds for $t = 2/2\alpha+1$ and (22) holds for $t = \alpha$. We also consider the weaker lacunarity condition given in (4).

We prove the following theorems.

THEOREM 9:

If the lacunarity condition (4) is satisfied

and

$$(23) \quad |f(x+h)-f(x)| \leq \frac{Ah^{\alpha}}{\left[\ell_1(h)\ \ell_2(h)\ldots\ \ell_m^{1+\epsilon}(h)\right]^{\frac{2\alpha+1}{2}}} \quad \text{in } I,$$

$0 < \alpha < 1$, then (21) holds for $t = 2/2\alpha+1$.

THEOREM 10:

Under the conditions of theorem 8, (22) holds for $t = \alpha$.

It may be remarked that if we omit the lacunarity condition (4) and take the interval $I = [-\pi, \pi]$, then theorems 9 and 10 reduce to the theorems proved by Zannen[2].

We prove these theorems for $m = 2$.

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1) Hardy [5]

2) Zannen [20]

PROOF OF THEOREM 9:

Let

$$r_{n_p}^2 = a_{n_p}^2 + b_{n_p}^2 .$$

Choosing the sequence M_k as in (10) and using the method of theorem 8, we get, as in (16) above

$$(24) \quad \sum_{n_k}^{2n_k} r_{n_p}^2 = \left(\frac{1}{n_k^{2\alpha} \left\{ \ell_1(\frac{\pi}{2n_k}) \; \ell_2^{1+\epsilon}(\frac{\pi}{2n_k}) \right\}^{2\alpha+1}} \right) .$$

Using (18) and (19), we get,

$$(25) \quad \sum_{2^k}^{2^{k+1}} r_{n_p}^2 \leq \frac{A}{2^{2k\alpha} \left\{ (k-1) \log^{1+\epsilon}(k-1) \right\}^{2\alpha+1}} ;$$

Now, we apply Hölder's inequality to get,

$$\sum_{2^k}^{2^{k+1}} r_{n_p}^t \leq \left(\sum_{2^k}^{2^{k+1}} r_{n_p}^2 \right)^{\frac{t}{2}} \left(\sum_{2^k}^{2^{k+1}} 1 \right)^{1-\frac{t}{2}}$$

$$\leq \frac{A}{2^{k\alpha t} \left\{ (k-1) \log^{1+\epsilon}(k-1) \right\}^{\frac{(2\alpha+1)t}{2}}} \cdot 2^{k(1-\frac{t}{2})}$$

$$= \frac{A}{(k-1)\log^{1+\epsilon}(k-1)}, \text{ for } t = \frac{2}{2\alpha+1} .$$

Therefore, the convergence of $\sum r_{n_p}^t$ follows from the convergence of

$$\sum_{k=3}^{\infty} \frac{1}{(k-1)\log^{1+\epsilon}(k-1)} .$$

Since $|a_{n_p}|^t$, as also $|b_{n_p}|^t$, do not exceed $r_{n_p}^t$, it follows that the series (21) is convergent.

PROOF OF THEOREM 10:

From (17), we have,

$$\sum_{2^k}^{2^{k+1}} (|a_{n_p}| + |b_{n_p}|) = O\left(\frac{2^{(\frac{1}{2} - \alpha)k}}{\ell_1(\frac{\pi}{2^{k+1}})\, \ell_2^{1+\epsilon}(\frac{\pi}{2^{k+1}})} \right) .$$

Therefore,

$$\sum_{2^k}^{2^{k+1}} n_p^{\alpha-\frac{1}{2}} (|a_{n_p}| + |b_{n_p}|) < A \frac{2^{(k+1)(\alpha-\frac{1}{2})} \cdot 2^{k(\frac{1}{2}-\alpha)}}{\ell_1(\frac{\pi}{2^{k+1}})\, \ell_2^{1+\epsilon}(\frac{\pi}{2^{k+1}})}$$

$$< \frac{A}{(k-1)\log^{1+\epsilon}(k-1)} \text{, by (18) and (19).}$$

Hence the convergence of (22), for $t = \alpha$, follows from the convergence of

$$\sum_{k=3}^{\infty} \frac{1}{(k-1)\log^{1+\epsilon}(k-1)} .$$

2 We shall need the following two definitions.

DEFINITION 1. Let E be the set of real numbers, and let $\alpha > o$. We say that $f(x) \in \text{Lip}\ \alpha$ in E, if

$$|f(x+h) - f(x)| = O(|h|^{\alpha})$$

uniformly for x in E, as $h \longrightarrow o$ through unrestricted real values.

DEFINITION 2. A subset E of $[-\pi, \pi]$, is said to have a positive spread if there is a number $d > o$ such that, for every integer $P > 1$, E contains P points $x_1, x_2, \ldots, x_p$; satisfying $|x_p - x_q| > dP^{-1}$, $(p \neq q)$.

Kennedy[1)] discussed the absolute convergence of the series (L) by replacing the subinterval I by a set, a subset E of positive spread. But, in doing so,

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1) Kennedy [7]

Noble's lacunarity condition (3) has been replaced by a stronger lacunarity condition. In fact, the following theorem is proved by Kennedy.

THEOREM D:

Let

(26) $$\lim_{k \to \infty} \frac{n_{k+1} - n_k}{n_k^{\beta} \log n_k} = \infty \quad , \; (o < \beta < 1).$$

Let $f(x) \in \text{Lip}\,\alpha$, $o < \alpha < 1$ in E, a subset of $[-\pi , \pi]$ of positive spread. Then

(27) $$a_{n_k} = O(1/n_k^{\alpha\beta}) ,$$

$$b_{n_k} = O(1/n_k^{\alpha\beta}) ;$$

and the series (L) is absolutely convergent if

(28) $$\alpha > \frac{1}{2} (\beta^{-1} - 1).$$

The author was unable to decide whether the conclusion of the theorem breaks down when

$$\alpha = \frac{1}{2} (\beta^{-1} - 1).$$

Here, we study a condition on $f(x)$ underwhich the series (L) is absolutely convergent for

$$\alpha = \frac{1}{2}(\beta^{-1} - 1).$$

Let P be a positive integer satisfying

$$(29) \quad P < \pi\,\delta^{-1}, \; A_2 P \exp(-A_3 \delta m) < \frac{1}{2} A_1\, \delta^{-1},$$

where $o < \delta < \pi$ and m is a positive integer such that (8) is satisfied.

Further, let the P points $x_1, x_2, \ldots, x_P$ in $[-\pi, \pi]$ satisfy

$$(30) \qquad |x_p - x_q| > 2\delta \quad (p \neq q),$$

and put

$$(31) \qquad S_m(x) = \frac{1}{P} \sum_{\ell=1}^{P} T_m(x - x_l),$$

where T_m is the trigonometric polynomial given in lemma 1.

We shall also need some results pertaining to $S_m(x)$ due to Kennedy[1] which we state in the form of lemmas.

LEMMA 3: (i) $S_m(x)$ is a trigonometric polynomial of degree m at most, with constant term 1 ;

(ii) $|S_m(x)| < A_1(P\,\delta)^{-1}$, for all x ,

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1) Kennedy [7]

(iii) $|S_m(x)| < A_2 \exp(-A_3\delta\, m)$, for all x ,

in $[-\pi , \pi]$ outside the union of the set of intervals $|x - x_\ell| < \delta$, ($\ell = 1,2,3,\ldots,P$);

(iv) $|S_m'(x)| < 2A_1 m(P\,\delta)^{-1}$, for all x.

LEMMA 4: Let P be even and let t_j be as in (5) and let x_ℓ be defined by

$$x_\ell = \frac{(2\ell - P)\pi}{P} , \quad (\ell = 1,2,\ldots P). \tag{32}$$

Then, we have,

$$S_m(x) = 1 + \sum t_j \cos jx , \tag{33}$$

where the summation is over all integers j which are multiple of P and satisfy $1 < j \leq m$.

Now we are in a position to prove the following theorem.

THEOREM 11:

If the lacunarity condition (26) holds and if

$$|f(x+h)-f(x)| = O\left(\frac{h^\alpha}{\ell_1(h)\,\ell_2(h)\ldots\ell_m^{1+\epsilon}(h)}\right) , \epsilon > o,$$

$h > o$, in E, as $h \to o$ through unrestricted real values, then the series (L) is absolutely convergent for

$$\alpha = \frac{1}{2}(\beta^{-1} - 1).$$

PROOF :

We shall prove the theorem for $m = 2$.

Let

(34) $$m_k = \left[A_3^{-1} \; n_k^{\beta} \log n_k\right] .$$

Let

(35) $$\delta_k = 3n_k^{-\beta} \; , \; P_k = \left[c \; n_k^{\beta}\right] ,$$

where $c > o$ is a constant.

If we let $m = m_k$ and $\delta = \delta_k$, then by (34) and (35), (8) is true for all sufficiently large k, since $\beta < 1$. If further, we take c small enough and put $P = P_k$, then, for all sufficiently large k, (29) is true and

(36) $$d \; P_k^{-1} > 2 \; \delta_k \; ,$$

where d is as in definition 2 for the set E.

Let $g_k(x)$ be as in (12). Then the Fourier series of $g_k(x)$ is given by (13).

Choose P_k points $\{x_\ell\}$ ($\ell = 1,2,\ldots.P_k$) from the set E, satisfying $|x_p - x_q| > 2\delta_k$, $(p \neq q)$: This is possible because E has positive spread and hence (36) holds for all sufficiently large k.

Let $S_{m_k}(x)$ be $S_m(x)$ as in (33), with $m = m_k$, $\delta = \delta_k$ and $P = P_k$.

Let

(37) $$m_k < (n_k - n_{k-1}, n_{k+1} - n_k).$$

Consequently, by the choice of m_k , if k is large enough, $g_k(x)\, S_{m_k}(x)$ has the Fourier coefficients α_{n_p} , β_{n_p} ; where

$$\alpha_{n_p} = 2b_{n_p} \sin \left(\frac{n_p\pi}{2n_k}\right),$$

$$\beta_{n_p} = -2a_{n_p} \sin \left(\frac{n_p\pi}{2n_k}\right),$$

for $n_p \geq n_k$.

Now, $f(x) \in L^2[-\pi , \pi]$, using lemma 2, because it results from the hypothesis that f is bounded in a closed subinterval I and consequently $f \in L^2(I)$. Hence $g_k(x)\, S_{m_k}(x) \in L^2\,[-\pi, \pi]$. Consequently, by Bessel's inequality

$$(38) \quad \sum_{n_k}^{2n_k}(a_{n_p}^2 + b_{n_p}^2)\sin^2\left(\frac{n_p\pi}{2n_k}\right) = \frac{1}{4}\sum_{n_k}^{2n_k}(\alpha_{n_p}^2 + \beta_{n_p}^2)$$

$$\leq \frac{1}{4\pi}\int_{-\pi}^{\pi} g_k^2(x) S_{m_k}^2(x)dx.$$

Let E_k be the union of the set of intervals $|x - x_\ell| < \delta_k$, ($\ell = 1, 2, \ldots\ldots, P_k$).

Outside E_k, we have

$$S_{m_k}(x) = O(n_k^{-\beta})$$

by lemma 3.(iii), (34) and (35); and

$$S_{m_k}(x) = O(1)$$

uniformly for x in E_k, by lemma 3(ii), and (35). Therefore

$$(39) \quad \int_{-\pi}^{\pi} g_k^2(x) S_{m_k}^2(x)dx = O\left(\int_{E_k} g_k^2(x)dx\right) + O(n_k^{-6}).$$

But if $|x - x_\ell| < \delta_k$, and k is large enough, then

$$| x \pm \frac{\pi}{2n_k} - x_\ell | < 2\delta_k ,$$

and so, by the condition satisfied by f(x) in E, since $x_\ell \in E$, we have

$$|g_k(x)| \leq |f(x + \frac{\pi}{2n_k}) - f(x_\ell)| + |f(x_\ell) - f(x - \frac{\pi}{2n_k})|$$

$$(40) \qquad = O\left(\frac{\delta_k^\alpha}{\ell_1(\delta_k)\, \ell_2^{1+\epsilon}(\delta_k)} \right)$$

Collecting (35), (38), (39), (40) and observing that

$$\ell_1(\delta_k) = O(\log n_k)$$

$$\ell_2(\delta_k) = O(\log\log n_k) ,$$

for k large enough, we get,

$$(41) \qquad \sum_{n_k}^{2n_k} (a_{n_p}^2 + b_{n_p}^2) = O\left(\frac{n_k^{-2\alpha\beta}}{(\log n_k \cdot \log\log^{1+\epsilon} n_k)^2} \right) ,$$

and from this it follows that

$$(42) \quad \sum_{2^k}^{2^{k+1}} (a_{n_p}^2 + b_{n_p}^2) = O\left(\frac{2^{-2k\alpha\beta}}{(k \cdot \log^{1+\epsilon_k})^2} \right), \text{ for}$$

for k large enough.

Using Cauchy's inequality and noting that, by the lacunarity condition, the number of nonvanishing terms in the sum on the left hand side of (42) is $O(2^{(1-\beta)k})$, we get,

$$(43) \quad \sum_{2^k}^{2^{k+1}} (|a_{n_p}| + |b_{n_p}|) = O\left(\frac{2^{-\alpha\beta k}}{k \cdot \log^{1+\epsilon_k}} \cdot 2^{(1-\beta)k/2} \right)$$

$$= O\left(\frac{2^{(-\alpha\beta + \frac{1-\beta}{2})k}}{k \log^{1+\epsilon} k} \right)$$

$$= O\left(\frac{1}{k \cdot \log^{1+\epsilon_k}} \right);$$

when $\alpha = \frac{1}{2}(\beta^{-1} - 1)$.

From this follows the absolute convergence of the series (L).

This completes the proof of the theorem.

We also prove the following theorems:

THEOREM 12:

Under the conditions of theorem D,

(44) $$\sum_{k=1}^{\infty}(|a_{n_k}|^t + |b_{n_k}|^t)$$

is convergent for

$$t > \frac{1-\beta}{\alpha\beta + (\frac{1-\beta}{2})} .$$

THEOREM 13:

If the lacunarity condition (26) is satisfied and if

(45) $$|f(x+h)-f(x)| = O\left(\frac{h^{\alpha}}{(\ell_1(h)\ \ell_2(h) \dots \ell_m^{1+\epsilon}(h))^{\frac{2\alpha\beta+(1-\beta)}{2(1-\beta)}}} \right),$$

$\epsilon > o$, $h > o$, in E, as $h \to o$ through unrestricted real values, then (44) is convergent for

$$t = \frac{1-\beta}{\alpha\beta + \frac{1-\beta}{2}} .$$

THEOREM 14:

Under the conditions of theorem D,

(46) $$\sum_{k=1}^{\infty} n_k^{t-\frac{1}{2}}(|a_{n_k}| + |b_{n_k}|)$$

is convergent for

$$t < \frac{1}{2}\beta + \alpha\beta.$$

THEOREM 15:

Under the conditions of theorem 11, the series (46) is convergent for

$$t = \frac{1}{2}\beta + \alpha\beta .$$

PROOF OF THEOREM 12:

Let

$$r_{n_p}^2 = a_{n_p}^2 + b_{n_p}^2 .$$

Under the conditions of the theorem, using ((4.7),p.204) of Kennedy[1], we have

$$(47) \qquad \sum_{2^k}^{2^{k+1}} r_{n_p}^2 = O(2^{-2k\alpha\beta}) .$$

Applying Hölder's inequality and noting that the number of nonvanishing terms in the above sum (47) are $O(2^{(1-\beta)k})$, we get,

$$\sum_{2^k}^{2^{k+1}} r_{n_p}^t \leq \left(\sum_{2^k}^{2^{k+1}} r_{n_p}^2 \right)^{t/2} . \quad 2^{(1-\beta)k(1-\frac{t}{2})}$$

$$= O(2^{-k\alpha\beta t + (1-\beta)(1-\frac{t}{2})k})$$

Therefore, the convergence of $\sum r_{n_p}^t$ follows from the convergence of

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1) Kennedy [7]

$$\sum_{k=1}^{\infty} 2^{\{(1-\beta)-(\alpha\beta+\frac{1-\beta}{2})t\}\, k}$$

which does converge when

$$t > \frac{1-\beta}{\alpha\beta + (\frac{1-\beta}{2})} .$$

From this, follows the convergence of the series (44).

PROOF OF THEOREM 13:

We shall prove this theorem for m = 2.

Using (39) and (45), we have,

$$\sum_{2^k}^{2^{k+1}} r_{n_p}^2 = O\left(\frac{2^{-2k\alpha\beta}}{(k\cdot\log^{1+\epsilon} k)^{\frac{2\alpha\beta + (1-\beta)}{(1-\beta)}}} \right)$$

Applying Holder's inequality, as in theorem 12, we get

$$\sum_{2^k}^{2^{k+1}} r_{n_p}^t = O\left(\frac{2^{-k\alpha\beta t}}{(k\cdot\log^{1+\epsilon} k)^{\frac{2\alpha\beta + (1-\beta)}{2(1-\beta)}\cdot t}} \cdot 2^{(1-\beta)k(1-\frac{t}{2})} \right)$$

$$= O\left(\frac{1}{k\cdot\log^{1+\epsilon} k} \right), \text{ when } t = \frac{1-\beta}{\alpha\beta + (\frac{1-\beta}{2})} .$$

From this follows the convergence of the series (44).

PROOF OF THEOREM 14:

From Kennedy[1], we have

$$\sum_{2^k}^{2^{k+1}} n_p^{t-\frac{1}{2}}(|a_{n_p}| + |b_{n_p}|) = O(2^{k(t-\frac{1}{2})} \cdot 2^{(\frac{1}{2} - \frac{1}{2}\beta - \alpha\beta)k})$$

$$= O(2^{(t - \frac{1}{2}\beta - \alpha\beta)k}),$$

and hence the series (46) converges when $t < \frac{1}{2}\beta + \alpha\beta$.

PROOF OF THEOREM 15:

We shall prove this theorem for $m = 2$.

Using (43), we have,

$$\sum_{2^k}^{2^{k+1}} n_p^{t - \frac{1}{2}}(|a_{n_p}| + |b_{n_p}|) = O\left(2^{k(t - \frac{1}{2})} \cdot \frac{2^{(-\alpha\beta + \frac{1 - \beta}{2})k}}{k \cdot \log^{1 + \epsilon} k}\right)$$

$$= O\left(\frac{2^{(t-\alpha\beta-\frac{\beta}{2})k}}{k \cdot \log^{1+\epsilon} k}\right)$$

$$= O\left(\frac{1}{k \cdot \log^{1+\epsilon} k}\right), \text{ when } t = \tfrac{1}{2}\beta + \alpha\beta,$$

and hence the series (46) converges.

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1) Kennedy ([7] , p.204)

3 Masako Sato[1] discussed the absolute convergence of the series (L) where the function f satisfy some continuity condition at a point, instead of in a small subinterval, and proved the following theorems.

THEOREM E:

Let $o < \alpha < 1$, and $o < \beta < \min(1-\alpha\ , \frac{2-\alpha}{3})$.

If

(48) $$k^{2/2-\alpha-2\beta} < n_k < e^{2k/2+\alpha+\beta} ,$$

(49) $$|n_{k\pm 1} - n_k| > 4ek\, n_k^{\beta} ,$$

(50) $$\frac{1}{h^{\beta}} \int_o^{h^{\beta}} |f(t)-f(t \pm h)|dt = O(h^{\alpha}) ,$$

(51) $$\frac{1}{\Gamma} \int_o^{\Gamma} |f(t)-f(t \pm h)|dt = O(1), \text{unif. in } \Gamma > h^{\beta},$$

then

(52) $$a_{n_k} = O(1/n_k^{\alpha}) ,$$

$$b_{n_k} = O(1/n_k^{\alpha}).$$

THEOREM F:

Let $\frac{1}{2} < a < \alpha < 1$, $o < \beta < (2 - \alpha)/3$,

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1) Masako Sato ([8] ; [9])

and $\beta/2 < \alpha - a \lesssim (2 - \alpha - \beta)/4$.

If

(53) $$k^{1/2\alpha-2a-\beta} < n_k < e^{2k/2+\alpha+\beta},$$

(49) is satisfied,

(54) $$\frac{1}{h^\beta}\int_0^{h^\beta} |f(t)-f(t \pm h)|^2 dt = O(h^{2\alpha}) \quad \text{as } h \to o,$$

(55) $$\frac{1}{\Gamma}\int_0^{\Gamma} |f(t)-f(t \pm h)|^2 dt = O(1) \text{ unif.in } \Gamma' > h^\beta,$$

then, the series (L) is absolutely convergent.

We discuss, here, the absolute convergence of the series (L) only under the conditions of theorem E. We are also able to cover a greater range of α i.e. $\frac{1}{2} \leq \alpha < 1$.

More precisely we prove the following theorem.

THEOREM 16:

Let $\frac{1}{2} \leq \alpha \lesssim 1$ and $o < \beta < \min(1-\alpha, \frac{2-\alpha}{3})$.

If the conditions (48),(49),(50) and (51) are satisfied, then the series (L) is absolutely convergent.

We need the following lemma for the proof of the theorem.

LEMMA :

If the sequence $\{n_k\}$ satisfies the condition (49), then, for all sufficiently large k,

(56) $$n_k > ck^{2+\beta} ,$$

where c is an absolute constant.

PROOF: From (49), we have,

$$n_{k+1} - n_k > 4ek\, n_k^{\beta} ,$$

and observing that $n_p > p$ on account of the lacunarity, we get,

$$n_{k+1} - n_1 = \sum_{p=1}^{k}(n_{p+1} - n_p)$$

$$> 4e \sum_{p=1}^{k} p\, n_p^{\beta}$$

$$> 4e \sum_{p=1}^{k} p^{1+\beta} .$$

Therefore

$$n_{k+1} > n_1 + 4e \int_1^k t^{1+\beta} dt$$

$$= n_1 + 4e \left\{ \frac{k^{2+\beta}}{2+\beta} - \frac{1}{2+\beta} \right\}$$

$$= k^{2+\beta} \left\{ \frac{4e}{2+\beta} + \frac{1}{k^{2+\beta}} \left(n_1 - \frac{1}{2+\beta} \right) \right\}$$

$$> \frac{8e}{2+\beta} \quad k^{2+\beta} ,$$

by choosing k large enough. Hence the lemma is proved.

PROOF OF THEOREM 16:

Under the hypothesis of this theorem

$$|a_{n_k}| + |b_{n_k}| = O\left(\frac{1}{n_k^{\alpha}}\right) ,$$

$$= O\left(\frac{1}{k^{2\alpha+\beta\alpha}}\right) , \text{ by the lemma.}$$

Now, our hypothesis implies that $2\alpha \geq 1$ and $\beta\alpha > o$, and hence $\sum\left(\frac{1}{k^{2\alpha+\beta\alpha}}\right)$ is convergent, which implies the convergence of $\sum_{k=1}^{\infty}(|a_{n_k}| + |b_{n_k}|)$. Hence the theorem is proved.

We also prove the following theorems.

<u>THEOREM 17</u>:

Let $o < \alpha < 1$. Under the hypothesis of theorem 16,

$$\sum_{k=1}^{\infty}(|a_{n_k}|^t + |b_{n_k}|^t)$$

is convergent for $t \geq 1/2\alpha$.

THEOREM 18:

Let $o < \alpha < 1$. Under the hypothesis of theorem 16,

$$\sum_{k=1}^{\infty} n_k^{t-\frac{1}{2}}(|a_{n_k}| + |b_{n_k}|)$$

is convergent for $t \leq \alpha$.

PROOF OF THEOREM 17:

We have, under the hypothesis of theorem 16,

$$(|a_{n_k}|^t + |b_{n_k}|^t) = O\left(\frac{1}{n_k^{\alpha t}}\right),$$

and using the lemma

$$(|a_{n_k}|^t + |b_{n_k}|^t) = O\left(\frac{1}{k^{2\alpha t + \beta\alpha t}}\right),$$

for all sufficiently large k.

Our hypothesis implies that $2\alpha t \geq 1$, and $\beta\alpha t > o$, and hence

$$\sum_{k=1}^{\infty} \frac{1}{k^{2\alpha t + \beta\alpha t}}$$

is convergent, which implies the convergence of

$$\sum_{k=1}^{\infty} (|a_{n_k}|^t + |b_{n_k}|^t).$$

PROOF OF THEOREM 18:

We have

$$n_k^{t-\frac{1}{2}}(|a_{n_k}| + |b_{n_k}|) = O(n_k^{t-\frac{1}{2}}/n_k^{\alpha})$$

$$= O\left(\frac{1}{n_k^{\alpha - t + \frac{1}{2}}}\right)$$

$$= O\left(\frac{1}{k^{(2+\beta)(\alpha - t + \frac{1}{2})}}\right),$$

for all sufficiently large k,

$$= O\left(\frac{1}{k^{1 + \frac{\beta}{2} + (\alpha - t)(2+\beta)}}\right).$$

Now, our hypothesis implies that $\alpha - t \geq o$, and $\beta/2 > o$, hence the series

$$\sum_{k=1}^{\infty} \frac{1}{k^{1 + \frac{\beta}{2} + (\alpha - t)(2+\beta)}}$$

is convergent which implies the convergence of

$$\sum_{k=1}^{\infty} n_k^{(t - \frac{1}{2})}(|a_{n_k}| + |b_{n_k}|).$$

4. In theorem 16, we discussed the absolute

convergence of the series (L) for the range $\frac{1}{2} \leq \alpha < 1$. This range can be extended in the discussion of the almost everywhere convergence of the series (L). In this connection we prove the following theorem.

THEOREM 19:

Under the hypothesis of theorem 16, the series (L) is almost everywhere convergent for $\frac{1}{4} \leq \alpha < 1$.

PROOF:

Under the hypothesis of the theorem 16, we have,

$$(a_{n_k}^2 + b_{n_k}^2) = O(1/n_k^{2\alpha})$$

$$= O(1/2^{4\alpha+2\alpha\beta}), \text{ by the lemma.}$$

Now, our hypothesis implies that $4\alpha \geq 1$ and $2\alpha\beta > 0$, which implies the convergence of

$$\sum_{k=1}^{\infty}(a_{n_k}^2 + b_{n_k}^2).$$

Hence $f \in L^2[-\pi, \pi]$. Then by Carleson[1)] theorem, the series (L) is almost everywhere convergent. This completes the proof of the theorem.

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1) Carleson [4]