CHAPTER 4

SIGNAL PROCESSING AND WAVELET TRANSFORMATIONS

Chapter 4. Signal Processing and Wavelet Transformations

4.1. Introduction

"Signal" is a formal representation of phenomenon evolving over time or space[45].

One can define Signal as a function, which is *dependent* on one or more independent variable.[46] When the function depends on a single variable, one can define the signal to be one-dimensional and when the function depends on two or more variables, the signal is describe to be multidimensional[47]. More precisely, "Signal" is a function of time[48] or space.

Given a function *f*, over a duration or time *t*, the values of the signal can be represented as *f(t),* where $t \in [t_0, tn]$ or $t \in [a, b]$ or $t \in [0,\infty)$.

Example 1:

$$f(t) = \cos\left(\frac{2\pi 15t}{16}\right)$$

Example 2:

$$f(t) = \text{Sin}\left(\frac{\pi t^2}{52}\right)$$

[45] Book, P. Prandoni , M. Vetterli, Signal Processing For Communication, EPFL Press, 2008

[46] http: // www.cdeep.iitb.ac.in/nptel/Electrical%20&%20Comm%20Engg/Signals%20and %20System/ Course_home1.2.html

[47] http://nptel.iitm.ac.in/courses/Webcourse-contents/IIT-KANPUR/Digi_Sign_Pro/ui/ Course_home1_1 .htm

[48] Lecture Notes, Stephen P. Boyd, www.stanford.edu/~boyd/ee102/signals.pdf

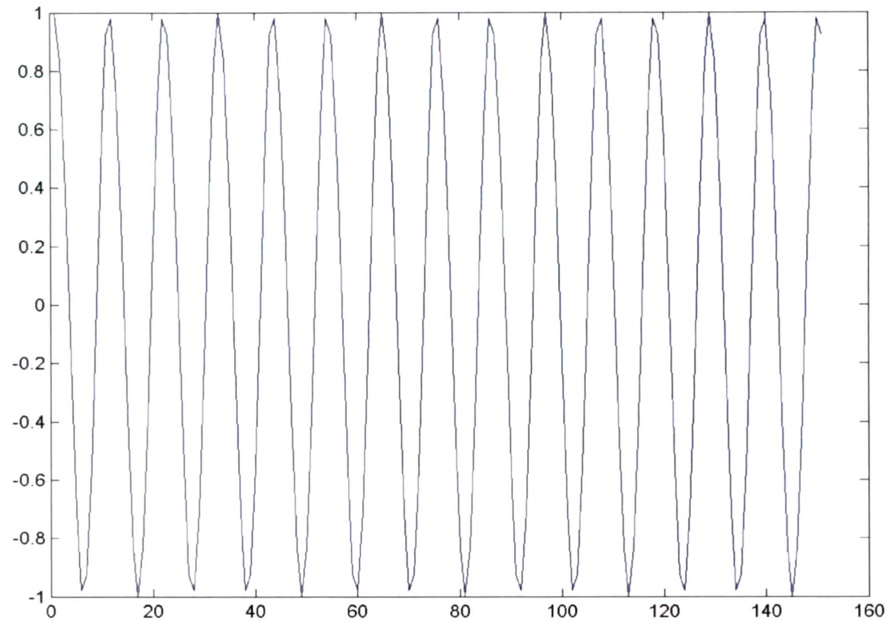

Figure 12. Signal Representation of the function stated in Example 1 - A Stationary Signal

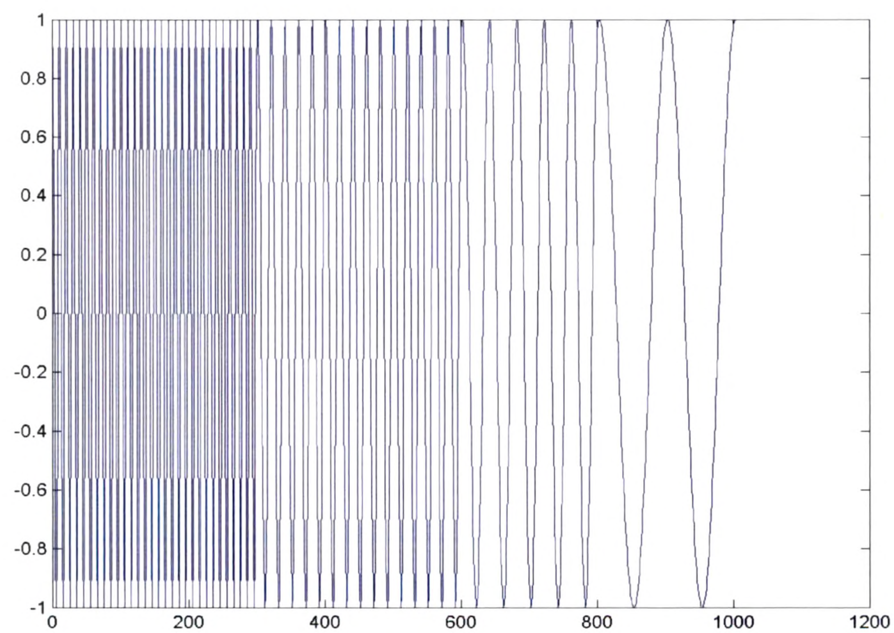

Figure 13. Signal Representation of the function stated in Example 2 - A Non-Stationary Signal

The raw format of the signal is usually represented in time-domain, which means that whatever the signal is measuring, is a function of time.

The plots of such signals, where one of the axes is time and other is the amplitude, give time-amplitude representation of the signal, as shown in Figure 12 and Figure 13.

The Figure 12 representing the function f(t) given in Example 1 is a signal whose frequency contents do not change in time. Such signals are called stationary signals, whereas, in Figure 13 representing function f(t) in Example 2, the frequency contents of the signals change with time. Such signals are known as the "chirp" or non-stationary signals.

The function *f(t)* which may be real or complex, is referred to as dependent variable over the time *t*. The time *t* is referred as an independent variable, which is a real value. The independent variable, *t* means sample time or epoch, not necessarily the actual time in seconds. The range of values of *t* is known as a domain of the signal[49]. If the function has a uniformly sampled points or is equally spaced, then for every t ϵ [t_0, tn], where t_0 is the initial value, the k^{th} element t_k, of a signal can be defined, as shown in [4.1]

$$tk = t0 + kh \quad \text{where, } k = 0, +1, +2, \ldots \qquad [4.1]$$ [50]

In Example 1 and 2 above, If we sample the functions *f(t)* at 0.1 unit time steps, the discrete values of the signals can be generated as shown in Figure 14 and Figure 15.

The sequences of these values can be regarded as discrete time signal or digital signal.

[49] http://nptel.iitm.ac.in/courses/Webcourse-contents/IIT-KANPUR/Digi_Sign_Pro/pdf/ch1.pdf

[50] Lecture Notes, Stephen P. Boyd, www.stanford.edu/~boyd/ee102/signals.pdf

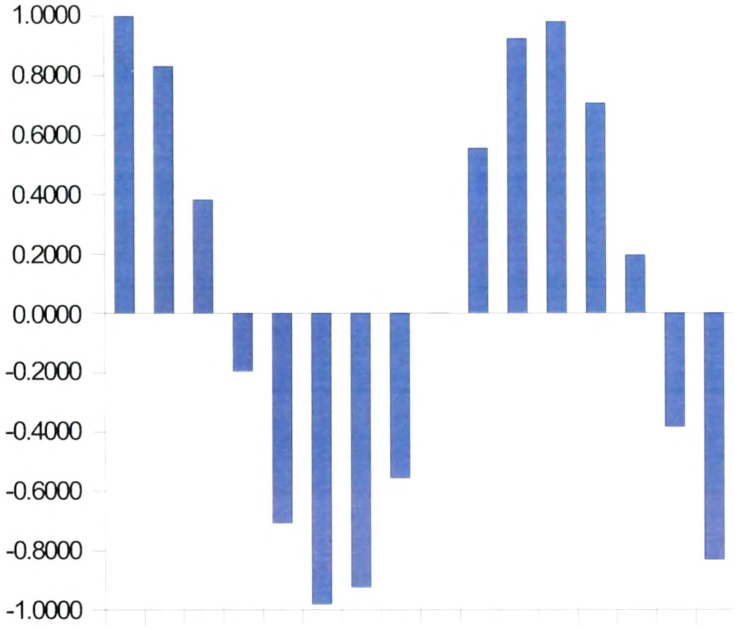

Figure 14. Discrete-Time Signal Representation of the function stated in Example 1

Figure 15. Discrete-Time Signal Representation of the function stated in Example 2

The term "digital" is derived from *digitus*, the Latin word for finger; which means countable; or representation as an integer number[51]. A digital signal, also known as discrete-time signal, is the representation of sequence of discrete values. It refers to the series of numerical values, drawn from the finite set of values permissible for a given application or that it takes values from the countable set[52]. In brief, a digital signal is a signal with amplitude that takes only a finite number of values. In a discrete-time signal, magnitudes are defined at specific instants of time only and are undefined elsewhere.

For example, the valid set of percentage of marks an undergraduate engineering student can score in four years of his study may be like {55, 75, 68, 90]. Here the values cannot be negative, nor can it be greater than 100, and we assume that percentage is given as an integer value. Thus, in the mentioned example, a digital signal f(t) can take values [55, 75, 68, 90]. The co-domain of the signal is [0, 100] i.e. $f(t) \in [0, 100]$ and the values of t = 1,2,3,4 are derived from equally spaced four academic years of engineering study. Here $t_0 = 1$ and each $t_k = t_0 + k*h$ where k = 1,2,3,4 is the element index and the step size *h* is 1. However, it is not necessary that all the academic years are of equal duration; in that case, the value of h may be varying.

Thus, a "discrete signal" is a function of time with values occurring at non-continuous or discrete positions. Equation [4.2] describes a discrete-time signal *f* by,

$$f(t) = f_t \quad \text{Where, } t \in \{0,1,2,...n\} \qquad [4.2]$$

Where, n is a positive integer, which we shall refer to as the length of f.

51 Book, P. Prandoni , M. Vetterli, Signal Processing For Communication , EPFL Press, 2008

52 http://nptel.iitm.ac.in/courses/Webcourse-contents/IIT-KANPUR/Digi_Sign_Pro/pdf/ch1.pdf

The values of f are the n real numbers $f0, f1, f2, . . . , fn$

These values are typically measured values of an analog signal g, measured at the time values $t = t_0, t_1, t_2, \ldots, t_n$.

When the values of f are an equally spaced sample values, it means the increment of time that separates each pair of successive time values is the same in a discrete signal[53].

4.2. Signal Processing, Transformation and Convolution

Raw signal is normally time-domain signal. Raw signal would hence provide limited information. To obtain further information of the signal, it is required to do some processing on the signal.

"Signal Processing" means operating on a signal using some function, to extract out the information preserved in the signal.

"Digital Signal Processing" is based on processing sequences of samples[54]. It refers to the processing of discrete-time signal represented as a sequence of numbers or symbols. Discrete time signals are usually periodic in nature.

Signal processing either continuous or discrete, deals with representation, transformation, or manipulation of a signal and its contained information[55].

"Transform of a signal" is just a different form or a method of representing the signal. Signal transformation does not alter the information content existing in the original signal.

A Transformation[56] can be represented mathematically as a function

[53] A Primer on Wavelets and their Scientific Applications, James S. Walker, Taylor & Francis Group, 2008

[54] ibid., Discrete-Time Signal Processing, Alan v. Oppenheim, Ronald W. Schafer, John R. Buck, Prentice Hall, 2nd Edition, 1998

[55] Discrete-Time Signal Processing, Alan v. Oppenheim, Ronald W. Schafer, John R. Buck, Prentice Hall, 2nd Edition, 1998.

F: X → X

or a function

$$F: X \to Y \qquad [4.3]$$

Or Transformation can also be represented as Y = f(X).

If X is a signal or a sequence consisting of {x(1), x(2), x(3),.....x(n)} i.e. {x(t)} , t ∈ [1,n], n denotes the length of the signal.

Then

$$Y = f(X) \Rightarrow y(t) = f[x(t)], \qquad [4.4]$$

Where,

t denotes the time or position of the signal.

Thus, *"Transformation"*, in terms of mathematics is, any function mapping a signal X to another signal or to itself. The signal X may have some algebraic properties or geometric structure and on "transformation", a function from X to itself, preserves this properties or structure. Briefly, a transformation refers to changing the form of data from one form to another, without changing the fundamental properties or behaviour of the data.

Transformation can be applied to a signal because several Arithmetic operations are possible on a signal[57] such as:

1) Product of two Signals:

$$Z = X.Y \Rightarrow z(t) = x(t).y(t), \text{ for } t \in [0, n-1]$$

2) Multiplication with a Scalar:

[56] Statistics and Computing - The Grammar of Graphics, Leland Wilkinson, Springer, 28-Jan-2006 - Computers - 708 pages

[57] Notes of : Professor David Heeger, Signals, Linear Systems, and Convolution, September 26, 2000

$Z = Z = \alpha X => z(t) = \alpha . x(t),\ \text{for}\ t \in [0.n-1]$

3) Addition of two Signals:

$Z = X + Y => z(t) = x(t) + y(t),\ \text{for}\ t \in [0, n-1]$

4) Shifting of a Signal:

$z(t) = x(t+s),\ \text{for}\ t \in [0, n-1], \text{where s is a shift unit}$

Thus, any digital signal X, may be represented as the sum of scaled and shifted unit impulses.

More precisely, since, an Integral is the limiting case of summation

$$\int_{t=-\infty}^{\infty} x(t).dt = lim_{\delta \to 0} \sum_{k=-\infty}^{\infty} x(k.\delta).\delta \qquad [4.5]$$

Equation [4.5] represents an analog signal as a limiting case of digital signal when the time increment approaches to zero.

A signal X can be expressed as an infinite sum of scaled and shifted unit impulses.

"Linear System" or a "Linear Transformation" is one which satisfies the rule of Homogeneity i.e. Scalar multiplication and Additivity i.e. Addition of two signals. The system which satisfies these two rules of Homogeneity and Additivity are also said to satisfy the *Principle of Superposition*[58].

Shift Invariance: A system is said to be a Shift-Invariant System if and only if,

58 Ibid : Notes of : Professor David Heeger, Signals, Linear Systems, and Convolution, September 26, 2000

$$y(t) = f[x(t)] \Rightarrow y(t-s) = f[x(t-s)] \qquad [4.6]$$

i.e. the output to the signal gets shifted in time when the corresponding shift in time is defined for the input signal, it is defined as the Shift-invariant Linear System.

Eg.: Differentiation is a shift-invariant linear operation.

The three conditions for a shift-invariant linear system for a function y given as

$$y(t) = \frac{d}{dt}x(t)$$

- Shift-invariance: $\frac{d}{dt}[x(t-s)] = y(t-s)$
- Homogeneity: $\frac{d}{dt}[\alpha x(t)] = \alpha . y(t)$
- Additivity: $\frac{d}{dt}[x1(t) + x2(t)] = \text{y}1(t) + y2(t)$

Thus, differentiation can be expressed as convolution.

By definition, the *"Convolution"* is the area of the overlap of two functions[59].

Let f (t) and g(t) be two functions. The convolution of f and g, denoted by f ∗ g, is the function of product of two functions f and g on t ≥ 0, given by,

$$(f * g)(t) = \int_{x=0}^{t} f(x)g(t-x)\, dx \qquad [4.7]$$

If we define f and g as the pair of bounded and integrable functions where f: R → C and g: R→C, then the convolution of the functions f and g, as defined above is given as f*g and can be also stated as in Equation [4.8]

[59] http://www.r2labs.org/references/Convolution.pdf

$$(f * g)(t) = \frac{1}{\sqrt{2\pi}} \int_R f(t-x) g(x) dx \qquad [4.8]$$

For any given signal, the concepts of Signal-processing can be applied in three steps[60]:

- Analysis
- Processing
- Synthesis

The Analysis phase decomposes the signal into its basic components. If we consider the space of all possible signals as a vector space then on decomposing it to cummulation of subspaces, each subspace captures a special feature of the signal.

The Processing phase performs the alteration to the basic components to make it relevant to the application or study the basic components to find the inferences in context of the given application.

The Synthesis phase does the reconstruction of the signal from its basic components, either in altered form or without alteration. If the reconstruction from the basic components happens in such a way that the reconstructed signal looks exactly same as the original signal, then it is said to be perfect reconstruction or lossless synthesis.

If some elements of the original signal are lost from the reconstructed signal, it is called lossy reconstruction. Lossy reconstruction would occur when some of the basic components, which are acquired after analysis phase, are discarded, due to its irrelevance from application point of view. Hence, perfect reconstruction cannot be guaranteed, in such cases.

[60] Lecture Notes : Willard Miller, Introduction to the Mathematics of Wavelets, May 3, 2006

4.2.1. Various types of Transformation of a signal:

- Fourier Transform
- Hilbert Transform
- Short – time Fourier Transform
- Radon Transform
- Wavelet Transform

Each transformation technique has its own purpose and areas of application alongwith advantages & disadvantages.

The Wavelet Transform (WT) is just one of the several ways of representing a signal from one form into another. Wavelet Transform provides the time-scale representation of the signal. Wavelet Transforms is one of the alternatives in signal-processing for analyzing the non-stationary signals. The other examples of transform which are widely used and can be applied for analysis of non-stationary signals are Fourier Transform and Short Time Fourier Transform. The frequency content of a signal can be acquired using these transforms. Most distinguished information is contained in the Frequency content of a signal hence; these transforms prove to be of worth.

4.3. Comparision between Wavelet Transform and Fourier Transform / Short Time Fourier Transform

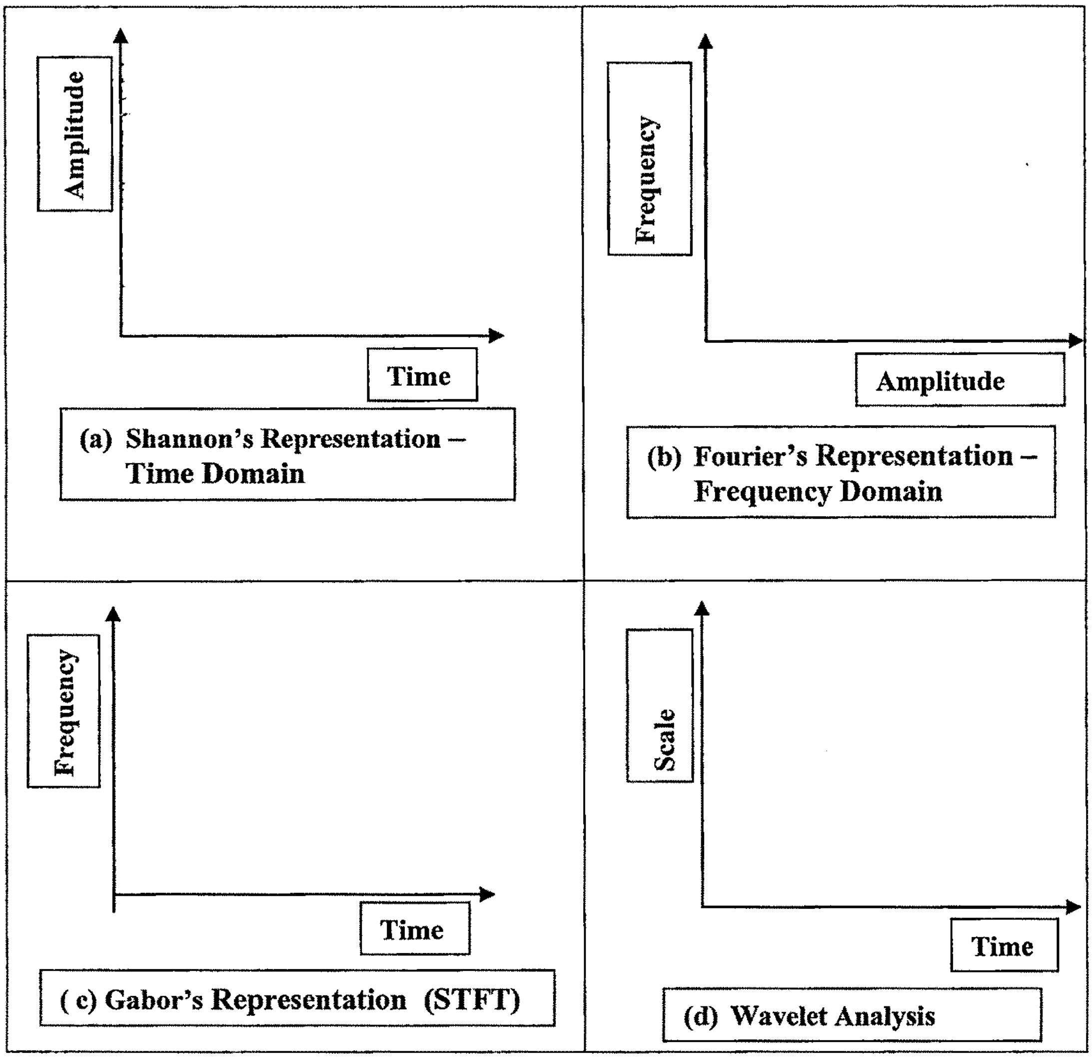

Figure 16. Various domains in which signals are represented

Figure 16. displays various domains in which the signals are represented. The signals can be given as:

- Shannon's Representation (Time Domain): Time vs. Amplitude
- Fourier's Representation (Frequency Domain): Frequency vs. Amplitude

- Gabor's Representation (Short Time Fourier Transform): Time vs. Frequency
- Wavelet Analysis: Time vs. Scale (Scale is an inverse of Frequency)

4.3.1. Similarities and Differences amongst Wavelet Transforms, Fourier Transforms and Short Time Fourier Transforms

- Development of Wavelet Transform happened to overcome the shortcoming of the Fourier Transform and Short-Time Fourier Transform (STFT), which analyze non-stationary signals, with long time intervals, rather than fixed sized small windows.
- Wavelet Transforms represent non-stationary signals in both frequency and time domains[61].
- Wavelet Transforms use functions that are localized in space whereas sine and cosine functions are not, which are used in Fourier Transforms[62].
- When applying Fourier analysis, the time information is lost as Fourier Transform converts a signal from time-domain to frequency-domain. Therefore, it is difficult to identify when a particular event had occurred. Whereas, the Wavelet Analysis, converts the time-amplitude signal into time-scale domain, thus, preserving the time information[63].

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61 "The Wavelet Tutorial", R. Polikar, *Rowan University*, 2001. Website: http://users.rowan.edu/~polikar/WAVELETS/WTtutorial.html

62 Amara Graps, An Introduction to Wavelets

63 http://www.mathworks.com [Matlab documentation].

- STFT, at all frequencies gives a constant resolution whereas the Wavelet Transform uses multi-resolution technique to analyze frequencies with diverse resolutions.
- Wavelet Transforms provides a variable resolution whereas Short-Time Fourier Transform (STFT) uses a fixed resolution at all times.
- Fourier Transform and STFT use waves, more precisely Sine or Cosine, to analyze signals whereas the Wavelet Transform uses wavelets of finite energy. A wave is a function of time or space and is oscillating and periodic function. Eg: Sine Waves. Wavelets conversely, are localized waves, which have their energy concentrated in time or space and are appropriate for analysis of transient signals Eg: Daubechies (db2 or db4) wavelets.
- Fourier analysis consists of breaking up of a signal into sine waves of various frequencies, whereas, Wavelet Analysis consists of breaking up of signal into shifted and scaled version of the mother wavelet.
- Both, the Fast Fourier transform (FFT) and the discrete wavelet transform (DWT) are linear operations that generate a data structure that contains $log_2 n$ segments of various lengths, usually filling and transforming it into a different data vector of length 2^n. Moreover, the mathematical properties of the matrices involved in the transforms are also similar[64].

[64] Amara Graps, An Introduction to Wavelets

- The wavelet analysis is similar to the STFT analysis. In STFT, one needs to multiply the window function with a signal to be analyzed, whereas in Wavelet Transform, the scalar product of signal to be analyzed and a wavelet function, is performed. The transform is computed for every generated segment.
- In STFT, size of the window remains invariable, whereas in Wavelet Transform, the width of the wavelet function, can be altered, with each spectral component.
- In Wavelet Analysis, high frequency components of signals have a short duration while low frequency components have a long duration.
- Discrete Wavelet Transform has the inverse transform matrix, which is the transform of the original signal, same as Fourier Transform. Hence, one can view both transforms as a rotation in function space to a different domain[65].

[65] ibid. Amara Graps, An Introduction to Wavelets

4.4. Wavelet Transformation

Wavelet Transformation is a linear transformation of a signal or data into co-efficients, on a basis of wavelet functions[66]. The wavelet transformations represent the data from one domain into another, from where hidden information can be explored. Wavelet Transform provides Time-Scale information[67].

A Basic Wavelet can be defined as a function $\psi \in L^2(R)$ which satisfies the admissibility conditions

$$C_\psi = \int_{-\infty}^{\infty} \frac{|(\hat{\psi}(x))|^2}{|w|} dw < \infty \qquad [4.9]$$

Where,

w refers to the frequency window

C_ψ refers to the continuous wavelet function

ψ is called a wavelet.

$\psi \in L^2(R)$ satisfies the admissibility condition as $t\,\psi(t) \in L^2(R)$ hence we can prove that $|t|^{1/2}\,\psi \in L^2(R)$ and $\psi \in L^1(R)$. Moreover, it proves that it is a continuous function, so the finiteness of C_ψ implies $\hat{\psi}(0) = 0$ or equivalently $\int_{-\infty}^{\infty} |\psi(t)|\, dt = 0$, as per Fourier Transforms.

Thus, ψ is called a wavelet.

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66 . J.K. Meher, M. R. Panigrahi, G. N. Dash, P. K. Meher, "Wavelet Based Lossless DNA Sequence Compression For Faster Detection Of Eukaryotic Protein Coding Regions ," I.J. Image, Graphics And Signal Processing 2012, 47-53

67 . I. Daubechies, "Ten Lectures On Wavelets", 1992

Wavelet Transforms apply a wavelet function that consists of two parameters, the translation parameter, and the scale parameter. The translation parameter *t* denotes the location of the wavelet function while it is shifted through the signal. Thus, it denotes the time or positional details in the Wavelet Transform.

The scaling parameter *s* corresponds to frequency information. Scaling either expands a signal, also known as dilation, or compresses a signal, also known as reduction. The scale parameter *s* is inversely proportional to the frequency of the signal i.e. *s α 1/frequency*. Large scales are associated with low frequencies and small scales are associated with high frequencies. Large scales expand the signal and provide detailed information hidden in the signal, while small scales condense the signal and provide global information about the signal. Low scales last for a short time in form of short bursts, in a signal, whereas large scales usually exist throughout the duration of the signal.

Thus, Wavelet Analysis deals with extraction of information from the signal at different positions/time and scales[68].

There are 3-ways of describing the Wavelets:

- Signal Processing - Convolution with Basis Functions (A. V. Haar)
- Multiresolution Analysis[69] (MRA) or Vector Space – (Ingrid Daubechies)
- Multirate Filtering / Filter Bank[70]

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68 Introduction to Wavelet Analysis, G. H. Watson, Paper presented at the RTO SC1 Lecture Series on 'Application of Mathematical Signal Processing Techniques to Mission Systems", held in Kiiln, Germany, 1-2 November 1999; Paris, France, 4-5 November 1999; Monterey, USA, 9-10 November 1999, and published in RTO EN-7.

69 D. Lee Fugal, Conceptual Wavelets in Digital Signal Processing, 2009 Space & Signals Technologies LLC, All Rights Reserved. www.ConceptualWavelets.com

Wavelet Transform of a signal performs the convolution of the signal to be analyzed with the basis function.

Thus, Wavelet Transform W_T can also be represented as in [4.10],

$$W_T = X.W, \quad [4.10]$$

Where,

X is the original signal

$$W = [\phi(t); \psi(t)]$$

Wavelet function $f(t)$ consists of convolution of the basis functions $\phi(t)$ and $\psi(t)$ as in [4.11]

$$W = f(t) = \phi(t) . \psi(t) \quad [4.11]$$

Relative to every basic wavelet the Continuous Wavelet Transform on $L^2(R)$ is defined by

$$(W_\psi f)(b,a) = |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t)\overline{\psi\left(\frac{t-b}{a}\right) d(t), f \in L^2 (R)} \quad [4.12]$$

Where,

$a, b \in R$, with $a \neq 0$

By assigning $\psi_{b,a} = |(a)|^{-\frac{1}{2}} \psi(\frac{t-b}{a})$

The Continuous Wavelet Transform can be defined using Equation [4.12] as

$$W_\psi f\,(b,a) = \langle f, \psi_{b,a} \rangle \quad [4.13]$$

[70] Ibid. D. Lee Fugal, Conceptual Wavelets in Digital Signal Processing, 2009 Space & Signals Technologies LLC, All Rights Reserved. www.ConceptualWavelets.com

Thus, the Wavelet Transform of a function f is the convolution of the function with the mother wavelet.

The series exapansion of a signal or function *f(x)* can often be better analyzed as a linear combination of expansion functions.

$$f(x) = \sum_k \alpha_k \phi_k(x) \qquad [4.14]$$

Where ,

k is an integer index of the finite or infinite sum

α_k are real valued expansion co-efficients

ϕ_k are real valued expansion functions or scaling functions.

As per Shannon's Sampling Theorem, every band limited signal can be perfectly recovered from its discrete sample, provided that the sampling period is sufficiently small.

If we consider the set of expansions composed of integer translates and binary scaling of the real, square integrable function $\phi_k(x)$ as the set of $\{(\phi_{j,k}(x)\}$

Where,

$$\phi_{j,k}(x) = 2^{\frac{1}{2}}\, \phi(2^j x - k) \;\; \forall j, k \in Z \text{ and } \phi(x) \in L^2(R) \qquad [4.15]$$

Where,

k determines the position of $\phi_{j,k}(x)$ along the X-axis

j determines width of $\phi_{j,k}(x)$ along the X-axis

$2^{\frac{1}{2}}$ controls the height or amplitude of $\phi_{j,k}(x)$

By selecting $\phi(x)$ appropriately, $\{\phi_{j,k}(x)\}$ can be made to span $L^2(R)$

A function f can be uniquely expressed as $f(x) = f_a(x) + f_d(x)$.

Where,

$f_a(x)$ represents an approximation of f(x) using scaling function

$f_d(x)$ represents the difference $f(x) - f_a(x)$

The two expansions divide the f(x), similar to a lowpass and highpass filters.

The low frequencies of f(x) are captured in $f_a(x)$ and the high frequencies details are expressed in $f_d(x)$.

Henceforth, we represent approximate co-efficients $f_a(x)$ as Ca and detail co-efficients $f_d(x)$ as Cd.

For a Discrete wavelet representation, the wavelet series expansion of function f(x) $\epsilon\, L^2(R)$, relative to the mother wavelet ψ(x) and scaling function ϕ(x) can be generated in accordance with decomposition $L^2(R)$ using Vector Space concept or Multi Resolution Analysis (MRA). For any scaling function ϕ(x) that meets the requirement of MRA, we can define the wavelet function ψ(x) that, alongwith its integer translates and binary scale, covers the difference between any two adjacent scaling subspaces.

Therefore, for any function $f(x) \in L^2(R)$, using Equation [4.13], [4.14] and [4.15] we can write

$$f(x) = \sum_{k \epsilon Z} C_{a_{j_0}}(k)\phi_{j_{0,k}}(x) + \sum_{j= j_0}\sum_{k \epsilon Z} C_{d_j}(k)\psi_{j,k}(x) \quad [4.16]$$

The $Ca_{j0}(k)$'s are called the approximation or scaling coefficients and $Cd_j(k)$'s are referred to as detail or wavelet coefficients, and can be obtained through convolution of a signal function with mother wavelet.

Thus, if the expansions functions form an orthonormal basis or tight frame, which is often the case, the expansion coefficients are calculated as:

$$C_{a_{j_0}}(k) = \langle f(x), \phi_{j_{0,k}}(x) \rangle = \int_{-\infty}^{\infty} f(x)\ \phi_{j_{0,k}}(x)\, dx \qquad [4.17]$$

$$C_{a_j}(k) = \langle f(x), \psi_{j,k}(x) \rangle = \int_{-\infty}^{\infty} f(x)\ \psi_{j,k}(x)\, dx \qquad [4.18]$$

If the function being expanded is a sequence of numbers, like samples of a continuous function f(x), the resulting coefficients are called the discrete wavelet transform (DWT) of f(x). In this Scaling and Wavelet coefficients becomes the DWT transform pair as

$$W_\phi(j_0, k) = \frac{1}{\sqrt{n}} \sum_x f(x) \phi_{j_{0,k}}(x) \qquad [4.19]$$

$$W_\psi(j, k) = \frac{1}{\sqrt{n}} \sum_x f(x) \psi_{j,k}(x) \qquad [4.20]$$

For $j \geq j_0$

$$f(x) = \frac{1}{\sqrt{n}} \sum_k W_\phi(j_0, k)(x)\ \phi_{j_{0,k}}(x) + \frac{1}{\sqrt{n}} \sum_{j=j_0}^{\infty} \sum_k W_\psi(j, k)(x) \psi_{j,k}(x)$$

[4.21]

In the Equation [4.21], the f(x), $\phi_{j_{0,k}}$(x) , $\psi_{j,k}$ (x) are functions of discrete variable x = 0, 1, 2, ... , (n-1). Usually, j_0 takes value 0 and n is selected to be the power of 2.

Thus, n = 2^J

Hence the summations are performed over the discrete values of the signal at

x = 0, 1, 2, ... (n-1)

For resolution levels as

j = 0, 1, 2, ... (J-1)

And co-efficient vectors at each resolution level as

k = 0, 1, 2, ... 2^j -1

Note that the integrations in the series expansions have been replaced by summations, and normalizing factor $\frac{1}{\sqrt{n}}$ has been added to both the forward and inverse expressions.

This factor could alternatively be incorporated only into one of the either forward or inverse transforms as $\frac{1}{n}$.

These equations are valid only for orthonormal basis and tight frame, and not for bi-orthogonal basis.

Thus, the wavelet transform is an inner product of the time series with the scaled and translated wavelet function.

ϕ(t) is called a scaling function and ψ(t) is known as a mother wavelet function. On performing convolution of these basis functions with the original signal X, wavelet co-efficients W_T are obtained. The scaling function ϕ(t) is utilized to obtain the approximation co-efficient C_a and the wavelet function ψ(t) is utilized to obtain the detail co-efficient C_d.

Since, the Wavelet Transform is used to decompose the signal into two sub-vectors – the approximation co-efficient vector and the detail co-efficient vector, this phase of Wavelet Transform is known as an "***Analysis Phase***". The wavelet co-efficients are later on used to synthesize the original signal.

When a Wavelet Transform is performed on a signal, the signal is decomposed into primarily two co-efficient vectors, the approximation co-efficient vector and the detail co-efficient vector. The stages of decomposition are known as "Resolution Level" The co-efficient vectors capture trends at different resolution levels[71]. These co-efficient vectors of different resolution levels represent the characteristics of the data, at each different scale.

In an analysis phase of Wavelet Transform, performing wavelet transform on a given signal X consists of passing through low-pass and high-pass decomposition filters and down sampling by 2 which in turn, generate two co-efficient vectors.

[71] Charu C. Aggarwal, "On The Use Of Wavelet Decomposition For String Classification," Springer - Data Mining And Knowledge Discovery, 10, 117-139, 2005

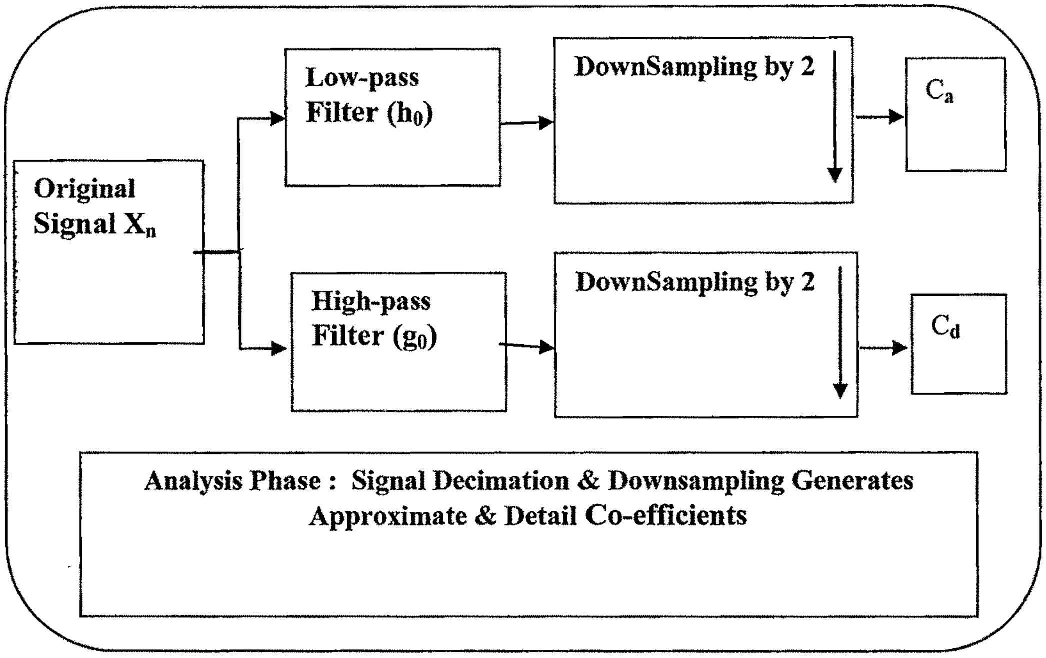

Figure 17. Wavelet Transform (Analysis Phase)

As shown in

Figure 17, the original signal is denoted by the sequence X_n, where n is an integer number referring to the number of elements in the signal X. The low-pass filter is denoted by h_0 while the high-pass filter is denoted by g_0. At each level, the low-pass filter associated with scaling function produces coarse approximations C_a, while the high-pass filter produces detail co-efficients C_d[72]. The number of co-efficients generated will be approximately half the length of the original input on passing the signal simultaneously through low-pass and high-pass filters and subsequently performing down-sampling.

72 MATLAB manual: http://www.mathworks.in/products/wavelet

When a discrete-time signal X passes through low-pass filters (scaling functions) and high-pass filters (wavelet functions) simultaneously, it is defined as performing the "Discrete Wavelet Transform" (DWT). In DWT, a time-scale representation of the digital signal is obtained on passing a signal through filters with different cut-off frequencies at different scales. ***"Filters"*** are the functions used in signal processing. The filtering operations performed on a signal determine the resolution of the signal. The resolution is a measure of the amount of detail information in the

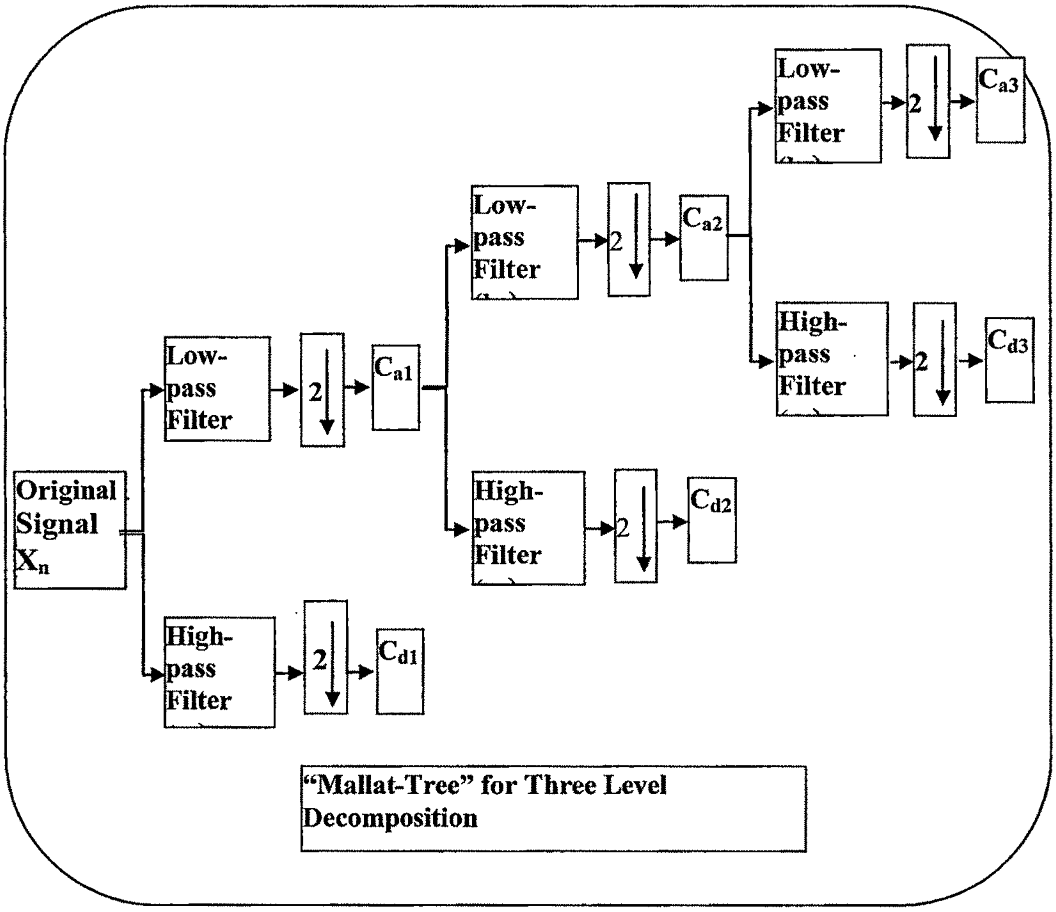

Figure 18. "Mallat-Tree" for Three Level Wavelet Decomposition

signal. The sub-sampling operations (i.e. up-sampling and down-sampling) determine the scale.

When a signal X_n of length ***n***, is transformed using the Discrete Wavelet Transform then, the transformation process consists of at most $\log_2 n$ stages[73]. Thus, there are at most $\log_2 n$ resolution levels in DWT.

If the length of each filter is equal to 2N and let n = length(X_n), then the signals H and G generated after passing through the low-pass filter and high-pass filter respectively, are of length n+ 2N - 1, and the length of co-efficients Ca_1 and Cd_1, as shown in [4.18] is

$$Length\ (Cd1) = \left\lfloor \frac{(n-1)}{2} + N \right\rfloor \quad [4.22]$$

Where,

n, is the length of the original signal

2N, is the length of a filter, N = {1, 2, 3....}

Since, the length of a signal and the length of the basis-vectors are different, multi-level decomposition is essential to decompose the entire signal, using the available basis vector[74]. The Mallat-Tree decomposition[75] is obtained when one computes Discrete Wavelet Transform by successive low-pass and high-pass filtering of the discrete time-domain signal.

4.5. Characteristics of Wavelet Transforms

[73] Ibid, MATLAB manual: http://www.mathworks.in/products/wavelet

[74] Musawir Ali, An Introduction to Wavelets and the Haar Transform (www.cs.ucf.edu/~mali/haar/)

[75]

The following characteristics of Wavelet Transforms have made it possible to use these transforms in variety of applications. The characteristics are:

- The Wavelet Transforms follow the Nyquist rule[76]. The Nyquist criterion states that, the minimum sampling rate that allows reconstruction of the original signal is 2ω radians, where ω is the highest frequency in the signal. Therefore, as the scale goes higher (lower frequencies), the sampling rate can be decreased thus reducing the number of computations. Thus, according to the Nyquist theorem, the highest frequency a signal can properly hold is half the number of samples per second in the signal[77].
- Multi-Resolution Analysis of Wavelet Transforms is possible. The resolution is the measure of amount of detail information in the signal. Wavelet Transforms can be acquired by applying iteration of filters with rescaling.
- The low pass filter associated with scaling function produces coarse approximations, a[n], whereas the high pass filter produces fine approximations.
- Wavelet Transforms has the ability to reduce distortion in the reconstructed signal while retaining all the significant features present in the signal.

[76] D. Lee Fugal, Conceptual Wavelets in Digital Signal Processing, Space and Signals Technologies, 2006

[77] http://courses.ae.utexas.edu/ase463q/design_pages/fall02/wavelet/4_wavelet_theory.htm

Compaction of Energy is possible through Wavelet Transforms[78]. Eg: The First level Haar Wavelet transform distributes the energy of a signal in such a way that over 98% of energy is concentrated into the sub-signal Ca which is just half the length of the original signal. Further level of decomposition to second and third level reduces the energy level to 90% and 88% respectively. The second level of transform reduces the length of the signal to one-fourth and third level to one-eigth of the length of the original signal[79].

Thus, Wavelet Transforms are suitable for compaction or reduction of data, without much loss of information. (The slight reduction in energy levels is due to **Heisenberg's Uncertainty Principle in Quantum Theory,** which describes that it is impossible to localize a fixed amount of energy into an arbitrarily small time interval.)

4.6. Haar Wavelet Transform in Decomposition Phase

The Haar Transform is the simplest and oldest compact, dyadic, orthonormal wavelet transform[80]. The Haar wavelet is discontinuous with a Haar function being an odd rectangular pulse pair, with compact support that provides good possibility for local analysis of signal[81].

Haar wavelets are conceptually simple, fast and memory efficient[82] [83], can be computed in place, without a temporary buffer, are exactly

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78 A Primer on Wavelets and their Scientific Applications, James S. Walker, Taylor & Francis Group, 2008

79 ibid. A Primer on Wavelets and their Scientific Applications, James S. Walker, Taylor & Francis Group, 2008

80 Moharir P.S., "Pattern recognition transforms,"New York: Wiley, 1992

81 Radomir S. Stankovic, Bogdan J. Falkowski, "The Haar wavelet transform: its status and achievements,"Elsevier

82 Ibid[28] Radomir S. Stankovic, Bogdan J. Falkowski, "The Haar wavelet transform: its status and achievements,"Elsevier

reversible and can be perfectly reconstructed, if no elements of co-efficient vectors are discarded at some threshold values. Since at each transform, only half the values of the given signal needs to be processed, its performance efficiency is linear in time and space, excluding other overheads.

Applying the Haar Transform on a discrete signal X_n involves passing the signal X_n through the two filters, a low-pass filter (h_0) and the high-pass filter (g_0) and consequent down-sampling by two. (*"Downsampling by 2"* means discarding every second signal sample. For example a sequence of numbers (signal) [6 4 2 1] becomes [6 2] or [4 1] depending on where one begins with). This is also referred as *"Decimation by 2"*[84] *in* wavelets' terminology). This decomposes the original signal X_n into two sub-signals of half its length. One sub-signal is a running average or trend (C_a); the other sub-signal is a running difference or fluctuation (C_d)[85].

$$X_n \xrightarrow[\rightarrow]{Haar\ Wavelet} (C_a \mid C_d)$$

Given a one-dimensional data vector

$$X[n] = [18,\ 16,\ 6,\ 6,\ 12,\ 20,\ 4,\ 12] \quad [4.23]$$

Where, X is having number of elements n = 8, which is usually the power of 2.

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83 Ruiz G, Michell JA, Buron A, "Switch-level fault detection and diagnosis environment for MOS digital circuits using spectral techniques," IEEE Proc Part E, 1992; 139(4):293-307

84 D. Lee Fugal, Conceptual Wavelets in Digital Signal Processing, Space and Signal Technologies, 2006

85 I. Daubechies, "Orthonormal Bases of Compactly Supported Wavelets,"Comm. Pure Appl. Math, Vol 41, 1988, pp. 906-966

Iteratively performing pairwise averaging and semi differencing[86] can compute Haar Wavelet Transform of the signal X.

The Approximate Coefficient can be obtained as shown in Equation [4.24]

$$Ca = (X_{2i-1} + X_{2i}) / \sqrt{2}, \quad for\ i \in [1,2,3,\ldots n/2] \quad [4.24]$$

And Detailed Coefficient can be obtained as,

$$Cd = (X_{2i-1} - X_{2i}) / \sqrt{2} \quad \mathbf{for\ i \in [1,2,3,\ldots n/2]} \quad \mathbf{[4.25]}$$

More precisely, Computing of Haar Wavelet Transform is done by convolving the Signal X with the basis vector $< \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} >$.

For a given signal, X_n, the first level decomposition will create the Approximate co-efficient at level A_1 and the Detailed co-efficient at level D_1. On further decompositions, we can get co-efficient at level A_2, D_2 and A_3, D_3 etc., which is presented in Table 5.

As shown in Table 5, we can observe that, the Haar Transform W_T of the original signal X_n as given in Equation [4.23] is:

$$W_T = [47/\sqrt{2},\ -1/\sqrt{2},\ 11,\ 8,\ 2/\sqrt{2},\ 0,\ -8/\sqrt{2},\ -8/\sqrt{2}] \quad [4.26]$$

As shown in Table 5, the decomposition is possible until a single element remains in the vector of approximate co-efficients, if n is in power of 2. If

86 Dimitris Sacharidis, "Constructing Optimal Wavelet Synopses", Proceedings of the 2006 International Conference on Current Trends in Database Technology EBDT '06, Pg 97-104, 2006 10.1007/11896548_10

n is not in the power of two, the decomposition is possible until the number n divisible by 2.

More appropriately, if n is the length of the original signal then if $2^j \approx n$ or $2^j <= n$ then signal can be decomposed or resolved upto j levels[87] i.e. the number of decompositions possible of this signal is j, if n is divisible by 2 for j times.

The output co-efficients are arranged in the vector W_T in the "Mallat Order"[88].

87 A Primer on Wavelets and their Scientific Applications, James S. Walker, Taylor & Francis Group, 2008

88 R

Table 5. Representation of Computations of Haar Wavelet Transform

Transformation Level or Decomposition Level (j)	Scale a = 2^j	Resolution r = 1/a)	Length of Signal (L)	Approximation Level (A_j),	Averages / Approximate Co-efficient (C_a)	Detail Co-efficient Level (D_j)	Differences / Detail Co-efficient (C_d)
(a)	(b)	(c)	(c)	(d)	(e)	(f)	(g)
Original Signal (0 level)	2^0 = 1	$1/2^0$ = 1	8	A_0	[18, 16, 6, 6, 12, 20, 4, 12]	D_0	-
1	2^1 = 2	$1/2^1$ = 1/2	4	A_1	[$34/\sqrt{2}$, $12/\sqrt{2}$, $32/\sqrt{2}$, $16/\sqrt{2}$	D_1	[$2/\sqrt{2}$, 0, $-8/\sqrt{2}$, $-8/\sqrt{2}$]
2	2^2 = 4	$1/2^2$ = 1/4	2	A_2	[23, 24]	D_2	[11, 8]
3	2^3 = 8	$1/2^3$ = 1/8	1	A_3	[$47/\sqrt{2}$]	D_3	[$-1/\sqrt{2}$]

In Haar Wavelet Transform, the values of these low-pass and high–pass filters are expressed as given in [4.29] to [4.32]

$$h(0) = \frac{1}{\sqrt{2}} \qquad [4.29]$$

$$h(1) = \frac{1}{\sqrt{2}} \qquad [4.30]$$

$$g(0) = \frac{1}{\sqrt{2}} \qquad [4.31]$$

$$g(1) = \frac{-1}{\sqrt{2}} \qquad [4.32]$$

Where,

h(0) and g(0) are analysis filters and h(1) and g(1) are synthesis filters.

In other words, these filter values can also be represented in terms of basis vectors as $< \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$ and $< \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} >$.

Since, the basis vectors used in Haar Discrete Wavelet Transform are the smallest possible basis vectors; it is not possible to do Haar Transform in one-pass. Thus, it becomes essential to recursively transform the input signal using these basis vectors[89]. This concept is shown clearly in column (e) of Table 5.

The scaling function in Haar Wavelet Transform is defined as in [4.33]

$$\phi(x) = \begin{cases} 1, & if, \ x \in [0,1) \\ 0, & if, \ x \notin [0,1) \end{cases} \qquad [4.33]$$

The mother wavelet function or translation function in Haar Wavelet Transform is defined as in [4.34],[90]

$$\psi(x) = \begin{cases} 1, & if, x \in [0, 0.5) \\ -1, & if, x \in [0.5,1) \\ 0, & if, x \notin [0,1) \end{cases} \qquad [4.34]$$

89 An Introduction to Wavelets and the Haar Transform, by Musawir Ali
90 I. Daubechies, "Ten Lectures On Wavelets", 1992

Thus, the Haar basis can be given using $\phi(x)$ the scaling function and $\psi(x)$ translational function or the wavelet function or wavelet basis that is associated with Haar Multiresolution Analysis.

Thus, if X(n) is a vector of size n, then, the approximation co-efficients C_a at approximation level A_j and detail co-efficients C_d at level D_j respectively are generated after transformation or decomposition of the vector X(n), at levels j $\in (1, \ldots \log_2(n))$.

As mentioned, X(n) is a vector of size n, the approximate length of the vector containing approximation co-efficient C_a is $\frac{n}{2}$ and similarly, the vector containing the detailed co-efficients C_d is also approximately of length $\frac{n}{2}$ i.e. the index i $\in \{0, 1, \ldots \frac{n}{2} - 1\}$, after performing one-level of Haar wavelet decomposition.

The ith element of the vector containing an approximate co-efficient C_{aj} and the vector containing the detail co-efficient C_{dj} at decomposition level j, can be given as shown in [4.35] and [4.36]

$$C_{aj} = \frac{1}{\sqrt{2}} X_j(2i-1) + \frac{1}{\sqrt{2}} X_j(2i) \qquad [4.35]$$

$$C_{dj} = \frac{1}{\sqrt{2}} X_j(2i-1) - \frac{1}{\sqrt{2}} X_j(2i) \qquad [4.36]$$

In Equations [4.35] and [4.36], the index i is positional information, in general, referred as time information and hence, i is same as t. As per specifications of Haar Wavelets, the length of original signal is expected to be of the power of 2. In that case, the length of the transformed vector containing the detailed co-efficient Cd, is usually 2j, where j is the decomposition level. The decomposition using Haar wavelets can be performed until the resolution (number of approximation co-efficient) becomes one or resolution level zero. Number of detailed co-efficients at each level j is equal to n/2j [91].

4.7. Inverse Wavelet Transform

Wavelet Transform aids in perfect reconstruction of the signal. The two set of wavelet-transformed co-efficients i.e. Ca and Cd, acquired from any level of decomposition, on up sampling by 2 and then on performing Inverse Wavelet Transform. The reconstructed signal is exactly same as the original signal, if no element is discarded from the two sets of decomposed co-efficients. The process of synthesis consists of two phases, the up-sampling by 2, to acquire the exact number of co-efficients i.e. n in each decomposed sets Ca and Cd. After up sampling the original signal can be acquired after performing inverse wavelet transform. *"Upsampling by 2"*[92] means inserting zeros between the existing data points. For example, A time-sequence of the numbers [6, 4, 3, 1] would become with upsampling by 2 [6, 0, 4, 0, 3, 0, 1] or in some cases [0, 6, 0, 4, 0, 3, 0, 1, 0] containing a leading and/or a trailing zeros.

In Synthesis phase, applying Inverse Wavelet Transform to given two co-efficient vectors ***Ca*** and ***Cd***, consists of performing up-sampling by 2 and then passing it through a low-pass and high-pass filters used for synthesis, an original signal can be reconstructed.

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91 J.K. Meher, M. R. Panigrahi, G. N. Dash, P. K. Meher, "Wavelet Based Lossless DNA Sequence Compression For Faster Detection Of Eukaryotic Protein Coding Regions ," I.J. Image, Graphics And Signal Processing 2012, 47-53

92 D. Lee Fugal, Conceptual Wavelets in Digital Signal Processing, Space and Signal Technologies, 2006

$$X_n \xleftarrow{\text{Inverse Haar Wavelet}} (C_a \mid C_d)$$

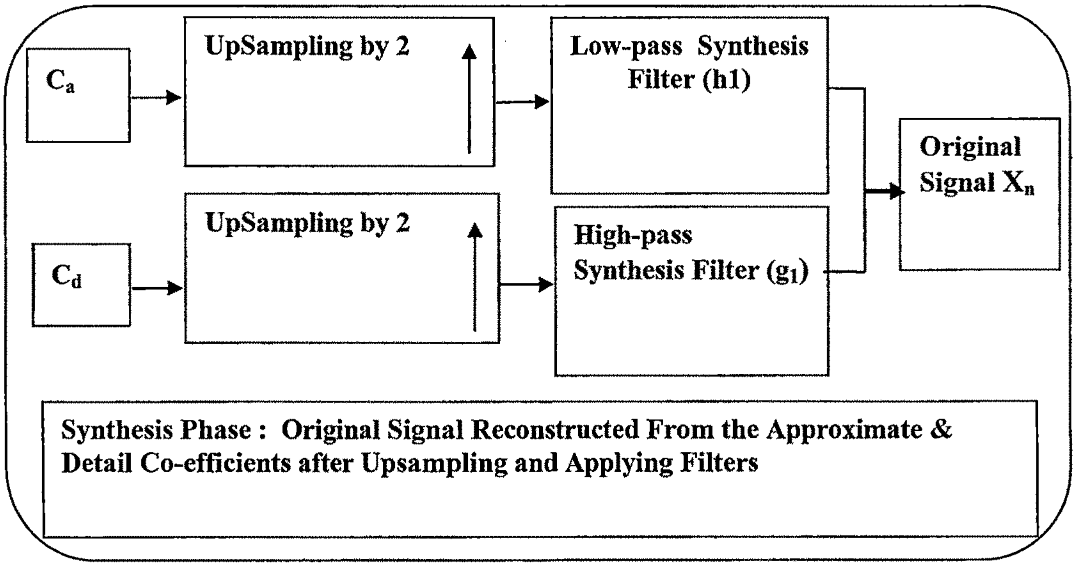

Figure 19. Inverse Wavelet Transform (Synthesis Phase)

As shown in Figure 19, the low-pass filter used in synthesis is denoted by h_1, while the high-pass filter is denoted by g_1. After up-sampling and applying appropriate filters, the original signal X_n can be reconstructed, where n is an integer number referring to the number of elements in the signal X.

The synthesis phase involves performing the Inverse Wavelet Transform. Performing Inverse Wavelet Transform means applying the convolution of the Wavelet Transform vector W_T and the basis vectors. Since, forward transformation was performed recursively; the inverse transform also needs to be performed recursively, to acquire the original signal X_n.

The "Mallat-Tree" is obtained when recursive Inverse Wavelet Transform is applied (As shown in **Error! Not a valid bookmark self-reference.**).

On performing Inverse Wavelet Transform to Level-3 coefficient vectors Ca_3 and Cd_3, generates the Level-2 approximate co-efficient vector Ca_2. This newly acquired vector Ca_2 can be inversely transformed using already available detail co-efficient vector of Level-2 Cd_2. On recursively performing this transform up to Level-1, helps in acquiring the original signal X. The perfect signal can be reconstructed, if no elements of any of the vectors, at any level of resolution were ever discarded.

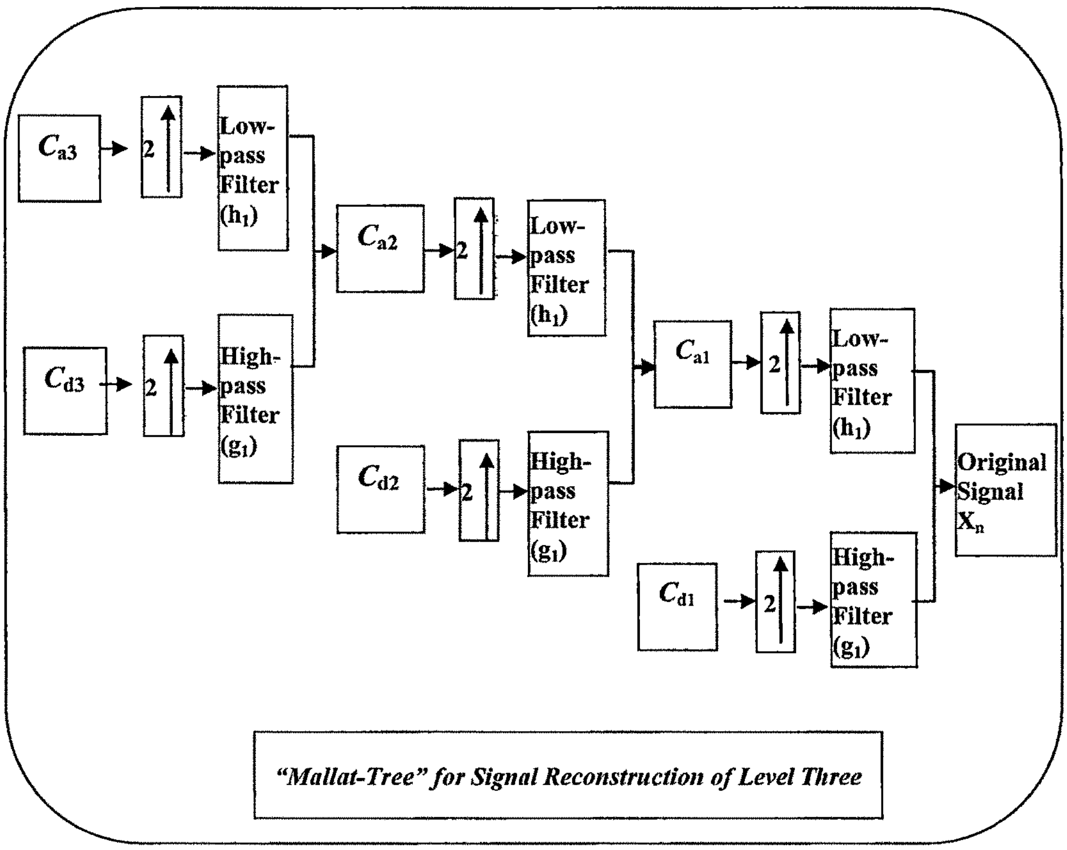

Figure 20. "Mallat Tree" for Synthesis Phase

Thus, as shown in The "Mallat-Tree" is obtained when recursive Inverse Wavelet Transform is applied (As shown in **Error! Not a valid bookmark self-reference.**).

On performing Inverse Wavelet Transform to Level-3 coefficient vectors Ca3 and Cd3, generates the Level-2 approximate co-efficient vector Ca2. This newly acquired vector Ca2 can be inversely transformed using already available detail co-efficient vector of Level-2 Cd2. On recursively performing this transform up to Level-1, helps in acquiring the original signal X. The perfect signal can be reconstructed, if no elements of any of the vectors, at any level of resolution were ever discarded.

Figure 20. Signal can be reconstructed perfectly; if we can retain the approximation co-efficients ***Ca*** of the last level of decomposition and the detail co-efficients of all levels of decomposition.

Since, the original signal

$X_n = C_{a1} + C_{d1}$

[4.37]

And $C_{a1} = C_{a2} + C_{d2}$,

[4.38]

Similarly,

$Ca2 = Ca3 + Cd3$

[4.39]

The original signal which has been decomposed up to level three can be reconstructed, if the co-efficients ***Ca3, Cd3, Cd2 and Cd1*** are given.

As a result, following can be acquired by performing,

$Ca3 + Cd3 + Cd2 + Cd1$

[4.40]

Using [4.37], [4.38] and [4.39], we can represent [4.40] as

$= Ca2 + Cd2 + Cd1$

$= Ca1 + Cd1$

$= Xn$

Where, "+" operator is an addition of two vectors.

An approximate signal, which is close to the original signal can definitely be acquired, in spite of discarding some of the elements using an appropriate threshold values. The threshold values used for discarding any elements should be relevant to the application, where one is trying to apply the Wavelet Transform.

4.8. Inverse Haar Wavelet Transform in Synthesis Phase

The synthesis phase involves performing the Inverse Haar Wavelet Transform. Performing Inverse Haar Wavelet Transform means, applying the convolution of the Haar transform W_T of the original signal X and the basis vectors. Since, forward transformation was performed recursively; the inverse transform also needs to be performed recursively, to acquire the original signal.

The basis vectors used for Inverse Haar Wavelet Transform are $<\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}>$.

Given the Wavelet Transform W_T of the signal X, it is possible to reconstruct the original signal, by convolving the vector containing wavelet co-efficients with the basis vector for Inverse Wavelet Transform.

Thus, given $W_T = [47/\sqrt{2}, -1/\sqrt{2}, 11, 8, 1/\sqrt{2}, 0, -8/\sqrt{2}, -8/\sqrt{2}]$ and Inverse Haar basis vector $<\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}>$,

On applying the convolution of two vectors recursively, we can obtain the original signal X[n] = [18, 16, 6, 6, 12, 20, 4, 12]

Considering the Haar Wavelet Transform as shown in Table 6 below,

Table 6. Representation of of Haar Wavelet Transform for Xn

Transformation Level or Decomposition Level (j)	Averages / Approximate Co-efficient (C_a) $C_a = (X_{2i-1} + X_{2i}) / \sqrt{2}$	Differences / Detail Co-efficient (C_d) $C_d = (X_{2i-1} - X_{2i}) / \sqrt{2}$
(a)	(b)	(c)
Original Signal (0 level)	$X_n = A_0$ = [18, 16, 6, 6, 12, 20, 4, 12]	-
1	A_1 = [$34/\sqrt{2}$, $12/\sqrt{2}$, $32/\sqrt{2}$, $16/\sqrt{2}$	D_1 = [$2/\sqrt{2}$, 0, -8 $/\sqrt{2}$, $-8/\sqrt{2}$]
2	A_2 = [23, 24]	D_2 = [11, 8]
3	A_3 = [$47/\sqrt{2}$]	D_3 = [$-1/\sqrt{2}$]

Thus, from the Table 6, we can observe that Wavelet Transform

W_T = [$47/\sqrt{2}$, $-1/\sqrt{2}$, 11, 8, $1/\sqrt{2}$, 0, $-8/\sqrt{2}$, $-8/\sqrt{2}$]

On convolution with Inverse Haar basis vector $<\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}>$

And operations as defined in Equation [4.40] would result in the original signal Xn.

The process of Inverse Haar Wavelet Transform can be stated as follows:

As stated in Equation [4.39]

C_{a3} +C_{d3} Will result in C_{a2}

Thus, Using Table 6,

$A_3 + D_3$

$\frac{1}{\sqrt{2}}(47/\sqrt{2}) + \frac{1}{\sqrt{2}}(-1/\sqrt{2}),\ \frac{1}{\sqrt{2}}(47/\sqrt{2}) + \frac{-1}{\sqrt{2}}(-1/\sqrt{2}),$

=23, 24

= A_2

Similarly, $A_2 + D_2$ would result in A_1

$\frac{1}{\sqrt{2}}(23) + \frac{1}{\sqrt{2}}(11),\ \frac{1}{\sqrt{2}}(23) + \frac{-1}{\sqrt{2}}(11),\ \frac{1}{\sqrt{2}}(24) + \frac{1}{\sqrt{2}}(8),\ \frac{1}{\sqrt{2}}(24) + \frac{-1}{\sqrt{2}}(8)$

$= [34/\sqrt{2}, 12/\sqrt{2}, 32/\sqrt{2}, 16/\sqrt{2}$

$= A_1$

And Thus, $A_1 + D_1$ would result in A_0, which is same as Original Signal

[18, 16, 6, 6, 12, 20, 4, 12]

$= A_0$

$= X_n$

4.9. Applications of Wavelet Transforms

Various applications of Wavelet Transforms are

- Image Compression (JPEG 2000)[93]
- Video Copression (MPEG 4)[94]
- Audio Compression
- FBI Fingerprint Compression
- Biometrics
- Mobile Applications - Speech Compression to reduce transmission time, Echo Cancellation
- Pattern Recognition – For Feature extraction, Edge detection
- Analysis of ElectroCardiogram
- Identification of Exons in Protein sequences
- Signal Processing – Compression, Encoding, Denoising

93 JPEG2000: Image Compression Fundamentals, Standards and Practice", D.Taubman, M. Marcellin, Kluwer Academic Publishers, 2001.
94 "The MPEG-4 Book", T. Ebrahimi, F. Pereira, Prentice Hall, 2002.