CHAPTER VI

SHADOWING PROPERTY ON TOPOLOGICAL SPACES

In this Chapter we define and study the notion of shadowing property on topological spaces. We recall the definition of δ–pseudo orbit for a continuous map f defined on a metric space (X, d). A sequence $\{x_i : i \geq 0\}$ of points of X is said to be a δ–pseudo orbit for f, if for each $i \geq 0$, $d(f(x_i), x_{i+1}) < \delta$. But this is equivalent to saying $(f(x_i), x_{i+1}) \in A_\delta$, where $A_\delta = \{(x,y) \mid d(x,y) < \delta\}$. Observe that A_δ is a subset of $X \times X$ containing the diagonal. Similarly we observe that if a δ–pseudo orbit for f is ε-traced by a point x in X, then for each $i \geq 0$, $d(f^i(x), x_i) < \varepsilon$ or equivalently $(f^i(x), x_i) \in A_\varepsilon$, where $A_\varepsilon = \{(x,y) \mid d(x,y) < \varepsilon\}$. Thus we can say that map f has shadowing property if for given $\varepsilon > 0$ there is a $\delta > 0$ such that for every A_δ -pseudo orbit $\{x_i : i \geq 0\}$ of f, there is a point x in X A_ε-tracing $\{x_i : i \geq 0\}$.

Above observations motivate definition of shadowing property for continuous maps on general topological spaces. In Section 1, we define the notion of shadowing property in topological setting, give examples and study properties of maps possessing A-shadowing property. In Section 2, we define the concept of positively expansive maps termed as positive A-expansive maps on a topological space X, study examples of such maps and observe their properties. Recently Wiseman and Richeson **[37]** have defined the concept of topologically positively expansive maps. We show that every topologically positively expansive map is positively A-expansive for some subset A of $X \times X$ containing the diagonal of X but converse need not be true. In Section 3, we define the concept of topological stability for a continuous map on a topological space. We recall that **[5]** topological Anosov maps (positive expansive maps having shadowing property) defined on compact metric space are always topologically stable in the class of homeomorphisms. We obtain similar results for positively A-expansive maps having A-shadowing property.

1. A-shadowing property: Definitions, examples and properties.

In this Section, we define the notion of shadowing property in topological setting, give examples and study various properties of such maps.

Definition 6.1.1. Let X be a topological space and let $f: X \to X$ be a continuous map. Suppose A and B are subsets of $X \times X$ containing the diagonal $\{(x,x): x \in X\}$ of X.

(a) A sequence $\{x_i : a<i<b\}$ is said to be B*-pseudo orbit* for f if for each i, $a < i < b-1$ we have $(f(x_i), x_{i+1}) \in B$.

(b) A B-pseudo orbit $\{x_i : a<i<b\}$ for f is said to be A*-traced* by a point x of X, if for each i, $a < i < b$, $(f^i(x), x_i) \in A$.

If f is bijective then $-\infty \le a < b \le \infty$, otherwise $0 \le a < b \le \infty$.

(c) Map f is said to have *A-shadowing property* if there is a subset B of $X \times X$ containing the diagonal such that every B-pseudo orbit for f is A-traced by a point of X.

If a continuous map f defined on a metric space (X,d) has shadowing property, then by definition for any $\varepsilon > 0$, f has A_ε-shadowing property. We observe through examples that converse need not be true. That is there are maps on a metric space which do not have shadowing property but has *A-shadowing property* for some subset A of $X \times X$ containing the diagonal. Before giving examples we discuss some properties of maps possessing A-shadowing property.

Theorem 6.1.2. *Let X be a topological space and let $f: X \to X$ be a continuous map. If f has the A-shadowing property, then f^k has the A-shadowing property for every $k > 0$.*

Proof. Choose $k > 0$ and fix it. Since f has the A-shadowing property, there is a subset B of $X \times X$ containing the diagonal such that every B-pseudo orbit for f is A-traced by a point of X. We show that every B-pseudo orbit for f^k is A-traced by some point of X Let $\{x_n : n \geq 0\}$ be a B-pseudo orbit for f^k. Then for each $n \geq 0, (f^k(x_n), x_{n+1}) \in B$. Construct sequence $\{y_i : i \geq 0\}$ as follows: $\{y_i : i \geq 0\}$ =

$\{x_0, f(x_0), f^2(x_0), \ldots\ldots, f^{k-1}(x_0), x_1, f(x_1), \ldots\ldots, f^{k-1}(x_1), \ldots\ldots, f^{k-1}(x_n), \ldots\ldots\}$, i.e.

$y_{kn+j} = f^j(x_n)$, for $0 \le j \le k-1$ and each $n \ge 0$. We show that $\{y_i : i \ge 0\}$ is a B-pseudo orbit for f. In fact, if $0 \le i \le nk-2, n > 0,$ then $(f(y_i), y_{i+1}) = (f^{j+1}(x_n), f^{j+1}(x_n)) \in B$, as B contains the diagonal and if $i = kn-1,$ then $(f(y_i), y_{i+1}) = (f^k(x_n), x_{n+1}) \in B$ because $\{x_n : n \ge 0\}$ is a B-pseudo orbit for f^k. Since f has the A-shadowing property, $\{y_i : i \ge 0\}$ is A-traced by a point of X, say y. Thus for each $i \ge 0$, $(f^i(y), y_i) \in A$. In particular for each $i = kn,\ n \ge 0$, $(f^{kn}(y), y_{kn}) \in A$. But $y_{kn} = x_n$ implies $((f^k)^n(y), x_n) \in A$. This proves $\{x_n : n \ge 0\}$ is A-traced by y.

Next result relates the A-shadowing property of a homeomorphism defined on a topological space with A-shadowing property of its inverse.

Theorem 6.1.3. *Let X be a Hausdorff topological space and let $f : X \to X$ be a homeomorphism. If f has the A-shadowing property, then so does f^{-1}.*

Proof. Since f has the A-shadowing property, there is a subset B of $X \times X$ containing the diagonal such that every B-pseudo orbit for f is A-traced by a point of X. We can assume without loss of generality that B is symmetric. We complete the proof by showing that every $(f^{-1} \times f^{-1})(B)$-pseudo orbit for f^{-1} is A-traced by a point of X. Let $\{x_i : i \in Z\}$ be a $(f^{-1} \times f^{-1})(B)$-pseudo orbit for f^{-1}. Then $(f^{-1}(x_i), x_{i+1}) \in (f^{-1} \times f^{-1})(B)$ which implies

$(x_i, f(x_{i+1}) \in B$ and hence for each $i \in \mathbf{Z}$, $(x_{-i}, f(x_{-i+1})) \in B$. Put $x_{-i} = y_i$ for each $i \in Z$ and observe that $(f(y_{i-1}), y_i) \in B$ since B is symmetric. This proves $\{y_i : i \in \mathbf{Z}\}$ is a B-pseudo orbit for f and hence is A-traced by some point y of X. Thus for each $i \in \mathbf{Z}$, $((f^{-1})^i(y), x_i) = (f^{-i}(y), y_{-i}) \in A$ and therefore $\{x_i : i \in Z\}$ is A-traced by y.

Following theorem shows that A-shadowing is a dynamical property since it is preserved under topological conjugacy.

Theorem 6.1.4. *Let X be a topological space and let $f : X \to X$ be a continuous map. Suppose Y is a topological space and $h : X \to Y$ is a homeomorphism. Then f has the A-shadowing property if and only if $g = hfh^{-1} : Y \to Y$ has the $(h \times h)(A)$-shadowing property.*

Proof. We denote $(h \times h)$ by H. Suppose f has the A-shadowing property. Then we show that $g = hfh^{-1}$ has $H(A)$-shadowing property. Since f has the A-shadowing property, there is a subset B of $X \times X$ containing the diagonal such that every B-pseudo orbit for f is A-traced by a point of X. We complete this part of the theorem by showing that every $H(B)$-pseudo orbit for g is $H(A)$-traced by a point of Y. Let $\{y_i : i \geq 0\}$ be a $H(B)$-pseudo orbit for g. Then for each $i \geq 0$

$$(g(y_i), y_{i+1}) \in H(B) \Rightarrow (fh^{-1}(y_i), h^{-1}(y_{i+1})) \in B .$$

Set $h^{-1}(y_i) = x_i$. Then for each $i \geq 0$, $(f(x_i), x_{i+1}) \in B$ and therefore $\{x_i : i \geq 0\}$ is A-traced by some point x of X. Hence for each $i \geq 0$,

$$(f^i(x), x_i) \in A \Rightarrow (hf^i(x), h(x_i)) \in (h \times h)(A) \Rightarrow (g^i(h(x)), y_i) \in H(A).$$

This proves pseudo orbit $\{y_i : i \geq 0\}$ for g is $H(A)$-traced by the point $h(x)$ of Y.

Example 6.1.5. (a) All the spaces consider here carry usual topology. Let $X = [0, 1)$ and define $f : X \to X$ by $f(x) = \dfrac{x}{4-3x}$. We show here that f has the A-shadowing property, where A is a subset of $X \times X$ containing the diagonal and A does not contain A_ε for any $\varepsilon > 0$. Observe that f is a homeomorphism. Consider the homeomorphisms $h : [0,\infty) \to [0,1)$ defined by $h(x) = \dfrac{x^2}{x^2+1}$ and $g : [0,\infty) \to [0,\infty)$ given by $g(x) = x/2$. Then g being a contraction map has the A_ε-shadowing property for any $\varepsilon > 0$. Set $A = (h \times h)(A_\varepsilon)$. Observe that by Theorem 6.1.4, $hgh^{-1} : [0,1) \to [0,1)$ has A-shadowing property. Since $hgh^{-1} = f$, f has A-shadowing property. Note that

$$A = (h \times h)(A_\varepsilon) = \{(u,v) : u = \frac{x^2}{x^2+1}, v = \frac{y^2}{y^2+1}, x - \varepsilon < y < x + \varepsilon,\ x, y \geq 0\}$$

does not contain A_ε for any $\varepsilon > 0$.

Recall that shadowing property of a map f defined on a compact metric space (X,d) is preserved under conjugacy. Example 6.1.5(b) justifies the necessity of compactness in the above phenomena. But the situation is different here. Following example shows that A-shadowing property can be preserved under conjugacy for non compact metric space, where A is a subset of $X \times X$ containing the diagonal and A does not contain A_ε for any $\varepsilon > 0$. Note that if f has A-shadowing property, then f has D-shadowing property for any $D \supseteq A$.

6.1.5(b) Consider the topological space R and $(-1, 1)$ with the Euclidean topology. Define $h: \mathrm{R} \to (-1, 1)$ by $h(x) = \begin{cases} -\dfrac{x}{x-1}, \text{ if } x \le 0 \\ \dfrac{x}{x+1}, \text{ if } x \ge 0 \end{cases}$ and map $g: \mathrm{R} \to \mathrm{R}$ by $g(x) = 2x$. Then g has the A_δ-shadowing property for every $\delta > 0$ as it has the shadowing property, where $A_\delta = \{(x,y) : x - \delta < y < x + \delta\}$. By Theorem 6.1.4, $hgh^{-1} : (-1, 1) \to (-1, 1)$ has the B_δ-shadowing property, where $B_\delta = (g \times g)(A_\delta)$. Note that $hgh^{-1}(x)$ is given by $hgh^{-1}(x) = \begin{cases} -\dfrac{2x}{x-1}, \text{ if } x \le 0 \\ \dfrac{2x}{x+1}, \text{ if } x \ge 0 \end{cases}$ and is a homeomorphism on $(-1, 1)$. Define $f : [-1, 1] \to [-1, 1]$ by $f(x) = \begin{cases} hgh^{-1}(x), & \text{if } x \in (-1, 1) \\ -1, & \text{if } x = -1 \\ 1, & \text{if } x = 1 \end{cases}$. Then f has the A-shadowing property, where $A = B_\delta = (g \times g)(A_\delta) \cup \{(1,1), (-1,-1)\}$. Now

$Fixf = \{-1, 0, 1\}$ and f satisfies the hypothesis of Theorem 3.3.1. Therefore f does not have the shadowing property but f has the A-shadowing property for some subset A of $[-1,1]\times[-1,1]$ -containing the diagonal of $[-1,1]$.

We now study condition under which product map has the A-shadowing property.

Theorem 6.1.6. *Let X, Y be topological spaces, $f: X \to X$ and $h: Y \to Y$ be continuous maps. Suppose f has A-shadowing property and h has B-shadowing property. Then $f\times h: X\times Y \to X\times Y$ has the $g^{-1}(A\times B)$-shadowing property, where $g:(X\times Y)^2 \to X^2\times Y^2$ is defined by $g(x,y,u,v) = ((x,u),(y,v))$.*

Proof. Since f has the A-shadowing property, there is a subset C of $X\times X$ containing the diagonal such that every C-pseudo orbit for f is A-traced by a point of X. Similarly B-shadowing property of h implies there exists a subset D of $Y\times Y$ containing the diagonal of Y such that every D-pseudo orbit for h is B-traced by a point of Y. In order to show that $f\times h$ has the $g^{-1}(A\times B)$-shadowing property, we show that every $g^{-1}(C\times D)$-pseudo orbit for $f\times h$ is $g^{-1}(A\times B)$-traced by a point of $X\times Y$. Let $\{z_i = (x_i, y_i): i\geq 0\}$ be a $g^{-1}(C\times D)$-pseudo orbit for $f\times h$. Then for each $i\geq 0$, $((f\times h)(z_i), z_{i+1}) \in g^{-1}(C\times D)$, which implies

$$g(f(x_i), h(y_i), x_{i+1}, y_{i+1}) \in C\times D$$

and hence for each $i \geq 0$

$$(f(x_i), x_{i+1}) \in C \text{ and } (h(y_i), y_{i+1}) \in D.$$

Thus $\{x_i : i \geq 0\}$ and $\{y_i : i \geq 0\}$ are C-pseudo orbit and D-pseudo orbit for f and h respectively. Hence there exist $x \in X$ and $y \in Y$ such that $\{x_i : i \geq 0\}$ is A-traced by x and $\{y_i : i \geq 0\}$ is B-traced by y. But this means that for each $i \geq 0$

$$(f^i(x), x_i) \in A \text{ and } (h^i(y), y_i) \in B$$

which implies

$$((f \times h)^i(x, y), (x_i, y_i)) \in g^{-1}(A \times B).$$

This completes the proof.

2. Positively A-expansive maps.

Recall that the notion of expansive homeomorphism is defined and studied on topological spaces in **[16]**. We define here the concept of positively expansive maps on topological spaces.

Definition 6.2.1. Let X be a topological space and let $f: X \to X$ be a continuous onto map. Map f is said to be positively A-expansive if for each pair of points $x, y \in X$ with $x \neq y$, there exists a non negative integer n such that $(f^n(x), f^n(y)) \notin A$.

Example 6.2.2. Consider the family of tent map $\{f_s : s \in [\sqrt{2}, 2]\}$, where $f_s : [0, 2] \to [0, 2]$ is defined by $f_s(x) = \begin{cases} sx, & \text{if } x \in [0, 1] \\ s(2-x) & \text{if } x \in [1, 2] \end{cases}$.

Let A_1 be the subset of $[0, 1] \times [0, 1]$ containing the diagonal of $[0, 1]$ and bounded by the curves $f_1(x) = \dfrac{s+1}{2s}x^2$ and $f_2(x) = \sqrt{\dfrac{2s}{s+1}x}$. Similarly A_2 is a subset of $[1, 2] \times [1, 2]$ containing the diagonal of $[1, 2]$ and bounded by the curves $f_3(x) = \dfrac{4s}{4s+2} + \dfrac{s+1}{4s+2}x^2$ and $f_4(x) = \sqrt{\dfrac{4s+2}{s+1}x - \dfrac{4s}{s+1}}$. For each s, $s \in [\sqrt{2}, 2]$, f_s is a positively A-expansive map, where $A = A_1 \cup A_2$. Infact $(x, f(x)) \in A_1$, for x such that $1 \le x \le 2$. Observing that whenever $(x, y) \in A_1$, then $y \ge \dfrac{s+1}{2s}x^2$ and $y^2 \le \dfrac{2s}{s+1}x$ we obtain an $n \ge \dot{0}$ such that $(f^n(x), f^n(y)) \notin A$, for $x \ne y, x, y \in [0, 2]$.

We now state some properties of positively A-expansive maps without proof as they can be proved easily **[16]**.

Theorem 6.2.3. (a) *Let X and Y be topological spaces and let $f : X \to X$ be a continuous onto map. Suppose $h : X \to Y$ is a homeomorphism. Then f is positively A-expansive if and only if $g = hfh^{-1} : Y \to Y$ is positively $(h \times h)(A)$-expansive, where $A \subseteq X \times X$.*

6.2.3. (b) *Let X, Y be topological spaces $f : X \to X$, $g : Y \to Y$ be positively A-expansive and positively B-expansive maps respectively. Then*

$f\times g: X\times Y\to X\times Y$ *is positively* C*-expansive, where* $C=h^{-1}(A\times B)$ *and* $h:(X\times Y)^2\to X^2\times Y^2$ *is defined by* $h(x,y,u,v)=((x,u),(y,v))$.

6.2.3. (c) *Let* X *be a paracompact Hausdorff space and let* Σ *be a uniformity on it consisting of all neighborhoods of the diagonal. Suppose* $h: X\to X$ *is a continuous onto map such that* h^m, $m\neq 0$ *is uniformly continuous with respect to* Σ. *Then* h *is positively* A*-expansive for some* A *in* Σ *implies* h^m, $m\neq 0$ *is positively* B*-expansive for some* B *in* Σ.

Let $f: X\to X$ be a continuous map on a topological space X. A set S is said to be invariant if $f(S)=S$. For a given subset A of X, let $InvA$ denote the maximal invariant subset of A then $InvA=\{x\in A\,|\,\text{there exists}....,x_{-1},x_0,x_1,....,\in A,$ where $x=x_0$ and $f(x_k)=x_{k+1}$ for each $k\in Z\}$ [37]. In the following result we show that if f is positively A-expansive, where A contains the diagonal, then the largest invariant set in A is the diagonal of X.

Proposition 6.2.4. *Let* X *be a topological space and* $f: X\to X$ *be a positively* A*-expansive map, where* A *is a subset of* $X\times X$ *containing the diagonal. Then the largest invariant subset of* A *is the diagonal of* X.

Proof. Set $F=f\times f$. Then the maximal invariant subset of A is given by

$InvA=\{(x,y)\in A\,|\,\text{there exists}...,(x_{-1},y_{-1}),(x_0,y_0),......,(x_k,y_k),....\in A$
with $(x,y)=(x_0,y_0)$ and $F((x_k,y_k))=(x_{k+1},y_{k+1})$ for each $k\in Z\}$.

Observe that for all $n\geq 0$

$$F((x_{n-1}, y_{n-1})) = (f^n(x), f^n(y)) = (x_n, y_n).$$

Suppose $(x, y) \in InvA$. Then for each $n \geq 0$, $(x_n, y_n) = (f^n(x), f^n(y)) \in A$. But f is positively A-expansive and therefore $x = y$. Hence $InvA \subseteq \Delta$, where Δ is the diagonal of X. The reverse containment is obvious.

Recall that **[37]** a continuous map f on a topological space X is said to be *topologically positively expansive* if for any neighborhood U of the diagonal, there exists a closed neighborhood $N \subset U$ with the property $N \subset IntF(N)$ and $InvN \subseteq \Delta$, where $F = f \times f$. In the following theorem we show that if f is topologically positively expansive, then f is always positively A-expansive f, for some subset A of $X \times X$ containing the diagonal.

Theorem 6.2.5. *Let X be a topological space and $f: X \to X$ be a continuous map. If f is topologically positively expansive, then f is always positively A-expansive for some subset A of $X \times X$ which is a closed neighborhood of the diagonal.*

Proof. Let U be a neighborhood of the diagonal. Since f is topologically positively expansive there exists a closed neighborhood N of the diagonal, $N \subset U$ with property $N \subset IntF(N)$ and $InvN \subseteq \Delta$. We show f is positively N-expansive. Let $x, y \in X$ such that $(f^n(x), f^n(y)) \in N$ for all $n \geq 0$. In particular, $x, y \in N \subset IntF(N)$. Therefore we get a sequence

$\ldots\ldots,(x_{-2},y_{-2}),(x_{-1},y_{-1}),(x_0,y_0)=(x,y)$ in N such that $F((x_k,y_k))=(x_{k+1},y_{k+1})$ for all $k<0$. Also by hypothesis for each $k\geq 0$, $(x_k,y_k)=(f^k(x),f^k(y))\in N$. This proves $(x_k,y_k)\in N$ for each integer k. Hence as discussed in the proof of Proposition 6.2.4, $(x,y)\in InvN$. But we know $InvN\subset\Delta$, therefore $(x,y)\in\Delta$. Hence $x=y$. This proves f is positively N-expansive.

Recall example 6.1.5(a). The map f defined there is a positively A-expansive map on the unit interval. But there do not exist any positively expansive maps on the non trivial interval. Also for compact metric spaces the notion of positively expansive maps and topologically positively expansive maps coincides **[37]**. Therefore there exists no topologically positively expansive map on a non-trivial interval but there exists a positively A-expansive map on the interval.

Following is an example of a positively A-expansive map which is also topologically positively expansive, where A does not contain A_δ for any $\delta>0$. The topology taken on $\boldsymbol{R}$ in the following example can be referred from **[40]**.

Example 6.2.6. Consider the set of real numbers. For each irrational number x, let $\{x_i\}$ be a sequence of rational numbers converging to x in the Euclidean topology. We define the topology τ on $\boldsymbol{R}$ by declaring each $\{x\}$, x a rational number open and basic open sets about irrational number x is given by $U_n(x) = \{x_i\}_n^\infty \cup \{x\}$. Then intersection of every open set with respect to τ with $\boldsymbol{R}-\boldsymbol{Q}$ is singleton. Therefore $\boldsymbol{R}-\boldsymbol{Q}$ is a discrete subset of $\boldsymbol{R}$. Consider the product space $R \times R$ with the product topology. Then for any neighbourhood U of the diagonal let N_δ denote the δ-neighbourhood of the diagonal in the usual topology on $\boldsymbol{R}$. The set $N = \Delta \cup (\boldsymbol{R}\times\boldsymbol{R}-\boldsymbol{Q}\times\boldsymbol{Q}) \cap N_\delta \cup (\boldsymbol{R}\times\boldsymbol{R}-\mathrm{Q}\times\phi) \cap N_\delta$ is a closed neighbourhood of the diagonal with respect to the topology τ on $\boldsymbol{R}$. Observe that $N \subset N_\delta$ and N does not contain points of N_δ whose both the coordinates are rational numbers. The map $f:\boldsymbol{R}\to\boldsymbol{R}$ defined by $f(x)=2x$ is an N_δ-expansive map and hence is N-expansive. Note that N does not contain N_δ for any $\delta > 0$.

3. Topological A - stability.

Recall that topological Anosov maps defined on compact metric spaces are always topologically stable in the class of homeomorphisms. A homeomorphism $f:X\to X$ defined on a compact metric space X is said to be topologically stable if for given $\varepsilon>0$ there is a $\delta>0$ such that for any homeomorphism $h:X\to X$ with $d(f(x),h(x))<\delta$, for each x in X, there exists a continuous map $g:X\to X$ satisfying $d(g(x),(x))<\varepsilon$ and $gh=fg$ i.e. for any homeomorphism h, $(f(x),h(x)) \in A_\delta$, where $A_\delta = \{(x,y):d(x,y)<\delta\}$ there exists a continuous map $g:X\to X$ satisfying $gh=fg$ and $(g(x),\ x)\in B_\varepsilon$, where $B_\varepsilon = \{(x,y):d(x,y)<\varepsilon\}$.

This motivates the following definition of topological A-stability. We define the concept for continuous onto maps.

Definition 6.3.1. Let X be a topological space and $f : X \to X$ be a continuous onto map. Suppose A is a subset of $X \times X$ containing the diagonal. Map f is said to be *topologically A-stable* if there is a subset B of $X \times X$ containing the diagonal such that for any continuous onto map $h : X \to X$ with $(f(x), h(x)) \in B$, $x \in X$, there exists a continuous map $g : X \to X$ satisfying $(g(x), x) \in A$, for each $x \in X$ and $gh = fg$.

We prove the following theorem which relates positively A-expansive maps having B-shadowing property B-stability of the map. Recall that in a topological space X, for subsets A and B of $X \times X$, $A \circ B = \{(x, z) :$ there exists $y \in X$ with $(x, y) \in B, (y, z) \in A\}$.

Theorem 6.3.2. *Let X be a first countable Hausdorff space and $\wp$ be a uniformity on X consisting of all neighbourhoods of the diagonal. Suppose for $A \in \wp$, f is positively A-expansive maps having B- shadowing property, where $B \in \wp$ in such that $B \circ B \circ B \subset A$ and B is a symmetric neighborhood of the diagonal, then f is topologically B- stable.*

Proof. Since f has the B-shadowing property, there is subset C of $X \times X$ containing the diagonal such that every C-pseudo orbit for f is B- traced by a point of X. We show that tracing point is unique. Suppose a C- pseudo orbit $\{x_n : n \geq 0\}$ is B- traced by points x and y. Then for each $n \geq 0$,

$(f^n(x), x_n) \in B$ and $(f^n(y), x_n) \in B$. But B is a symmetric subset of $X \times X$ containing the diagonal. Therefore,

$$(f^n(x), x_n) \in B \text{ and } (x_n, f^n(y)) \in B$$
$$\Rightarrow (f^n(x), f^n(y)) \in B \circ B$$

Therefore for each $n \geq 0$, $(f^n(x), f^n(y)) \in B \circ B \subset A$. By positive A-expansivity of f, we have $x = y$. Thus, a C-pseudo orbit for f is uniquely traced by a point of X. Now we show that f is topologically B-stable. Let $h: X \to X$ be a continuous onto map such that for each $x \in X$, $(f(x), h(x)) \in C$. Therefore for each $n \geq 0$, $(f(h^n(x)), h^{n+1}(x)) \in C$, which implies $\{h^n : n \geq 0\}$ is a C-pseudo orbit for f. But f has the B-shadowing property. Therefore there is a unique point $p(x) \in X$ such that for each $n \geq 0$, $(f^n(p(x)), h^n(x)) \in B$. This is true for all $x \in X$. Define map $p: X \to X$ by $x \mapsto p(x)$. Since the tracing point of $\{h^n(x)\}$ is unique p is well defined. Now for each $x \in X$ and all $n \geq 0$, $(f^n(p(x)), h^n(x)) \in B$. In particular for $h(x) \in X$ and for each $n \geq 0$, $(f^n(p(x)), h(h^n(x))) \in B$ which implies $(f^n(p(x)), h^{n+1}(x)) \in B$.

Also for each $n \geq 0$, $(f^{n+1}(p(x)), h^{n+1}(x)) \in B$ i.e. $(f^n(f(p(x))), h^{n+1}(x)) \in B$. This further implies $(f^n(p(x)), f^n f(p(x))) \in B \circ B \subset C$. Since f is positively A- expansive, we get $p h(x) = f(p(x)$ for all $x \in X$ and thus $ph = fp$. Again, for each $n \geq 0$ $(f^n(p(x)), h^n(x)) \in B$. Therefore in particular for $n = 0$, $(p(x), x) \in B$. Finally we prove that p is continuous. In order to show that p is continuous we show that if $\{x_\alpha\}$ is a net in X such that $x_\alpha \to x$ then $p(x_\alpha) \to p(x)$. Since $x_\alpha \to x$ and h is a continuous map we have $h(x_\alpha) \to h(x)$. This implies that there exists β such that $h(x_\alpha) \in B[h(x)]$ for all $\alpha \geq \beta$, which implies $(h(x_\alpha), h(x)) \in B$. Now, consider

$$\begin{aligned}(f(p(x_\alpha), f(p(x)) &= (p(h(x_\alpha)), p(h(x)) \\ &= (p\ (h\ (x_\alpha)), h(x_\alpha)) \circ (h(x_\alpha), h(x)) \\ &\quad \circ (h(x), p(h(x)), \quad \text{for every } \alpha \geq \beta \\ &\in B \circ B \circ B \subset A, \quad \text{for every } \alpha \geq \beta\end{aligned}$$

Since $\wp$ is a uniformity of all neighbourhoods of diagonal, for $x \in X$, $U \in \wp$, $U(x)$ is a neighbourhood base at x. Therefore, if $U \in \wp$ is such that $A \subset U$, then $f(p(x_\alpha)) \in A[f(p(x))] \subset U(f(p(x)))$, for all $\alpha \geq \beta$. This implies $f(p(x_\alpha)) \to f(p(x))$. But f is continuous, therefore $p(x_\alpha) \to p(x)$.

Thus whenever $x_\alpha \to x$ then $p(x_\alpha) \to p(x)$. Hence p is continuous.