SUMMARY

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ON

DYNAMICAL PROPERTIES OF MAPS ON TOPOLOGICAL SPACES AND G-SPACES

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SUMMARY

A *dynamical property* of a map is that property which is invariant under topological conjugacy. The set of dynamical properties of a map includes the topological but not the differentiable invariants of the underlying space. Since the invariant measures are preserved by topological conjugacy, the Ergodic Theory of a map is included in its dynamics.

Introduced by Anosov in **[2]**, the notion of *shadowing property* (or *pseudo orbit tracing property*) turned out to be one of the very important and useful dynamical properties of continuous maps on metric spaces. Since its inception, this notion has been extensively studied by several Mathematicians including Bowen **[5]**, Walters **[15]**, Morimoto **[8]**, Shub **[13]** and Aoki **[3]**. In recent years theory of shadowing has become a significant part of qualitative theory of dynamical systems containing a lot of interesting and deep results. It plays an important role in the investigation of the stability theory.

For a given real number $\delta > 0$ a sequence of points $\{x_i : a < i < b\}$ of a metric space (X, d) is called a δ*-pseudo orbit of* a continuous map $f : X \to X$ if $d(f(x_i), x_{i+1}) < \delta$, for each $i \in (a, b-1)$. Given $\varepsilon > 0$, a δ-pseudo orbit $\{x_i\}$ is said to be ε*-traced* by a point $\hat{x} \in X$ if $d(f^i(x), x_i) < \varepsilon$ for every $i \in (a, b)$. Here the symbols a and b are taken as $-\infty \leq a < b \leq \infty$, if f is bijective and as $0 \leq a < b \leq \infty$ if f is not bijective. We say that f has the

shadowing property (or *pseudo orbit tracing property*) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo orbit of f can be ε-traced by some point of X. The notion of δ-pseudo orbit is quiet a natural notion since on account of rounding errors, a computer will actually calculate a pseudo orbit rather than an orbit. Moreover, ε-tracing shows that a pseudo orbit is uniformly approximated by a genuine orbit. If X is compact then the shadowing property of $f: X \to X$ is independent of the choice of metric d compatible with the topology of X **[3]**.

Several problems including properties of maps possessing shadowing property and its relation with other dynamical properties have been studied in detail. Moreover, one of the basic problems studied in the theory of shadowing is finding class of maps possessing / not possessing shadowing property.

Various generalizations of shadowing property have been obtained and studied. For example, Lipschitz shadowing property, Limit shadowing property and shadowing property for maps on Banach spaces are defined and studied in detail. Moreover, the concepts of s-limit shadowing **[1]**, rotation shadowing **[4]**, asymptotic shadowing **[6]**, weak shadowing, strong shadowing **[10]**, average shadowing **[10]**, uniform pseudo-orbit tracing property **[7]**, shadowing property for flows **[14]** etc are defined and studied in detail. Interrelations of some of these different notions of shadowing properties have also been studied in Sakai **[10]**.

Studying the available literature, it appeared to us that the notion of shadowing property and some other related concepts have not been defined and studied for continuous maps on metric G-spaces and on general topological spaces. Analyzing definitions carefully on metric spaces and studying several related examples and results, we could successfully formulate definitions on metric G-spaces, on topological spaces and obtain interesting examples and results.

The present thesis is the outcome of the researches carried out by the author mainly along these lines. There are six chapters in the thesis and the present chapter aims at providing introduction to the subject matter of the thesis through the recent development in the area.

In Chapter 2 we define the notion of shadowing property on a metric G-space. Let (X, d) be a metric G–space and $f: X \to X$ be a continuous map. For a positive real number δ, a sequence of points $\{x_i : a<i<b\}$ in X is said to be $\delta-G$ *pseudo orbit* for f if for each i, $a<i<b-1$, there exists a $g_i \in G$ such that $d\,(g_i f(x_i), x_{i+1})<\delta$. Let a given $\varepsilon>0$, a $\delta-G$ pseudo orbit $\{x_i : a<i<b\}$ for f is said to ε–*traced* by a point x of X if for each i, $a<i<b$, there exists a $p_i \in G$ such that $d(f^i(x), p_i x_i)<\varepsilon$. Note that if f is bijective we take $-\infty \leq a<b \leq \infty$, other wise $0 \leq a<b \leq \infty$. Map f is said to

have the G*–shadowing property* if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that every $\delta - G$ pseudo orbit for f is ε–traced by a point of X.

Through various examples it is observed that the notion of shadowing property neither implies not is implied by the notion of G-shadowing property. Besides studying general properties of maps possessing G-shadowing property we obtain characterizations for maps to possess the G-shadowing property. A necessary and sufficient condition is obtained for the identity map to have the G-shadowing property.

Theorem 1. *Let X be a compact metric G–space, where G is compact. Then the identity map f on X has the G–shadowing property if and only if the orbit space X/G of X is totally disconnected.*

The following result gives a necessary and sufficient condition for a map to have the G-shadowing property.

Theorem 2. *Let X be a compact metric G–space and $f : X \to X$ be a pseudoequivariant map. Suppose the orbit map $\pi : X \to X/G$ is a covering map. Then f has the G–shadowing property if and only if the induced map $\hat{f} : X/G \to X/G$ has the shadowing property.*

In Chapter 3 we study the G-shadowing property of the shift map on the inverse limit space generated by a continuous onto map defined on a compact metric G-space X.

Theorem 3. *Let f be an equivariant onto-self-map defined on a compact metric G-space X, where G is compact. If f has the G-shadowing property then the shift map σ on the inverse limit space X_f has the G-shadowing property.*

Converse of Theorem 3 is true if the map f is a local homeomorphism. Examples are provided to justify the necessity of local homeomorphism.

In Chapter 4 we define the notion of positively expansive map on a metric G-space X. Let (X,d) be a metric $G-$space. A continuous onto map $f:X \to X$ is said to be *positively $G-$expansive*, if there exists a positive real number c such that for all x, y in X with $G(x) \neq G(y)$, there exists a non-negative integer n such that

$$d(f^n(u), f^n(v)) > c \text{, for all } u \in G(x) \text{ and } v \in G(y);$$

c is then called a *$G-$expansive constant* for f. Besides proving general properties of a positively G-expansive we give class of maps on a continuum which are not positively G-expansive maps.

Theorem 4. *Let $f: X \to X$ be a pseudo equivariant minimal open map defined on a compact metric $G-$space X, where G is compact and action of G on X is non-transitive. Then f is not a positively $G-$expansive map.*

In the following result we obtain a necessary and sufficient condition for a positively expansive map to have the G-shadowing property.

Theorem 5. *Let X be a compact metric $G-$space with G compact and let $f: X \to X$ be a positively $G-$expansive pseudoequivariant map. Then f has the $G-$shadowing property if and only if for every open set U of X and for each x in U, there exists a $\delta > 0$ and a $g \in G$ such that*

$$gU_{\delta}(f(x)) \subset f(U) \qquad (*)$$

where $U_{\delta}(x)$ denotes the $\delta-$neighbourhood of x.

We also define the concept of non wandering and chain recurrent points for a continuous map on a metric G-space X. Let X be a metric $G-$space and $f: X \to X$ be a continuous onto map. A point x in X is said to be $G-$*non wandering point* of f if for every neighbourhood U of x, there exists an integer $n > 0$ and a $g \in G$ such that $gf^{n}(U) \cap U \neq \varphi$. We shall denote the set of all $G-$non wandering points of f by $\Omega_G(f)$. For $x, y \in X$ and $\delta > 0$, x is said to be $\delta - G$ *related* to y (denoted by $x \overset{\delta}{\sim}_G y$) if there are finite $\delta - G$ pseudo orbits $\{x = x_0, x_1, \ldots, x_k = y\}$ and $\{y = y_0, y_1, \ldots, y_n = x\}$ for f. If for every $\delta > 0$, x is $\delta - G$ *related* to y; then x is said to be G related to

y (denoted by $x \sim_G y$). A point x is said to be $G-$*chain recurrent point* of f if $x \sim_G x$. We shall denote the set of all $G-$chain recurrent points of f by $CR_G(f)$.

Theorem 6. *Let X be a compact metric $G-$space where G is compact. Suppose a pseudoequivariant continuous onto map f defined on X has the $G-$shadowing property. Then*

(i) $f(\Omega_G(f)) = \Omega_G(f)$.

(ii) $CR_G(f) = \Omega_G(f)$.

Following result gives a decomposition for $\Omega_G(f)$.

Theorem 7. *If a pseudoequivariant map f defined on a compact metric G-space X has the G-shadowing property, then $\Omega_G(f)$ can be written as a disjoint union of sets B_β, where each B_β is an open subset of $\Omega_G(f)$.*

In the following Theorem we obtain an analogue of Aoki's theorem **[3]**.

Theorem 8. *Let $f: X \to X$ be a pseudoequivariant onto map defined on a compact metric G-space X, where G is compact. If f has the G-shadowing then so does $f_{|\Omega_G(f)}$.*

We give an application of G-shadowing property in the following result. Relevant concept defined in G-setting are as follows:

Let X be a compact metric G-space and $f: X \to X$ be a continuous onto map. Then f is said to be G*-minimal* if for each $x \in X$, $cl(\bigcup_{g \in G} O_f(gx)) = X$.

Theorem 9. *Let X be a compact connected metric G- space with metric d and having more than one point, where G is compact then a pseudo equivariant G-minimal homeomorphism does not posses the G-shadowing property.*

In Chapter 6 we define the notion shadowing property on a topological space. Let X be a topological space and let $f: X \to X$ be a continuous map. Suppose A and B are subsets of $X \times X$ containing the diagonal $\{(x,x): x \in X\}$ of X. A sequence $\{x_i : a<i<b\}$ is said to be B*-pseudo orbit* for f if for each i, $a < i < b-1$ we have $(f(x_i), x_{i+1}) \in B$. A B-pseudo orbit $\{x_i : a<i<b\}$ for f is said to be A*-traced* by a point x of X, if for each i, $a < i < b$, $(f^i(x), x_i) \in A$. If f is bijective then $-\infty \leq a < b \leq \infty$, otherwise $0 \leq a < b \leq \infty$. Map f is said to have A*-shadowing property* if there is a subset B of $X \times X$ containing the diagonal such that every B-pseudo orbit for f is A-traced by a point of X.

We define the notion of positively expansive map on a topological space X and obtain its relation with the concept of topologically A-stable, defined as follows:

Let X be a topological space and $f: X \to X$ be a continuous onto map. Suppose A is a subset of $X \times X$ containing the diagonal. Map f is said to be *topologically* A*-stable* if there is a subset B of $X \times X$ containing the diagonal such that for any continuous onto map $h: X \to X$ with $(f(x), h(x)) \in B$, $x \in X$, there exists a continuous map $g: X \to X$ satisfying $(g(x), x) \in A$, for each $x \in X$ and $gh = fg$.

Theorem 11 *Let* X *be a first countable Hausdorff space and* $\wp$ *be a uniformity on* X *consisting of all neighbourhoods of the diagonal. Suppose for* $A \in \wp$, f *is positively* A*-expansive maps having* B*- shadowing property, where* $B \in \wp$ *in such that* $B \circ B \circ B \subset A$ *and* B *is a symmetric neighborhood of the diagonal, then* f *is topologically* B *- stable.*

Major results of the Chapter 2 in its original form are published in the JP Journal of Geometry and Topology, Volume **3**, 2003. Some of the results of Chapter 4 are accepted for publication in the Journal of the Indian Mathematical Society.

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