CHAPTER I

INTRODUCTION

For topological spaces X and Y, two continuous maps $f: X \to X$ and $g: Y \to Y$ are said to be *topologically conjugate* if there exists a homeomorphism $h: X \to Y$ such that $h \circ f = g \circ h$.

A *dynamical property* of a map is that property which is invariant under topological conjugacy. The set of dynamical properties of a map includes the topological but not the differentiable invariants of the underlying space. Since the invariant measures are preserved by topological conjugacy, the Ergodic Theory of a map is included in its dynamics.

Introduced by Anosov in **[2]**, the notion of *shadowing property* (or *pseudo orbit tracing property*) turned out to be one of the very important and useful dynamical properties of continuous maps on metric spaces. Since its inception, this notion has been extensively studied by several Mathematicians including Bowen **[8]**, Walters **[48]**, Morimoto **[29]**, Shub **[43]** and Aoki **[3]**. In recent years theory of shadowing has become a significant part of qualitative theory of dynamical systems containing a lot of interesting and deep results. It plays an important role in the investigation of the stability theory. Shadowing property has also been used to give global error estimates for numerically computed orbits of dynamical systems and to rigorously prove the existence of periodic orbits and chaotic behaviour. Several problems including properties of maps possessing shadowing property and its relation with other dynamical properties have been studied in detail. Moreover, one of the basic problems studied in the theory of shadowing is finding class of maps possessing / not possessing shadowing property. It is known that **[5]** every Anosov diffeomorphism and every Anosov differentiable map on a closed smooth manifold possess shadowing property and an endomorphism $\psi : T^n \to T^n$ of the n-Torus T^n has shadowing property if and only if ψ is hyperbolic **[5]**. In **[30]**, Morimotto has proved that every isometry of a compact Riemann manifold of positive dimension does not have the shadowing property. In **[4]**, Aoki has proved that every distal homeomorphism of a compact connected space does not have shadowing property. In **[15]**, Coven, Ian and Yorke have classified the family of tent maps on [0, 2] possessing shadowing property. Liang **[10]** has characterized uniformly piecewise maps on closed interval of $\mathbb{R}$ possessing the shadowing property. Gedeon and Kuchta have obtained necessary and sufficient condition for maps of the type 2^n to have the shadowing property. Dateyma **[19]** has studied the shadowing property for homeomorphisms of Cantor Set. In **[5]**, Aoki has showed that every group automorphisms of a zero dimensional compact metric group has the shadowing property. In **[38]**, Sakai has proved that if f is an open ε-local expansion then f has shadowing property.

Various generalizations of shadowing property have been obtained and studied. For example, Lipschitz shadowing property, Limit shadowing property and shadowing property for maps on Banach spaces are defined and studied in detail **[34]**. Moreover, the concepts of s-limit shadowing **[1]**, rotation shadowing **[6]**, bishadowing **[34]**, asymptotic shadowing **[11]**, weak shadowing **[14]**, strong shadowing **[34]**, average shadowing **[39]**, uniform pseudo-orbit tracing property **[27]**, shadowing property for flows **[44]** etc are defined and studied in detail. Interrelations of some of these different notions of shadowing properties have also been studied in Sakai **[39]**.

Studying the available literature, it appeared to us that the notion of shadowing property and some other related concepts have not been defined and studied for continuous maps on metric G-spaces and on general topological spaces. Analyzing definitions carefully on metric spaces and studying several related examples and results, we could successfully formulate definitions on metric G-spaces, on topological spaces and obtain interesting examples and results.

The present thesis is the outcome of the researches carried out by the author mainly along these lines. There are six chapters in the thesis and the present chapter aims at providing introduction to the subject matter of the thesis through the recent development in the area.

For a given real number $\delta > 0$ a sequence of points $\{x_i : a < i < b\}$ of a metric space (X, d) is called a δ*-pseudo orbit of* a continuous map $f : X \to X$ if $d(f(x_i), x_{i+1}) < \delta$, for each $i \in (a, b-1)$. Given $\varepsilon > 0$, a δ-pseudo orbit $\{x_i\}$ is said to be ε*-traced* by a point $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ for every

$i \in (a,b)$. Here the symbols a and b are taken as $-\infty \le a < b \le \infty$, if f is bijective and as $0 \le a < b \le \infty$ if f is not bijective. We say that f has the *shadowing property* (or *pseudo orbit tracing property*) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo orbit of f can be ε-traced by some point of X. The notion of δ-pseudo orbit is quiet a natural notion since on account of rounding errors, a computer will actually calculate a pseudo orbit rather than an orbit. Moreover, ε-tracing shows that a pseudo orbit is uniformly approximated by a genuine orbit. If X is compact then the shadowing property of $f: X \to X$ is independent of the choice of metric d compatible with the topology of X **[5]**

We define necessary terminologies and state related results.

Recall that a topological group is a triple $(G,\tau,\cdot)$, where $(G,\cdot)$ is a group and τ is a Hausdorff topology on G such that the map $h: G \times G \to G$ defined by $h(x, y) = xy^{-1}$ is continuous. Some of the standard examples of topological groups are: the additive groups R of real numbers and $\mathbf{Z}$ of integers with usual topology, the additive group $\mathbf{Z}_m$ of residue classes modulo m with discrete topology, the orthogonal group $O(n)$ (respectively $SO(n)$) of all $n \times n$ real orthogonal matrices having determinant 1 or -1 (respectively 1) under multiplication and with the subspace topology of R^{n^2}, the multiplicative group $U(n)$ of n^{th} roots of unity with usual subspace topology of the complex plane, and so on.

By a topological *transformation group* or a G*-space* X, we mean **[9]** a triple (X, G, θ) consisting of a topological space X, a topological group G and an action θ of G on X i.e., a continuous map $\theta: G\times X \to X$ satisfying $\theta(e,x)=x$ and $\theta(g_1,\theta(g_2,x))=\theta(g_1g_2,x)$, where e is the identity of G, $x\in X$ and g_1, $g_2\in G$. An action θ of G on X is called *trivial* if $\theta(g,x)=x$ for each g in G and x in X. By a *metric* $G-$*space* X we mean a metric space X on which a topological group G acts. For g in G and x in X, we denote $\theta(g,x)$ by $g\cdot x$ (or simply by gx) and for $A\subseteq X$, let $gA=\{ga \mid a\in A\}$. A subset A of a G-space X is called G*-invariant* if $\theta(G\times A)\subseteq A$ and for x in X, the set $G(x)=\{gx \mid g\in G\}$ is called the G*-orbit* of x in X. Notice that the relation '~' defined on X as $x\sim y$ if and only if $x=gy$ for some g in G, where $x,y\in X$, is an equivalence relation. Therefore these G-orbits form a partition of X. The quotient space X/G of X having G-orbits as its members is called the *orbit space* of X; and the quotient map $\pi: X\to X/G$, sending x to $G(x)$, is called the *orbit map* which is clearly open and continuous **[9]**. Given G-spaces X and Y, a continuous map $f: X\to Y$ satisfying $f(gx)=gf(x)$ for all g in G and x in X is called an *equivariant map.* In case an equivariant map is a homeomorphism, then f^{-1} is also equivariant. If f is such that for each $x\in X$, $f(G(x))=G(f(x))$ then f is said to be *pseudoequivariant.* Obviously every equivariant map is a pseudoequivariant map. But the converse need not be true. Every pseudoequivariant map f induces a continuous map $\hat{f}:X/G \to Y/G$ given by $\hat{f}(G(x)) = G(f(x))$ **[17]**.

A metric d on a metric G-space X is called an *invariant metric* if $d(x,y) = d(gx, gy)$ for each $g \in G$. If X is a metric G-space with G compact then there exists an invariant metric d on X which induces a metric d_1 on X/G given by $d_1(G(x), G(y)) = \inf\{d(gx, ky) \mid g, k \in G\}$ **[9]**.

We have the following characterizations for identity maps on compact metric spaces to possess shadowing property. This result was pointed by Fujii.

Theorem. 1.1. *Let X be a compact metric space and f be the identity map on X. Then f has the shadowing property if and only if X is totally disconnected.*

In general, composition of maps possessing shadowing property need not possess shadowing property. For example, consider the self-map f defined on the space of real numbers by $f(x) = \frac{x}{2}$ and $g: \mathrm{R} \to \mathrm{R}$ by $g(x) = 2x$. Then $f \circ g$ is the identity map R which does not have the shadowing property.

In case of a homeomorphism, we have the following:

Theorem 1.2. *Let X be a compact metric space. If $f:X\to X$ is a homeomorphism with the shadowing property then f^{-1} also has the shadowing property.*

Following theorem gives the condition under which the shadowing property is preserved under the conjugancy.

Theorem 1.3. *Let X, Y be compact metric spaces, $f:X\to X$ be a continuous map and $h:X\to Y$ be a homeomorphism. Then $g = hfh^{-1}:Y\to Y$ has the shadowing property if and only if f has the shadowing property.*

The following result show that shadowing property is preserved by product of two maps having shadowing property and vice versa.

Theorem 1.4. *Let X and Y be metric spaces and $X\times Y$ be the product space with metric $d((x, y), (x', y')) = \max\{d_1(x,x'), d_2(y, y')\}$ where d_1 and d_2 are metrics for X and Y respectively. Let $f:X\to X$ and be continuous maps and let $f\times g$ be the map defined by $(f\times g)(x, y) = (f(x), g(y))$, $(x,y)\in X\times Y$. Then $f\times g$ has the shadowing property if and only both f and g have the shadowing property.*

Observing that every metric space X is a metric G-space under the trivial action of a group G on X, we analyze the definition of shadowing property for a continuous map f on X we define the concept of shadowing property for a continuous map on a metric G-space and call it the G-shadowing property. Under the trivial action of G on X both the notions coincide. We provide examples to conclude that G-shadowing neither implies nor is implied by the shadowing under non-trivial actions of G on X. Also, examples provided show that the notion of G-shadowing property depends on the action of G i.e. a map f may have shadowing property with respect to one group but need not possess with respect to another group. Besides studying properties of maps possessing G-shadowing property we obtain a characterization for the identity map on a compact metric G-space to possess the G-shadowing property. We also obtain conditions under which G-shadowing property of a continuous map f on a compact metric G-space X implies shadowing property of induced map $\hat{f}$ on X/G and vice versa.

Inverse limit spaces have been studied extensively in dynamical system as well as in continuum theory as a tool to deal with many unsolved problems **[24]**. We first recall necessary definitions and results related to shadowing property on inverse limit spaces. Let (X, d) be a compact metric space. Consider $X^{\mathbf{Z}}$, the compact metric space of all two sided sequences $(x_i)_{i=-\infty}^{\infty}$, endowed with the product topology. A compatible metric $\widetilde{d}$ defined on $X^{\mathbf{Z}}$ is given by

$$\widetilde{d}((x_i),(y_i))=\sum_{i=-\infty}^{\infty}\frac{d(x_i,y_i)}{2^{|i|}}.$$

For a continuous onto map $f:X\to X$ consider the subspace $X_f=\{(x_i)_{i=-\infty}^{\infty}: f(x_i)=x_{i+1}$, for each $i\in Z\}$ of X^Z. Then X_f is a closed subspace of X^Z, known as the *inverse limit space* generated by f. The map $\sigma:X_f\to X_f$ defined by $\sigma((x_i))=(y_i)$, where $y_i=x_{i+1}=f(x_i)$ for each $i\in Z$ is called the *shift map* induced by f. Following result relates the shadowing property of the shift map σ on X_f with the shadowing property of f **[5]**.

Theorem 1.5. *Let X be a compact metric space. If a continuous onto $f:X\to X$ has the shadowing property then the shift map $\sigma:X_f\to X_f$ has the shadowing property.*

Converse of the above result is not true. Following results gives the condition under which the converse is true. We first recall definition of local homeomorphism. A continuous onto map $f:X\to Y$ is a local homeomorphism if for $x\in X$, there is an open neighbourhood U_x of x in X such that $f(U_x)$ is open in Y and $f_{|U_x}:U_x\to f(U_x)$ is a homeomorphism.

Theorem 1.6. *Let X be a compact metric space and $f:X\to X$ be a continuous map such that the shift map $\sigma:X_f\to X_f$ has the shadowing property. If f is a local homeomorphism then f also has the shadowing property.*

In **[11]**, Chen and Li have introduced the concept of the asymptotic shadowing property on a space X. For continuous onto maps the notion of shadowing and asymptotic shadowing are equivalent. Using this they obtained the equivalence of the shadowing property of f and the shadowing property of σ without using the local homeomorphism condition on f. In this paper they have also obtained a class of maps on the closed unit interval I of R, possessing the shadowing property. The result is as follows:

Theorem 1.7. *If* $f : \mathrm{I} \to \mathrm{I}$ *is continuous and has fixed points only the end point of the interval, then* f *has the shadowing property.*

Since shadowing property is preserved under conjugancy, if the space is compact, the above result holds for any closed interval $[a,b]$, $a<b$ in $\mathbf{R}$.

Another characteristics for maps of I to possess the shadowing property is discussed in **[10]** by Chen. We recall certain definitions. Let f be a continuous map of a compact metric space (X,d) to itself and ε be a positive number. A point $x \in X$ is said to be ε*-linked* to a point $y \in X$ by f if there exists an integer $m \geq 1$ and a point $z \in B_\varepsilon(x)$ such that $f^m(z) = y$ and

$$d(f^j(x), f^j(z)) \leq \varepsilon \text{ for } 0 \leq j \leq m.$$

Here $B_\varepsilon(x)$ denotes the closed ball of radius ε about x. A point $x \in X$ is said to be *linked* to $y \in X$ by f if x is ε-linked to a point $y \in X$ by f for every $\varepsilon > 0$. A subset C of X is *linked* by f if every point $c \in C$ is linked to some point in C by f. For a piecewise monotone map f of a compact interval I to itself, let $C(f)$ be the *turning point set* of f i.e. $C(f)=\{x \in \mathrm{I} : f$ has local extrema at $x\}$. A map f has the linking property if $C(f)$ is linked by f. Further, a piecewise monotone map f on $[a,b]$ is uniformly piecewise linear if there are $s>1$ and $\{\alpha_i\}_{i=1}^{m}$ such that $f(x)=\alpha_i \pm sx$ for $x \in [a_{i-1},a_i]$ where $a_0=a<a_1<a_2<...a_m=b$ are the turning points of f and the sign involved depends only on $i=0,1,2,...,m$.

Theorem 1.8. *Suppose f is a map that is conjugate to a continuous uniformly piecewise linear map of an interval to itself. Then f has the shadowing property if and only if it has the linking property.*

As observed above, the dynamics of the shift map σ is closely related to the dynamics of the map f. Hence it is interesting to study the notion of G-shadowing property in this context. In chapter 3 we relate the G- shadowing property of σ and the G- shadowing property of f. First we observe that if X is a metric space $f:X \to X$ is equivariant then X_f is a G-space under the diagonal action of G on X_f defined by $g(x_i)_{i=-\infty}^{\infty}=(g.x_i)_{i=-\infty}^{\infty}$. We obtain the results similar to Theorem 1.5 and Theorem 1.6 for the G-shadowing property. We also give necessary examples to justify the hypothesis. Further in this chapter we obtain a class of maps on the closed unit interval I of $\mathbb{R}$ possessing shadowing property. Using this results and Theorems 1.7, 1.8 we discuss certain class of maps possessing or not possessing the $\mathbb{Z}_2$ shadowing property on $I=[0,1]$. We illustrate through graphs of some maps on I, possessing or not possessing the shadowing property. These graphs again reflect that the notion of shadowing property and the G- shadowing property are independent under the non-trivial action of group on the space.

Like shadowing property another important dynamical property is expansivity of a map. The notion of expansive homeomorphism on a metric space was introduced by Utz **[45]** in 1950. Such homeomorphisms have also been studied extensively. They have lot of applications in Ergodic Theory, Topological Dynamics, Symbolic Dynamic, Continuum Theory, etc. R. F. Williams **[49]** and M. Eisenberg **[18]** have introduced this concept for continuous onto maps on metric space and termed it as positively expansive maps. We recall the definition: Let X be a metric space with metric d. A continuous onto map $f : X \to X$ is said to be *positively expansive* if there is a real number $e > 0$ such that if $x \neq y$ then $d(f^n(x), f^n(y)) > e$ for some non-negative integer n; e is then called an *expansive constant* for f. For compact spaces, the notion is independent of the choice of compatible metric used, though the expansive constant is not independent.

Positive expansivity is not a topological property. For there exists a positively expansive map on $\mathbb{R}$, the usual space of real numbers, but there does not exist any positively expansive map on $(0,1)$. Properties of positively expansive maps have been studied in detail **[4, 5]**. Following result shows positive expansive homeomorphism behave in a different way from expansive homeomorphism on a compact metric space.

Theorem 1.10. *If a homeomorphism of a compact metric space is positively expansive, then the space is a set consisting of finitely many points.*

Theorem 1.10 was first proved by Keynes and Robertson **[25]**. Later, it was proved by Hiraide **[5]** and recently Richeson and Wiseman **[36]** proved the theorem, using elementary topological arguments.

The following result gives the relation between the positive expansivity of a map f and the expansivity of the shift map on the inverse limit space X_f generated by f. The result is as follows **[5]**.

Theorem 1.11. *Let $f: X \to X$ be a continuous onto map defined on a compact metric space X. If f is positively expansive, then the shift map $\sigma: X_f \to X_f$ is positively expansive homeomorphism.*

Relation of positive expansivity with other dynamical properties is also studied. In particular its relation to the shadowing property is extensively studied. For example in **[39]** K. Sakai has shown that various types of shadowing properties such as shadowing property, Lipschitz shadowing property, Limit shadowing property, strong shadowing property are all equivalent for positively expansive open maps of a compact metric space.

The following result gives a characterization for a positively expansive map to possess the shadowing property **[5]**.

Theorem 1.12. *Let X be a compact metric space. A positively expansive map $f:X \to X$ has the shadowing property if and only if f is an open map.*

The following result reflects the behaviour of fixed point for a positively expansive map **[5]**.

Theorem 1.13. *Let $f:X \to X$ be a positively expansive map of a compact connected metric space X. If f is an open map then f has fixed points in X.*

Some of the stability problems in dynamical systems are studied with the help of shadowing property together with related concepts of various recurrent properties such as periodicity, recurrence and non wanderingness. Morimotto **[29]** suggested the problem of which recurrent sets can accept the shadowing property of homeomorphisms. Aoki **[3]** proved that if f is a homeomorphism with the shadowing property on a compact metric space then its restriction to non wandering set also has the shadowing property. Before stating the theorem we define certain terminologies.

Let f be a continuous onto self map defined on a metric space X. A point $x \in X$ is said to be a *non wandering point* of f if for every neighbourhood U of x, there exists an integer $n>0$ such that $f^n(U) \cap U \neq \phi$. The set of all non wandering points of f is denoted by $\Omega(f)$. If the space is compact then $\Omega(f)$ is a non-empty closed subset of X. An interesting problem is to know which maps f are non wandering maps i.e. every point of a metric space X is a non wandering point of f. Such problems are studied in detail in **[13, 23, 43]**.

Certain recurrent points are defined using pseudo orbit of a map f. One such is a chain recurrent point. Let f be a continuous onto map of a metric space to itself. A point x is *related to itself* if for every $\delta > 0$ there is a finite δ-pseudo orbit for f from x to itself. It is denoted by $x \sim x$. Let $CR(f)$ denote the set of all chain recurrent points of f i.e. $CR(f) = \{x \in X : x \sim x\}$. Again as $\Omega(f)$, $CR(f)$ is a non-empty closed subset of X if X is a compact metric space. It is observed that ~ is an equivalence relation on $CR(f)$ and $\Omega(f)$ is a subset of $CR(f)$. Both $CR(f)$ and $\Omega(f)$ are f-invariant subsets of X if f is a homeomorphism. But in general it is not true. Following theorem gives the condition under which they are always f-invariant and $\Omega(f)$ is same as $CR(f)$ **[5]**.

Theorem 1.14. *Let X be a compact metric space and $f : X \to X$ be a continuous onto map having the shadowing property. Then following holds*

(1) $\Omega(f) = CR(f)$ (2) $f(\Omega(f)) = \Omega(f)$.

In **[3]**, Aoki has proved the following result, thus answering a question raised by Morimotto.

Theorem 1.15. *Let X be a compact metric space and $f:X \to X$ be a continuous onto map. If f has the shadowing property then so does $f_{|\Omega(f)}$. Moreover, if f is positively expansive then the set of periodic points are dense in $\Omega(f)$.*

As stated above it will be interesting to know which homeomorphisms are non wandering homeomorphisms. For example minimal homeomorphism on a compact metric space is always non-wandering homeomorphism. We recall the definition of a minimal map. A continuous onto map $f:X \to X$ defined on a compact metric space is said to be *minimal* if for each x in X, the f-orbit $O_f(x)=\{f^n(x):n \geq 0\}$ of x in dense in X i.e. $cl(O_f(x))=X$. A minimal homeomorphism $f:X \to X$ is characterized as the map having largest closed invariant sets as the empty set and the whole set X. Following theorem relates the shadowing property and minimal homeomorphism **[5]**.

Theorem 1.16. *Let $f:X \to X$ be a homeomorphism of a compact metric space X. Suppose X is connected and not one point. If f is minimal then f does not have the shadowing property.*

Theorem 1.16 is considered to be one of interesting application of shadowing property in topological dynamics. Another good application of expansive homeomorphisms having the shadowing property is specification. We recall the definition of specification.

Definition 1.17. Let $f:X \to X$ be a homeomorphism of a compact metric space X. The homeomorphism f is said to have *specification* if for any $\varepsilon > 0$ there exists $M = M(\varepsilon) > 0$ such that for any finite sequence of points $x_1, x_2, \ldots, x_k \in X$, and for j with $2 \leq j \leq k$, choosing any sequence of integers $a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_k \leq b_k$, such that $a_j - b_{j-1} \geq M (2 \leq j \leq k)$ and an integer p with $p \geq M + (b_k - a_1)$ there exists a point $x \in X$ with $f^p(x) = x$ so that $d(f^i(x), f^i(x_j)) < \varepsilon$ for $a_j \leq i \leq b_j$ and $1 \leq j \leq k$.

We note here that specification together with expansivity of map gives interesting results in Ergodic Theory **[21]**. Following result relates topological Anosov maps (i.e. expansive homeomorphism having the shadowing property) and specification **[5]**.

We first recall the definition of topologically mixing. A continuous onto map $f:X \to X$ of a metric space X is said to be a *topologically mixing* if for non-empty open set U, V there is $N > 0$ such that $U \cap f^n(V) \neq \phi$ for all $n \geq N$.

Theorem 1.18. *Let $f : X \to X$ be a homeomorphism of a compact metric space. If f is a topologically mixing, expansive homeomorphism having shadowing property then f has the specification.*

The notion of expansive homeomorphism is defined and studied in detail on a metric G- space. Let $h : X \to X$ be a homeomorphism. Then h is called G*-expansive* if there exists a $\delta > 0$ such that whenever $x, y \in X$ with $G(x) \neq G(y)$, there exists an integer n satisfying $d(h^n(u), h^n(v)) > \delta$ for all $u \in G(x)$ and $v \in G(x)$; δ is then called a G- expansive constant for h **[16]**.

In chapter 4 we define the notion of positive expansivity on a metric G-space and term it as positively G-expansive map. Through several examples, we conclude that positive G-expansivity neither implies nor is implied by positive expansivity. We study properties of G-expansive maps in detail. Observing that the concept of G-shadowing property and positively G-expansive maps are independent of each other we obtain a necessary and sufficient condition for positively G-expansive maps to possess the G-shadowing property. We also obtain a class of maps on a compact connected metric G-space that are not positively G-expansive. Further, in this chapter, we define the notion of non wandering points and chain recurrent points of a continuous onto self map defined on a metric G-space X. We discuss several examples and properties of such sets. We obtain result similar to Theorem 1.13 for maps possessing G-shadowing property.

In chapter 5 we continue with our study of chain recurrent points for G-shadowing property. We first obtain result similar to Theorem 1.14 in G-setting. We further define the concept of G-periodic points of a map f defined on a metric G-space X and show that set $Per_G(f)$ of all G-periodic points of f is dense in $\Omega_G(f)$, for a positively G-expansive maps f having the G-shadowing property. Necessary examples are provided to strengthen the hypothesis of different results obtained at each stage. We also define the concept of G-specification and relate it to the G-expansive homeomorphism having the G-shadowing property. We define the notion of minimal maps on a metric G-space X. Several examples of such maps are studied and we obtain a result similar to Theorem 1.16.

We recall the notion of topological stable homeomorphism. Let X be a compact metric space and $f: X \to X$ be a homeomorphism. If for given $\varepsilon > 0$ there exists $\delta > 0$ such that for any homeomorphism $h: X \to X$ with $d(h(x), f(x)) < \delta$ for all $x \in X$, there is a continuous map $p: X \to X$ satisfying $d(p(x), x) < \varepsilon$ for all $x \in X$ and $fg = gh$ then f is called *topologically stable* in the class of homeomorphism.

Recently in **[37]** Richeson and Wiseman introduced the concept of topologically positively expansive maps on a metric space X and showed that the concept is equivalent to the notion is positively expansive maps for compact metric spaces.

Walter **[48]** in 1978 showed that topologically Anosov maps are always topologically stable in the class of homeomorphism. This fact established that shadowing property is a weak form of topological stability. Following theorem is proved in **[48]**.

Theorem 1.18. *Let X be a compact metric space and $f: X \to X$ be an expansive homeomorphism having the shadowing property. Then f is always topological stable in the class of homeomorphism.*

The notion of expansive homeomorphism is defined and studied on a topological space in **[16]**. We recall the definition. Let X be a topological space and let A be a subset of $X \times X$, then a homeomorphism h on X is called A-expansive if for x, y in X with $x \neq y$, there exists an integer n such that $(h^n(x), h^n(y)) \notin A$. An example of an A-expansive homeomorphisms on $\mathrm{I} = [0,1]$ is given which does not contain A_ε, where $A_\varepsilon = \{(x,y) : d(x,y) \leq \varepsilon\}$, for any $\varepsilon > 0$.

In chapter 6 we define the notion of shadowing property on topological space and give examples of maps on I which does not have the shadowing property but has the A-shadowing property for some subset A of $\mathrm{I} \times \mathrm{I}$ containing the diagonal. Besides studying properties of maps having A-shadowing property we also obtain results similar to Theorem 1.1, 1.2 and 1.3.

We define the notion of topological stability and positively expansive maps on a topological space. We also compare the notion of positively A -expansive maps with notion of topologically positively expansive maps and show that every topologically positively expansive map is positively A -expansive. But converse need not be true. Finally we obtain an analogue of Theorem 1.18 for positively A - expansive maps.