CHAPTER III

G - SHADOWING ON INVERSE LIMIT SPACES

The notion of inverse limit spaces have been studied extensively in the context of dynamical systems as well as in continuum theory and has been used as a tool to deal with many unsolved problems **[24]**. For instance one can recall that Williams **[50]** provides an early demonstration of the utility of the inverse limit constructions in dynamical systems. On the other hand in continuum theory many surprising and complicated examples have been constructed using inverse limits **[24]**.

The inverse limit technique is particularly useful in those cases where it allows one to build complicated but useful structures out of simpler ones. Moreover, some dynamical properties of a continuous map $f : X \to X$ of a space X can be obtained by investigating the topological properties of the inverse limit space (X_f, σ), where σ is the shift map, and vice versa.

In this Chapter we relate the G-shadowing property of a continuous map f on a metric G-space with G-shadowing property of shift map σ (under the diagonal action of G). We also study some maps on the closed unit interval $I = [0,1]$ of R possessing / not possessing shadowing / $\mathbf{Z_2}$-shadowing property. In Section 1, we study the G-shadowing property of the shift map σ on X_f induced by a continuous map f on a metric G-space X

having G-shadowing property. In Section 2, observing that converse of the main result in Section 1 is not true, we find condition under which converse is true. In the last section, using results obtained earlier we illustrate through graphs some maps on the closed unit interval $\mathrm{I}=[0,1]$ of $\mathbf{R}$ possessing / not possessing shadowing / $\mathbf{Z_2}$-shadowing property. We also use these graphs to show that the notions of shadowing property and G-shadowing property are independent of each other.

1. G-shadowing of σ from G-shadowing of f.

In this section we study the G-shadowing property of the shift map σ on the inverse limit space X_f generated by a continuous map f on a G-space X. Let X be a metric G-space with metric d. Recall that a continuous map $f: X \to X$ is said to be a *equivariant map* if $f(gx) = g f(x)$, for each $g \in G$ and $x \in X$. Map f being equivariant, we consider the diagonal action of G on X_f, defined by $g(x_i) = (gx_i)$, for all $g \in G$ and $(x_i) \in X_f$. In the following theorem we derive the G-shadowing property of the shift map $\sigma: X_f \to X_f$ from the G-shadowing of property f. We first observe the following lemma.

Lemma 3.1.1. *Let (X, d) be a compact metric G-space, where G is compact. Then a continuous onto map $f: X \to X$ satisfying the following property has the G-shadowing property.*

"For each $\varepsilon > 0$, there is a $\delta > 0$ such that if the points $\{x_0, x_1, \ldots, x_k\}$ satisfy that for each i, $0 \le i \le k-1$, there exists $g_i \in G$ such that $d(g_i f(x_i), x_{i+1}) < \delta$,

for some, then there exists an $x \in X$ such that for each i, $0 \le i \le k-1$, there exists $p_i \in G$ satisfying $d(f^i(x), p_i x_i) < \varepsilon$".

Proof. Let $\varepsilon > 0$ be given. By the hypothesis there is a $\delta > 0$ such that if $\{x_0, x_1, \ldots, x_k\}$ satisfy that for each i, $0 \le i \le k-1$, there exists $g_i \in G$ such that $d(g_i f(x_i), x_{i+1}) < \delta$, then there is an x in X such that $d(f^i(x), p_i x_i) < \frac{\varepsilon}{2}$, for some $p_i \in G$, $0 \le i \le k$. In order to show that f has the G-shadowing property we show that every δ-G pseudo orbit $\{x_j : j \ge 0\}$ for f is ε-traced by a point of X. Let $\{x_j : j \ge 0\}$ be a δ-G pseudo orbit for f. Then for each $j \ge 0$, there exists $g_j \in G$ such that $d(g_j f(x_j), x_{j+1}) < \delta$. Fix $k > 0$ and put $z_i = x_i, 0 \le i \le k$. Then $\{z_i : 0 \le i \le k\}$ is a δ-G pseudo orbit for f. From the hypothesis there exists a point z_k in X such that for each $i, 0 \le i \le k$, there exists $p_i^k \in G$ with

$$d(f^i(z_k), p_i^k z_i) < \frac{\varepsilon}{2} \qquad (*)$$

Since (*) is true for each $k > 0$, we get a sequence $\{z_k : k > 0\}$ in the compact metric space X and hence there exists a convergent subsequence say $\{z_{k_n} : n \ge 0\}$ of $\{z_k : k > 0\}$. Suppose $z_{k_n} \to z$ as $n \to \infty$. We show that $\{x_j : j \ge 0\}$ is ε-traced by the point z of X. As $z_{k_n} \to z$, for each $j \ge 0$, we have $f^j(z_{k_n}) \to f^j(z)$ as $n \to \infty$. This implies that for each $j \ge 0$ there is a positive integer N_j satisfying

$$d(f^j(z_{k_n}), f^j(z)) < \frac{\varepsilon}{2}, \text{ for all } n \geq N_j, \qquad \textbf{(I)}$$

and for any fixed $k_n, n \geq N_j$, corresponding to each $i, 0 \leq i \leq k_n$, there exists $p_i^{k_n} \in G$ satisfying

$$d(f^i(z_{k_n}), p_i^{k_n} z_i) < \frac{\varepsilon}{2}. \qquad \textbf{(II)}$$

Choose k_n for each $j \geq 0$ such that $n \geq N_j$ and $j \leq k_n$. Then **(I)** and **(II)** gives

$$d(p_j^{k_n} x_j, f^j(z)) \leq d(f^j(z_{k_n}), p_j^{k_n} z_j) + d(f^j(z_{k_n}), f^j(z))$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This implies $\{x_j : j \geq 0\}$ is ε-traced by the point z of X. Therefore f has the G-shadowing property.

Theorem 3.1.2. *Let f be an equivariant onto self-map defined on a compact metric G-space X, where G is compact. If f has the G-shadowing property then the shift map σ on the inverse limit space X_f has the G-shadowing property.*

Proof. Let $\alpha = diam X$ and let $\varepsilon > 0$ be given. Choose $N > 0$ such that $0 < \frac{\alpha}{2^{N-2}} < \frac{\varepsilon}{4}$. From uniform continuity of $f^j, 0 \leq j \leq 2N$, there exists $\gamma > 0$ such that

$$d(x,y) < \gamma \Rightarrow d(f^j(x), f^j(y)) < \frac{\varepsilon}{8}, \quad 0 \leq j \leq 2N \qquad \textbf{(I)}$$

Since f has the G-shadowing property, there is a $\tau > 0$ such that every τ-G pseudo orbit for f is γ-traced by a point of X. Choose δ such that $0 < 2^N \delta < \tau$. In view of Lemma 3.1.1, to show that σ has the G-shadowing property it is sufficient to show that every finite δ-G pseudo orbit for σ is ε-traced by a point of X_f. Let $\{(x_i^n) : 0 \le n \le k\}$ be a finite δ-G pseudo orbit for σ. This implies that for each n, $0 \le n \le k-1$, there exists $g_n \in G$ satisfying $\tilde{d}(g_n \sigma(x_i^n), x_i^{n+1}) < \delta$ which is equivalent to

$$\sum_{i=-\infty}^{\infty} \frac{d(g_n f(x_{-i}^n), x_{-i}^{n+1})}{2^{|i|}} < \delta$$

and hence

$$\frac{d(g_n f(x_{-N}^n), x_{-N}^{n+1})}{2^N} < \delta .$$

This proves $\{x_{-N}^n : 0 \le n \le k\}$ is a τ-G pseudo orbit for f. Since f has the G-shadowing property, $\{(x_{-N}^n) : 0 \le n \le k\}$ is γ-traced by a point of X, say, y. Hence for each n, $0 \le n \le k$, there exists $k_n \in G$ such that

$$d(f^n(y), k_n x_{-N}^n) < \gamma \qquad \textbf{(II)}$$

Put $y_{i-N} = f^i(y)$, if $i \ge 0$ and $y_{i-N} \in f^{-1}(y_{i+1-N})$, for $i < 0$. Then, $y_{-N} = y$, $y_{-N+1} = f(y), \ldots\ldots, y_N = f^{2N}(y), \ldots\ldots$. Also for $i < 0$, $y_{i-N} \in f^{-1}(y_{i+1-N})$ implies $f(y_{i-N}) = y_{i+1-N}$. Therefore, $\tilde{y} = (y_i) \in X_f$. We show that $\{(x_i^n) : 0 \le n \le k\}$ is ε-traced by the point $\tilde{y}$ of X_f. For each $n, 0 \le n \le k$, $k_n \in G$, satisfying **(II)** also satisfies

$$\tilde{d}(\sigma^n(\tilde{y}), k_n(x_i^n)) = \sum_{i=-\infty}^{\infty} \frac{d(f^n(y_i), k_n x_i^n)}{2^{|i|}}$$

$$= \sum_{i=-\infty}^{-N-1} \frac{d(f^n(y_i), k_n x_i^n)}{2^{|i|}} + \sum_{i=-N}^{N} \frac{d(f^n(y_i), k_n x_i^n)}{2^{|i|}}$$

$$+ \sum_{i=N+1}^{\infty} \frac{d(f^n(y_i), k_n x_i^n)}{2^{|i|}}.$$

Observe that

$$\sum_{i=-\infty}^{-N-1} \frac{d(f^n(y_i), k_n x_i^n)}{2^{|i|}} \le \sum_{i=-\infty}^{-N-1} \frac{\alpha}{2^{|i|}} < \frac{\varepsilon}{8}$$

and

$$\sum_{i=N+1}^{\infty} \frac{d(f^n(y_i), k_n x_i^n)}{2^{|i|}} < \frac{\varepsilon}{8}.$$

Moreover (I) and (II) imply that for each $n, 0 \le n \le k,$ and $0 \le j \le 2N$, $d(f^{n+j}(y), f^j(k_n x_{-N}^n)) < \frac{\varepsilon}{8}$. Using successively, $f(x_i^n) = x_{i+1}^n$, we obtain $f^j(k_n x_{-N}^n) = k_n x_{-N+j}^n$, $0 \le j \le 2N$ and hence

$$\sum_{i=-N}^{N} \frac{d(f^n(y_i), k_n x_i^n)}{2^{|i|}}$$

$$= \frac{d(f^n(y_{-N}), k_n x_{-N}^n)}{2^N} + \frac{d(f^n(y_{-N+1}), k_n x_{-N+1}^n)}{2^{N-1}} + \ldots\ldots +$$

$$+ \frac{d(f^n(y_0), k_n x_0^n)}{2^0} + \ldots\ldots + \frac{d(f^n(y_N), k_n x_N^n)}{2^N}$$

$$= \frac{d(f^n(y), k_n x_{-N}^n)}{2^N} + \frac{d(f^n(f(y)), k_n x_{-N+1}^n)}{2^{N-1}} + \ldots\ldots +$$

$$+ \frac{d(f^n(f^N(y)), k_n x_0^n)}{2^0} + \ldots\ldots + \frac{d(f^n(f^{2N}(y)), k_n x_N^n)}{2^N}$$

$$< \frac{\varepsilon/8}{2^N} + \frac{\varepsilon/8}{2^{N-1}} + \ldots\ldots + \frac{\varepsilon/8}{2^0} + \ldots\ldots + \frac{\varepsilon/8}{2^N} < \frac{3\varepsilon}{8}$$

Therefore for each $n, 0 \le n \le k,$ there exists $k_n \in G$ such that

$$\tilde{d}(\sigma^n(\tilde{y}), k_n(x_i^n)) < \frac{\varepsilon}{8} + \frac{3\varepsilon}{8} + \frac{\varepsilon}{8} < \varepsilon$$

Thus $\{(x_i^n): 0 \le n \le k\}$ is ε-traced by the point $\tilde{y} \in X_f$. Hence σ has the G-shadowing property

2. G-shadowing of f from G-shadowing of σ.

Using the following result, we give an example of a map f not possessing G-shadowing property but σ, generated by f, possessing G-shadowing property.

Lemma 3.2.1. *Let f be a continuous self map on $\mathrm{I} = [0, 1]$ and let I_f be the corresponding inverse limit space generated by f. Suppose the shift map $\sigma : \mathrm{I}_f \to \mathrm{I}_f$ has only two fixed points, say, $\tilde{p}$ and $\tilde{q}$ with $\tilde{p} = (p_i)_{i=-\infty}^{\infty}$ and $\tilde{q} = (q_i)_{i=-\infty}^{\infty}$, such that p_0 and q_0 are the end points of $f(\mathrm{I})$. Then the shift map σ has the shadowing property.*

Proof. Since $\tilde{p} = (p_i)_{i=-\infty}^{\infty}$ and $\tilde{q} = (q_i)_{i=-\infty}^{\infty}$ are fixed points of σ, $\sigma((p_i)_{i=-\infty}^{\infty}) = (p_i)_{i=-\infty}^{\infty}$ implies $f(p_i) = p_i$ for each $i \in \mathrm{Z}$. In particular, $f(p_0) = p_0$ implies p_0 is a fixed point of f. Note that $p_i = p_0$, for each $i \in \mathrm{Z}$. Similarly, for the fixed point $\tilde{q} = (q_i)_{i=-\infty}^{\infty}$, $q_i = q_0$, for each $i \in \mathrm{Z}$, and q_0 is a fixed point of f. Therefore $p_0, q_0 \in Fixf$. In fact $Fixf = \{p_0, q_0\}$. For if $z_0 \in Fixf$, then that will create a third fixed point of σ which is not possible. Also, by hypothesis p_0 and q_0 are the end points of $f(\mathrm{I})$. Therefore by

Theorem 1.7 $f_{|f(\mathrm{I})} : f(\mathrm{I}) \to f(\mathrm{I})$ has the shadowing property. We consider the following two cases.

Case 1. Suppose f is an onto map. Then $f(\mathrm{I}) = \mathrm{I}$ and $p_0 = 0, q_0 = 1$. Thus by Theorem 1.7 f has the shadowing property which further implies by Theorem 1.5 that σ has the shadowing property.

Case 2. Suppose f is a non onto map. We show that there exists no $(x_i) \in \mathrm{I}_f$ such that $x_i \in \mathrm{I} - f(\mathrm{I})$. For, if such an i exists, then there exists no $y \in \mathrm{I}$ such that $f(y) = x_i$. But since $(x_i) \in \mathrm{I}_f$, there is an i such that $x_{i-1} \in \mathrm{I}$ with $f(x_{i-1}) = x_i$. Therefore $(x_i) \in \mathrm{I}_f$, implies $x_i \in f(\mathrm{I})$, for each $i \in \mathbf{Z}$ But $f_{|f(\mathrm{I})} : f(\mathrm{I}) \to f(\mathrm{I})$ has the shadowing property, therefore σ has the shadowing property.

Example 3.2.2. Consider the map $f : \mathrm{I} \to \mathrm{I}$ defined by

$$f(x) = \begin{cases} \frac{x}{2} + \frac{1}{8}, & \text{if } 0 \le x \le \frac{1}{4} \\ 4x - \frac{3}{4}, & \text{if } \frac{1}{4} \le x \le \frac{5}{16} \\ \frac{13}{16} - x, & \text{if } \frac{5}{16} \le x \le \frac{3}{8} \\ \frac{x}{2} + \frac{1}{4}, & \text{if } \frac{3}{8} \le x \le \frac{5}{8} \\ \frac{19}{16} - x, & \text{if } \frac{5}{8} \le x \le \frac{11}{16} \\ 4x - \frac{9}{4}, & \text{if } \frac{11}{16} \le x \le \frac{3}{4} \\ \frac{x}{2} + \frac{3}{8}, & \text{if } \frac{3}{4} \le x \le 1 \end{cases}$$

and suppose $G = \mathbf{Z}_2$ act on I by the usual action. That f is an equivariant map follows because $-1f(x) = f(1-x)$ for each $x \in \mathrm{I}$. Further observe that f is not a local homeomorphism as f is not an open map. Next, we show that f does not have the shadowing property. For an ε, $0 < \varepsilon < \frac{1}{6}$, and any $\delta > 0$, we choose a δ-pseudo orbit $\{x_i : i \geq 0\}$ as follows: $x_0 = 0,\ x_1 \in U_\delta(f(x_0))$ such that $x_1 > f(x_0)$. Continue this process of obtaining x_i's such that $x_i > x_{i-1}$, $x_i \in U_\delta(f(x_{i-1})), i = 0, 1, \ldots, k-1$ and $x_k \in U_\delta((\frac{1}{4}))$ for some k. Take $x_{k+1} = \frac{1}{4}$. Choose $x_{k+2} \in U_\delta(f(x_{k+1}))$ such that $\frac{1}{2} > x_{k+2} > f(x_{k+1})$. Continue this process of obtaining x_i's such that $\frac{1}{2} > \ldots > x_m > \ldots > x_{k+1}$, $x_i \in U_\delta(f(x_{i-1}))$, $|x_m - \frac{1}{4}| > \frac{\varepsilon}{4}$ for some m. The δ-pseudo orbit $\{x_i\}$ thus obtained is not $\frac{\varepsilon}{4}$-traced by any point of I. Therefore f does not have the shadowing property. In fact we have shown that $f|_{[0, \frac{1}{2}]}$ does not have the shadowing property. This further implies that the induced map $\hat{f} : \mathrm{I}/\mathbf{Z}_2 \to \mathrm{I}/\mathbf{Z}_2$ does not have the shadowing property as it is conjugate to $f|_{[0, \frac{1}{2}]}$. Hence the map f does not have the $\mathbf{Z}_2$-shadowing property.

Finally, we show that the shift map σ has the $\mathbf{Z}_2$-shadowing property. It is sufficient to show $\hat{\sigma}$ has the shadowing property. Since f is a non-onto map on I, $(x_i) \in \mathrm{I}_f$ implies $x_i \in f(\mathrm{I})$, for each $i \in \mathbf{Z}$. Also f is a equivariant map implies σ is an equivariant map and for $\sigma = \sigma_f$ we have $\hat{\sigma}_f = \sigma_{\hat{f}}$. But the only fixed points of $\sigma_{\hat{f}}$ are $\mathbf{Z}_2(\tilde{\frac{1}{4}})$ and $\mathbf{Z}_2(\tilde{\frac{1}{2}})$, where $\tilde{p} = (\ldots, p, p, p, \ldots)$ for any $p \in \mathrm{I}$. Also $\mathrm{Z}_2(\frac{1}{4})$ and $\mathrm{Z}_2(\frac{1}{2})$ are fixed points of $\hat{f}$. Therefore by Lemma 3.2.1, $\hat{\sigma}$ has the shadowing property and σ has the $\mathbf{Z}_2$-shadowing property.

Observe that the map f in above example is not a local homeomorphism. In the following result we obtain the conditions under which the converse of Theorem 3 1 2 is true

Theorem 3.2.3. *Let f be an equivariant onto local homeomorphism defined on a compact metric G-space X, where G is compact. If the shift map σ on X_f has the G-shadowing property then f has the G-shadowing property.*

Proof. Let $\varepsilon > 0$ be given. Since σ has G-shadowing property, there exists a $\delta > 0$ such that every δ-Gpseudo orbit for σ is ε-traced by a point of X_f. Choose $N > 0$ such that $0 < \frac{\alpha}{2^{N-2}} < \frac{\delta}{4}$, where $\alpha = diam X$. Since f is a local homeomorphism on a compact metric space, there exists γ, $0 < \gamma < \frac{\delta}{8}$ such that for each $x \in X$, if U is a γ-neighborhood of x, then $f_{|U} : U \to f(U)$ is a homeomorphism. Also, for each j, $1 \le |j| \le N$, f^j is uniformly continuous. Therefore there is an $\eta > 0$ such that

$$d(x,y) < \eta \Rightarrow d(f^j(x), f^j(y)) < \gamma \qquad \textbf{(I)}$$

In order to show that f has the G-shadowing property we show that every η-Gpseudo orbit for f is ε-traced by a point of X. Let $\{x^n : n \ge 0\}$ be a η-Gpseudo orbit for f. Then for each $n \ge 0$, there exists $g_n \in G$ such that

$$d(g_n f(x^n), x^{n+1}) < \eta \qquad \textbf{(II)}$$

Consider the sequence $\{\widetilde{y}^n : n \geq 0\}$ constructed as follows:

$\widetilde{y} = (y_i^n) = (\ldots, x^n, f(x^n), \ldots\ldots)$ i.e. $y_i = f^i(x^n)$ if $i \geq 0$ and $y_i \in f^{-1}(y_{i+1})$ if $i < 0$

For each $n \geq 0$, $g_n \in G$ satisfying (II) satisfies

$$\widetilde{d}(g_n \sigma(\widetilde{y}^n), \widetilde{y}^{n+1}) = \sum_{i=-\infty}^{\infty} \frac{d(g_n f(y_i^n), y_i^{n+1})}{2^{|i|}}$$

$$= \sum_{i=-\infty}^{-N-1} \frac{d(g_n f(y_i^n), y_i^{n+1})}{2^{|i|}} + \sum_{i=-N}^{N} \frac{d(g_n f(y_i^n), y_i^{n+1})}{2^{|i|}} + \sum_{i=N+1}^{\infty} \frac{d(g_n f(y_i^n), y_i^{n+1})}{2^{|i|}}$$

We observe

$$\sum_{i=-\infty}^{-N-1} \frac{d(g_n f(y_i^n), y_i^{n+1})}{2^{|i|}} \leq \sum_{i=-\infty}^{-N-1} \frac{\alpha}{2^{|i|}}$$

$$= \frac{\alpha}{2^{N+1}} + \frac{\alpha}{2^{N+2}} + \ldots\ldots = \frac{\alpha}{2^N} < \frac{\delta}{8}$$

Similarly, $\sum_{i=N+1}^{\infty} \frac{d(g_n f(y_i^n), y_i^{n+1})}{2^{|i|}} < \frac{\delta}{8}$. Moreover from (I) and (II) for any $n \geq 0$

and for each j with $|j| \leq N$, $d(f^j(g_n f(x^n)), f^j(x^{n+1})) < \gamma < \frac{\delta}{8}$. Since f is an

equivariant map, we obtain

$$d(g_n f^j(f(x^n)), f^j(x^{n+1})) < \frac{\delta}{8} \text{---------(III)}$$

Now consider

$$\sum_{i=-N}^{N} \frac{d(g_n f(y_i^n), y_i^{n+1})}{2^{|i|}} =$$

$$\frac{d(g_n f(f^{-N}(x^n)), f^{-N}(x^{n+1}))}{2^N} + \frac{d(g_n f(f^{-N+1}(x^n)), f^{-N+1}(x^{n+1}))}{2^{N-1}} +$$

$$\ldots+\frac{d(g_n f(f^0(x^n)), f^0(x^{n+1}))}{2^0}+\ldots\ldots+\frac{d(g_n f(f^N(x^n)), f^N(x^{n+1}))}{2^N}$$

$$<\frac{\delta/8}{2^N}+\frac{\delta/8}{2^{N-1}}+\ldots+\frac{\delta/8}{2^0}+\ldots+\frac{\delta/8}{2^N}<\frac{3\delta}{8}$$

This implies that $\{\tilde{y}^n : n \geq 0\}$ is a δ-G pseudo orbit for σ. But σ has the G-shadowing property. Therefore $\{\tilde{y}^n : n \geq 0\}$ is ε-traced by a point of X_f, say, $\tilde{y}$. Therefore for each $n \geq 0$, there is a $p_n \in G$ satisfying $\tilde{d}(\sigma^n(\tilde{y}), p_n \tilde{y}^n) < \varepsilon$ which implies $\sum_{i=-\infty}^{\infty} \frac{d(f^n(y_i), p_n y_i^n)}{2^{|i|}} < \varepsilon$ and hence we obtain $\frac{d(f^n(y_0), p_n y_0^n)}{2^0} < \varepsilon$. But $y_0^n = x_n$. Therefore, $\{x^n : n \geq 0\}$ is ε-traced by the point y_0 of X. Thus f has the G-shadowing property.

3. Maps on [0, 1] possessing / not possessing Z_2-shadowing/shadowing property.

We study here some maps on the closed unit interval I of R possessing / not possessing the shadowing / G-shadowing property. First we give the following result which gives a class of maps on I not having the shadowing property.

Theorem 3.3.1. *Let f be a continuous map defined on* I *with* $Fixf$ *an at most countable nowhere dense set having at least three elements. For* $c_i \in Fixf$, *suppose* $f[c_i, c_{i+1}] \subset [c_i, c_{i+1}]$, *for some consecutive values of* i, *say* n *and* $n+1$. *If either* $f(x) > x$ *or* $f(x) < x$ *for all* $x \in (c_n, c_{n+1}) \cup (c_{n+1}, c_{n+2})$, *then f does not have the shadowing property.*

Proof. It is sufficient to consider the case when $Fixf = \{c_1, c_2, c_3\}$. Without loss of generality we can assume that $c_1 < c_2 < c_3$. Suppose $f(x) > x$ for all $x \in (c_1, c_2) \cup (c_2, c_3)$. Choose an $\varepsilon > 0$ such that $\varepsilon < \min\{c_2 - c_1, c_3 - c_2\}$. For a given $\delta > 0$ choose a δ-pseudo orbit $\{x_i : i \geq 0\}$ as follows: Choose $x_0 = c_1$, $x_1 \in U_\delta(f(x_0))$, $x_2 \in U_\delta(f(x_1))$,.., such that $c_2 > x_2 > f(x_1) > x_1 > f(x_0) > x_0$. This way we obtain an increasing sequence $\{x_i\}$ converging to c_2 and therefore we find $k > 0$ such that $x_k \in U_\delta(c_2)$. Thus we have points $x_0, x_1,.....x_k$ such that $x_i \in U_\delta(f(x_{i-1}))$, $0 \leq i \leq k$. We also have an i, say l satisfying $|x_l - c_1| > \frac{\varepsilon}{4}$. Next, take $x_{k+1} = c_2$ and continue the process of selecting x_i's. Observe that we shall obtain m satisfying $|x_m - c_2| > \frac{\varepsilon}{4}$. We first prove that the δ-pseudo orbit $\{x_i : i \geq 0\}$ obtained in this manner is not $\frac{\varepsilon}{4}$-traced by any point of I.

Obviously $\{x_i : i \geq 0\}$ is not $\frac{\varepsilon}{4}$-traced by any point of $\mathrm{I} - U_{\frac{\varepsilon}{4}}(c_1)$, because $|t - c_1| > \frac{\varepsilon}{4}$ if $t \in \mathrm{I} - U_{\frac{\varepsilon}{4}}(c_1)$. We consider other possible cases:

<u>Case</u> (1) Suppose $t \in [c_1, c_1 + \frac{\varepsilon}{4})$. Then there is x_m in $\{x_i : i \geq 0\}$ which is at a distance greater than $\frac{\varepsilon}{4}$ from $f^m(t)$ for any $t \in [c_1, c_1 + \frac{\varepsilon}{4})$

<u>Case</u> (2) Suppose $t \in (c_1 - \frac{\varepsilon}{4}, c_1]$. If $f^n(t) \in [0, c_2]$, for all $n \geq 0$, then there is x_m in $\{x_i : i \geq 0\}$ which is at a distance greater than $\frac{\varepsilon}{4}$ from $f^m(t)$. If $f^n(t) \in [c_2, c_3]$, for all $n \geq 0$, then there is x_l in $\{x_i : i \geq 0\}$ which is at a distance greater than $\frac{\varepsilon}{4}$ from $f^l(t)$.

Therefore f does not have shadowing property.

Note. Let $f: \mathrm{I} \to \mathrm{I}$ be a pseudoequivariant map such that the induced map $\hat{f}$ satisfies the hypothesis of the Theorem 3.3.1. Then the induced map $\hat{f}$ does not have the shadowing property and therefore f does not have the Z_2-shadowing property.

The following result gives a family of maps on I having Z_2-shadowing property. Recall the Theorem of Chen and Li **[11]** which says that if the only fixed points of a continuous map f on I are the end points of I then f has the shadowing property. Also, it is known that a uniformly piecewise linear map has the shadowing property if and only if it has the f linking property **[10]**.

Theorem 3.3.2. *Let f be a pseudoequivariant map defined on a* $\mathbf{Z_2}$*-space* I. *Then f has the* $\mathbf{Z_2}$*-shadowing property in each of the following case:*

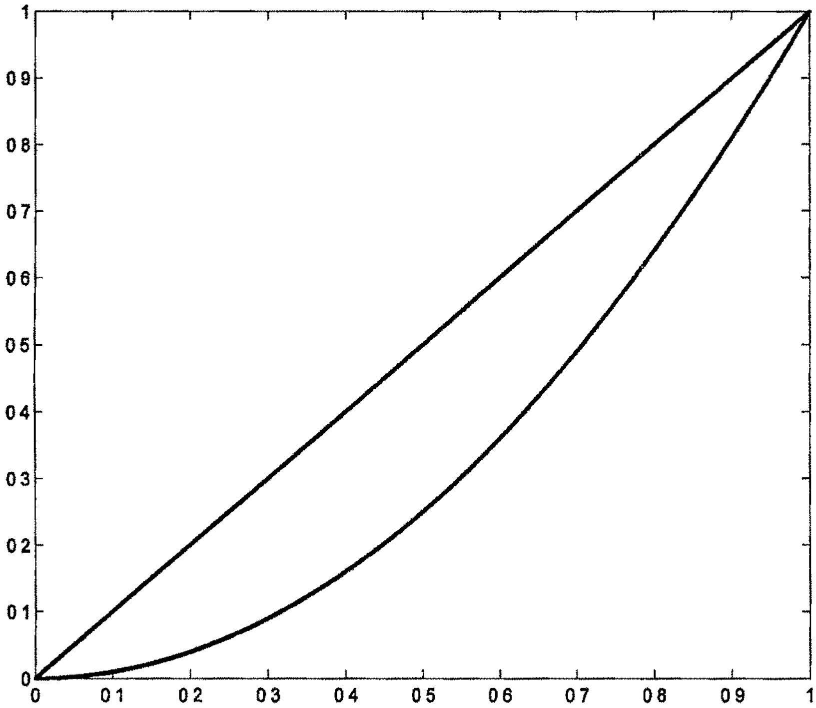
1. *The only fixed points of the induced map $\hat{f}: \mathrm{I}/Z_2 \to \mathrm{I}/Z_2$ are the end points of the induced space I/Z_2.*
2. *The map $\hat{f}$ is uniformly piecewise linear continuous having the linking property.*

Proof. 1. Since the only fixed points of $\hat{f}$ are the end points of $\mathrm{I}/\mathbf{Z_2}$, by Theorem 1.7, $\hat{f}$ has the shadowing property. Therefore f has the $\mathbf{Z_2}$-shadowing property.

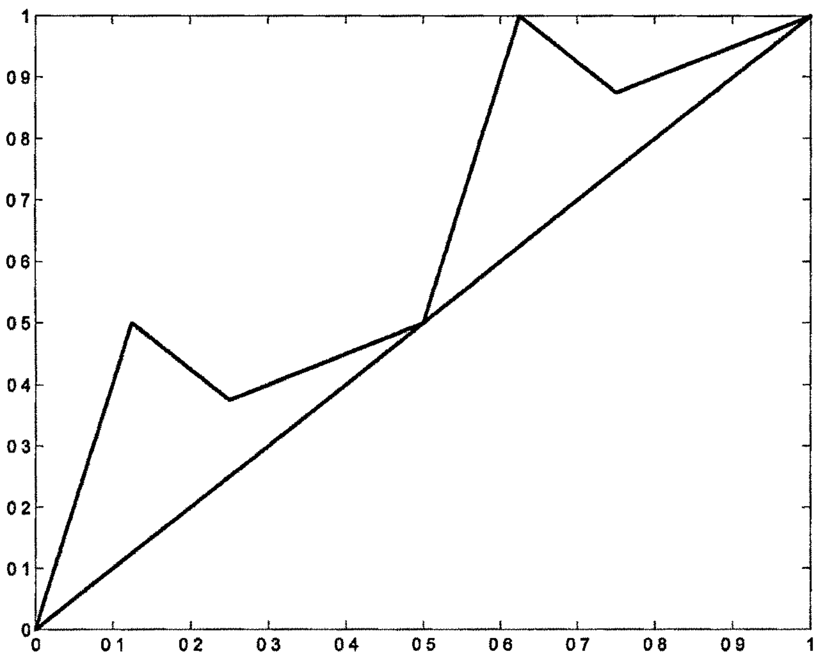
2. Since $\hat{f}$ is uniformly piecewise linear map and has the linking property, therefore by Theorem 1.8 $\hat{f}$ has the shadowing property and hence f has the $\mathbf{Z_2}$-shadowing property.

We now study graphs of some maps on I possessing / not possessing $\mathbf{Z_2}$-shadowing / shadowing property.

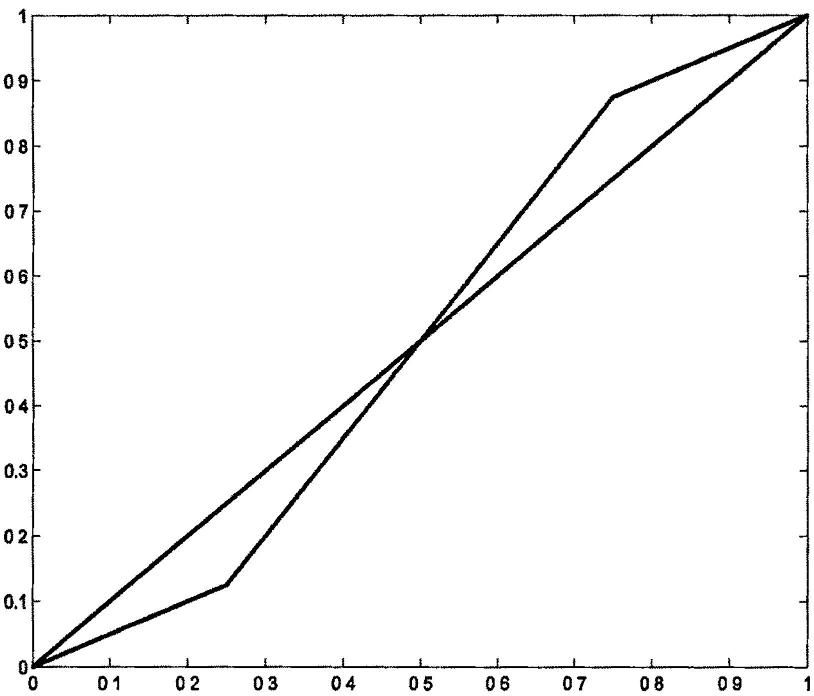
A. Consider the following graph of the map f on I defined by $f(x) = x^2$. Since the only fixed points of f are end points of I, by Theorem 1.7, f has the shadowing property. Also f has the $\mathbf{Z_2}$-shadowing property. Therefore the shift map on the corresponding inverse limit space has the shadowing property as well as $\mathbf{Z_2}$-shadowing property.



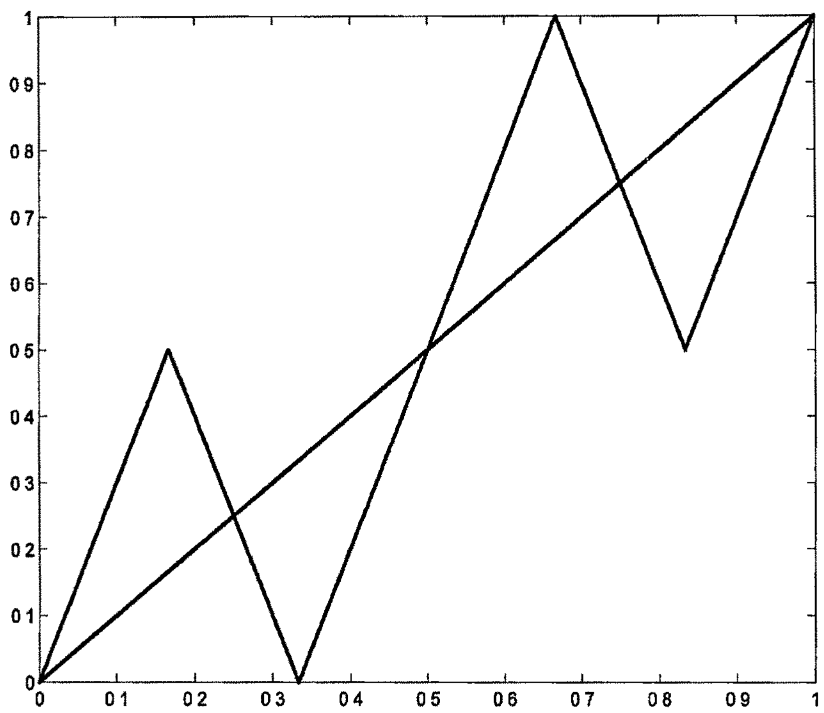
B. Observe that the map in the following graph has three fixed points and satisfies the hypothesis of Theorem 3.3.1. Therefore it does not have the shadowing property



C. Next is graph of an equivariant map f. Since only fixed points of the induced map $\hat{f}$ are the end points of I/Z_2, therefore by Theorem 3.3.2, f has the $\mathbf{Z}_2$-shadowing property. From Theorem 3.1.2, we conclude that the corresponding shift map on the inverse limit space has the $\mathbf{Z}_2$-shadowing property.



D. Map in the following graph is uniformly continuous pseudoequivariant. Since the turning point $\frac{1}{6}$ is not linked to any other turning point of f and hence by Theorem 1.8 f does not have the shadowing property. Since the induced map has the linking property, by Theorem 3.3.2 f has the $\mathbf{Z}_2$-shadowing property. Theorem 3.1.2 now implies that the shift map on the corresponding inverse limit space has the $\mathbf{Z}_2$-shadowing property.



E. As observed in Example 3 2.2, map in the following graph has neither shadowing nor $\mathbf{Z}_2$-shadowing property. But the shift map has the $\mathbf{Z}_2$-shadowing property.