CHAPTER V

SOME APPLICATIONS OF G–SHADOWING PROPERTY

In this chapter we continue with our study of G-non wandering points, G-chain recurrent points of a continuous map f on a metric G-space X and obtain certain applications of G-shadowing property. Recall that if f is a continuous onto map defined on a compact metric space X and f has the shadowing property, then the set of non wandering points gets decomposed into smaller sets, called as *basic sets*.

In Section 1, we obtain such a decomposition for the set of G-non wandering points and use it to obtain a result similar to that of Theorem 1.15 proved by Aoki **[3]**.

In Section 2, we define the concept of periodic points on a metric G-space and call it as G-periodic points. We mainly study the behavior of the set of G-periodic points of a positively G-expansive map having the G-shadowing property.

In Section 3, we introduce notion of specification for homeomorphisms on metric G-spaces. We find conditions under which a homeomorphism on a compact metric G-space having G-shadowing has G-specification.

In section 4, we study relation of G-shadowing of a homeomorphism with minimality on G-space.

1. Decomposition Theorem.

In this section we obtain a decomposition for the set of G-non wandering point of a continuous onto map defined on a compact metric G-space X and use it to show that $f_{|\Omega_G(f)}$ has the G-shadowing property, if f has the G-shadowing property.

The following result gives decomposition for the set of G-non wandering points.

Theorem 5.1.1 *If a pseudoequivariant map f defined on a compact metric G-space X has the G-shadowing property, then $\Omega_G(f)$ can be written as a disjoint union of sets B_β, where each B_β is an open subset of $\Omega_G(f)$.*

Proof. Since f has the G-shadowing property, by Theorem 4.4.11 $\Omega_G(f)=CR_G(f)$. For a given $\delta>0$, $x \overset{\delta}{\sim}_G y$ is an equivalence relation on $CR_G(f)$ and $\Omega_G(f)=CR_G(f)=\{B_\beta : \beta\in\Lambda\}$ is the set of equivalence classes.

Let $x\in B_\beta$ then for every $y\in B_\beta$, $x \overset{\delta}{\sim}_G y$. Take any $y\in B_\beta$. Then there exists finite δ-Gpseudo orbits from x to y and y to x, say, $\theta_1=\{x=x_0, x_1, x_2, \ldots\ldots\ldots, x_k=y\}$ and $\theta_2=\{y=y_0, y_1, y_2, \ldots\ldots\ldots, y_l=x\}$. Since θ_1 is a finite δ-Gpseudo orbit for f, there is $g_0\in G$, such that $d(g_0 f(x_0), x_1)<\delta$ or $d(g_0 f(x), x_1)<\delta$. This implies $g_0 f(x)\in U_\delta(x_1)$. Continuity of map f implies there exists γ, $0<\gamma<\frac{\delta}{3}$, such that $f(U_\gamma(x))\subset g_0^{-1}U_\delta(x_1)$. In order to show B_β is open we show that

$U_{\gamma}(x)\cap\Omega_G(f)\subset B_\beta$. Let $z\in U_{\gamma}(x)\cap\Omega_G(f)$. We complete the proof by showing that $y \overset{\delta}{\sim}_G z$ i.e. there is a finite δ-G pseudo orbit for f from y to z and z to y. Let $y\in B_\beta$. Since θ_1 is a finite δ-G pseudo orbit for f from x to y and $z\in U_{\gamma}(x)\cap\Omega_G(f)$, therefore $\theta_1'=\{z=x_0', x_1, x_2, \ldots\ldots\ldots, x_k=y\}$ is a finite δ-G pseudo orbit for f from z to y.

Now θ_2 is a finite δ- pseudo from y to x. Therefore there is a $p_{l-1}'\in G$ such that

$$
\begin{aligned}
&d(p_{l-1}'f(y_{l-1}), x)<\delta\\
&\Rightarrow d(f(p_{l-1}y_{l-1}), x)<\delta, \text{ for some } p_{l-1}\in G\\
&\Rightarrow f(p_{l-1}y_{l-1})\in U_\delta(x)
\end{aligned}
$$

Observe that every point of θ_1 and θ_2 are in $\Omega_G(f)$. For, consider a point x_i of θ_1, then $\{x_i, x_{i+1}, x_{i+2},\ldots, x_k=y=y_0, y_1,\ldots, y_l=x=x_0, x_1,\ldots\ldots, x_i\}$ is a finite δ-G pseudo orbit for f from x_i to itself. Hence $x_i\in\Omega_G(f)$. Now $y_{l-1}\in\Omega_G(f)$ and f has the G-shadowing property, therefore $f(p_{l-1}y_{l-1})\in\Omega_G(f)$. Hence $f(p_{l-1}y_{l-1})\in U_\delta(x)\cap\Omega_G(f)$. Since $\gamma<\frac{\delta}{3}$, we consider the following two cases:

Case1. $f(p_{l-1}y_{l-1})\in cl[U_\gamma(x)]\cap\Omega_G(f)$

Then either $f(p_{l-1}y_{l-1})\in U_\gamma(x)$ or $f(p_{l-1}y_{l-1})$ is a limit point of $U_\gamma(x)$. If $f(p_{l-1}y_{l-1})\in U_\gamma(x)$, then $\{y=y_0, y_1, y_2,\ldots\ldots\ldots, y_{l-1}, z\}$ is a finite δ-G pseudo orbit from y to z. If $f(p_{l-1}y_{l-1})$ is a limit point of $U_\gamma(x)$ then there is $t\in U_\gamma(x)$ such that $d(f(p_{l-1}y_{l-1}), t)<\gamma$. This implies

$$d(f(p_{l-1}y_{l-1}), z) \le d(f(p_{l-1}y_{l-1}), t) + d(t, z) < \gamma + \gamma < \delta$$

Therefore, $\{y = y_0, y_1, y_2, \ldots\ldots\ldots, y_{l-1}, z\}$ is again a finite δ-G pseudo orbit from y to z.

Case 2. $f(p_{l-1}y_{l-1}) \notin cl[U_\gamma(x)] \cap \Omega_G(f)$

Since $\Omega_G(f)$ is a compact subset of X, therefore $cl[U_\gamma(x)] \cap \Omega_G(f)$ is a compact subset of $\Omega_G(f)$. Because $f(p_{l-1}y_{l-1}) \in \Omega_G(f)$ but $f(p_{l-1}y_{l-1}) \notin cl[U_\gamma(x)] \cap \Omega_G(f)$, it follows that

$$d(f(p_{l-1}y_{l-1}), cl[U_\gamma(x)] \cap \Omega_G(f)) > 0.$$

Also, there is $t \in cl[U_\gamma(x)] \cap \Omega_G(f)$ such that

$$d(f(p_{l-1}y_{l-1}), t) < \delta \qquad \text{(I)}$$

Therefore, $t \in cl(U_\gamma(x))$ and $z \in U_\gamma(x)$ implies

$$d(t, z) \le d(t, x) + d(z, x) < 2\gamma$$

Since $t \in cl[U_\gamma(x)] \cap \Omega_G(f)$, $t \overset{\gamma}{\sim}_G t$. Therefore there is a finite γ-G pseudo orbit $\theta_3 = \{t = t_0, t_1, \ldots\ldots, t_n, t\}$. This implies that there is $q_n \in G$ such that $d(q_n f(t_n), t) < \gamma$. Consider

$$d(q_n f(t_n), z) \le d(q_n f(t_n), t) + d(t, z) < \gamma + 2\gamma = 3\gamma < \delta$$

i.e. $d(q_n f(t_n), z) < \delta$ (II)

From (I) and (II) we have $\{y = y_0, y_1, \ldots\ldots, y_{l-1}, t, t_1, \ldots\ldots, t_n, z\}$ as a finite δ-G pseudo orbit y to z. Therefore in each case there is a finite δ-G pseudo orbit from y to z. Hence $z \in B_\beta$ which implies B_β is open

In the following theorem we show that if f has the G-shadowing property then so does $f_{|\Omega_G(f)}$.

Theorem 5.1.2. *Let $f : X \to X$ be a pseudoequivariant onto map defined on a compact metric G-space X, where G is compact. If f has the G-shadowing then so does $f_{|\Omega_G(f)}$.*

Proof. Let $\varepsilon > 0$ be given. Since f has the G-shadowing property, there is a $\delta > 0$ such that every δ-Gpseudo orbit for f is $\frac{\varepsilon}{2}$-traced by a point of X. Also, G-shadowing of f implies $\Omega_G(f) = CR_G(f) = \bigcup_{\beta \in \Lambda} B_\beta$, where $B_\beta \cap B_\alpha = \phi, \alpha \neq \beta$. By Theorem 5.1.1, each B_β is open. Therefore, $\{B_\beta : \beta \in \Lambda\}$ is an open cover of $\Omega_G(f)$. Compactness of $\Omega_G(f)$ implies there is a finite sub cover $B_1, B_2, \ldots\ldots, B_n$ of $\Omega_G(f)$. Hence $\Omega_G(f)$ is a disjoint union of open sets B_i and therefore each B_i is a closed subset of $\Omega_G(f)$ and hence compact. Thus $\delta_{ij} = d(B_i, B_j) > 0$, for all i, j, $i \neq j$. Let $\delta_1 = \min\{\delta_{ij} : i \neq j, 1 \leq i, j \leq n\}$. Choose α such that $0 < \alpha < \min\{\delta_1, \delta\}$. In order to show that $f_{|\Omega_G(f)}$ has the G-shadowing property we show that every α-Gpseudo orbit for $f_{|\Omega_G(f)}$ in $\Omega_G(f)$ is ε-traced by a point of $\Omega_G(f)$. Let $\{x_i : i \geq 0\}$ be an α-Gpseudo orbit for $f_{|\Omega_G(f)}$ in $\Omega_G(f)$. Then by the choice of α, $\{x_i : i \geq 0\} \subset B_k$, for some k. Take $x_p, x_q \in \{x_i : i \geq 0\}$ such that $p < q$.

Then $x_p,\ x_q \in \Omega_G(f)$ implies $x_q \overset{\delta}{\sim}_G x_p$. Therefore there exist finite δ-G pseudo orbits

$$\theta_1 = \{x_p = y_0, y_1, \ldots, y_{k_1} = x_q\}$$

and

$$\theta_2 = \{x_q = z_0, z_1, \ldots, z_{k_2} = x_p\}$$

say. Put $k = k_1 + k_2$ and construct a δ-G pseudo orbit $\{t_i : i \geq 0\}$ such that $t_{ki} = x_p$, $t_{ki+j} = y_j$, $0 \leq j \leq k_1$, $t_{k_1+ki} = x_q$ and $t_{k_1+ki+j} = z_j$, $0 \leq j \leq k_2$. Since f has the G-shadowing property $\{t_i : i \geq 0\}$ is $\frac{\varepsilon}{2}$-traced by a point of X, say, $x_{p,q}$. This implies that for each $i \geq 0$, there exists $l_i \in G$ such that

$$d(f^i(x_{p,q}), l_i t_i) < \frac{\varepsilon}{2} \text{---------(I)}$$
$$\Rightarrow d(f^{ki+j}(x_{p,q}), l_{ki+j} t_{ki+j}) < \frac{\varepsilon}{2}, \text{ for all } j,\ 0 \leq j \leq k$$

Consider the set $T = cl(\{f^{ki}(x_{p,q}) : i \geq 0\})$. Following are the two possible cases.

Case 1. Suppose T is discrete. Then T is finite being closed subset of a compact set Therefore there is $r > 0$ such that $f^r(x_{p,q}) = x_{p,q}$. Thus, $x_{p,q} \in \Omega_G(f)$.

Case 2. Suppose T is not discrete. Then by compactness of T, there is a subsequence $\{f^{ki_n}(x_{p,q})\}$ which is convergent. Let it converge to $x'_{p,q}$ i.e.

$$f^{ki_n}(x_{p,q}) \to x'_{p,q} \text{ as } n \to \infty$$
$$\Rightarrow \text{there exists } N > 0 \text{ such that } d(f^{ki_n}(x_{p,q}), x'_{p,q}) < \frac{\varepsilon}{2}, \text{ for all } n \geq N.$$

From (I) for each n, $n \in N$ there is $p_{ki_n} \in G$ satisfying

$$d(p_{ki_n}t_{ki_n}, x'_{p,q}) \leq d(p_{ki_n}t_{ki_n}, f^{ki_n}(x_{p,q})) + d(f^{ki_n}(x_{p,q}), x'_{p,q})$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

But $t_{ki_n} = x_p$. Thus, there is $p_{ki_n} \in G$ such that $d(p_{ki_n}x_p, x'_{p,q}) < \varepsilon$. We show that $x'_{p,q} \in \Omega_G(f)$. Let $\beta > 0$ be given. Uniform continuity of f implies there exists an $\eta > 0$ such that $d(x, y) < \eta \Rightarrow d(f(x), f(y)) < \beta$. Also, $f^{ki_n}(x_{p,q}) \to x'_{p,q}$ implies there exists $N_1 > 0$ and $N_2 > 0$ such that $d(f^{ki_n}(x_{p,q}), x'_{p,q}) < \eta$, for all $i_n \geq N_1$ and $d(f^{ki_n}(x_{p,q}), x'_{p,q}) < \beta$ for all $i_n \geq N_2$. This implies $\{x'_{p,q}, f^{ki_n}(x_{p,q}), x'_{p,q}\}$ is a finite β-G pseudo orbit for f. Therefore $x'_{p,q} \in \Omega_G(f)$.

Thus for $x_p, x_q \in \{x_i : i \geq 0\}$ there is a point $x'_{p,q} \in \Omega_G(f)$. Since $x_p, x_q \in \{x_i : i \geq 0\}$ is arbitrary, it follows that for each $x_p, x_q \in \{x_i : i \geq 0\}$ with $p < q$, there exists $x'_{p,q} \in \Omega_G(f)$. Consider the sequence $\{x'_{p,q}\}$ in $\Omega_G(f)$. Since $\Omega_G(f)$ is compact, a subsequence of $\{x'_{p,q}\}$ will converge. Suppose it converges to y. Since $\Omega_G(f)$ is a closed subset of X, it follows that $y \in \Omega_G(f)$. Since $f^{ki_n}(x_{p,q}) \to x'_{p,q}$ and $x'_{p,q} \to y$, it follows that $\{x_i : i \geq 0\}$ is ε-traced by the point y of X. Therefore $f_{|\Omega_G(f)}$ has the G-shadowing property.

Let X be a compact metric G-space and $f: X \to X$ be a pseudoequivariant map having the G-shadowing property. Then by Theorem 5.1.1 we have $\Omega_G(f) = \bigcup_{\beta\in\Lambda} B_\beta$, where $\alpha \neq \beta \Rightarrow B_\beta \cap B_\alpha = \phi$. With the above assumptions we have the following theorem.

Theorem 5.1.3. *If U and V are open subsets of B_β then there is an $n > 0$ and $g \in G$ such that $U \cap g f^n(V) \neq \phi$.*

Proof. Let $x, y \in B_\beta$ such that $x \in U$ and $y \in V$. It is sufficient to show that for any $\varepsilon > 0$, there exists $n > 0$ and $g \in G$ such that $(U_\varepsilon(x)) \cap gf^n(U_\varepsilon(y)) \neq \varphi$. By Lemma 4.3.2 there exists an η, $0 < \eta < \varepsilon$, such that for all $x \in X$ and $g \in G$, $U_\eta(gx) \subset gU_\varepsilon(x)$. By Theorem 5.1.2, $f_{|\Omega_G(f)}$ has the G-shadowing property, therefore there exists a $\delta > 0$ such that every δ-Gpseudo orbit in B_β is η-traced by a point of B_β. Now $x, y \in B_\beta \subset \Omega_G(f)$ implies $x \overset{\delta}{\sim}_G y$. Therefore there is a finite δ-Gpseudo orbit $\theta = \{y = y_0, y_1, \ldots\ldots y_k = x\}$ from y to x. Since $f_{|\Omega_G(f)}$ has the G-shadowing property, it therefore follows θ is η-traced by a point of B_β, say, z. This implies that for each $i, 0 \leq i \leq k$, there exists $p_i \in G$ such that $d(f^i(z), p_i y_i) < \eta$. In particular for $i = 0$ and $i = k$, there are $p_0, p_k \in G$ such that $d(z, p_0 y) < \eta$ and $d(f^k(z), p_k x) < \eta$. But $d(z, p_0 y) < \eta$ implies $z \in U_\eta(p_0 y) \subset p_0 U_\varepsilon(y)$. This implies

$$f^k(z) \in f^k(p_0 U_\varepsilon(y)).$$

Pseudoequivariancy of f implies there exists a $p \in G$ such that

$$f^k(z) \in p f^k(U_\varepsilon(y)) \qquad \textbf{(I)}$$

Also, $d(f^k(z), p_k x) < \eta$ implies $f^k(z) \in U_\eta(p_k x) \subset p_k U_\varepsilon(x)$ i.e.

$$f^k(z) \in p_k U_\varepsilon(x) \qquad \textbf{(II)}$$

Therefore from **(I)** and **(II)**

$$p f^k(U_\varepsilon(y)) \cap p_k U_\varepsilon(x) \neq \varphi$$

or $\quad g f^k(U_\varepsilon(y)) \cap U_\varepsilon(x) \neq \varphi$, for some $g \in G$.

2. G - periodic points.

In this section, we define the concept of G-periodic point of f a continuous map defined on a metric G–space X and relate it with a G–non wandering point of f.

Definition 5.2.1. Let X be a metric G-space and $f: X \to X$ be a continuous map. A point x of X is said to be a G*-periodic point* of f if there is an integer $n > 0$ and $g \in G$ such that $f^n(x) = gx$.

Smallest such positive integer n is said to be the G *- period of* x. We denote the set of all G - periodic points of f by $Per_G(f)$.

Obviously, every periodic point of f is a G - periodic point of f. But converse need not be true. Refer Example 5.2.2(a) and 5.2.2(b).

Example 5.2.2.(a) Consider the space, map and group G_1 of Example 2.1.2(c). It is observe there that $X/G=\{G_1(0),G_1(1),G_1(-1),G_1(\frac{1}{2}),G_1(\frac{1}{3})\}$. Observe that $\frac{1}{2}$ is not a periodic point of f. Also, $f^2(\frac{1}{2})=\frac{1}{4}, h^2(\frac{1}{4})=\frac{1}{2}$. Therefore, $f^2(\frac{1}{2})= h^{-2}(\frac{1}{2})$. This implies $\frac{1}{2}\in Per_G(f)$. Infact $f^2(G(\frac{1}{2}))=G(\frac{1}{2}))$. Therefore by Proposition 5.2.4, every point of $G(\frac{1}{2})$ is a G-periodic point. Similarly, every point of $G(\frac{1}{3})$ is a G-periodic point.

5.2.2.(b)* Note that $G=\left\{e^{\imath 0},e^{\imath\frac{\pi}{2}},e^{\imath\pi},e^{\imath\frac{3\pi}{2}}\right\}$. Here $e^{\imath 0}\in Per_G(f)$ and $G(e^{\imath\theta})=\left\{e^{\imath 0},e^{\imath\frac{\pi}{2}},e^{\imath\pi},e^{\imath\frac{3\pi}{2}}\right\}$. We observe that $e^{\imath\frac{\pi}{2}}$ is not a G-periodic point of f.

In fact for any n, $f^n(e^{\imath\frac{\pi}{2}})=e^{\imath\frac{\pi}{2^{n+1}}}$ and for no $n\in\mathbb{N}$, $f^n(e^{\imath\frac{\pi}{2}})=ge^{\imath\frac{\pi}{2}}$, for any $g\in G$, i.e. $f^n(e^{\imath\frac{\pi}{2}})\neq ge^{\imath\frac{\pi}{2}}$ for any $n\in\mathbb{N}$ and any $g\in G$. In fact $Per_G(f)=\{e^{\imath 0},e^{\imath\pi}\}$ and $Per\,(f)=\{e^{\imath 0}\}$. Note that f is not a pseudoequivariant map.

5.2.2.(c) Consider S^1 and suppose $G=S^1$ by the usual action complex multiplication. Then for each $e^{\imath\theta}\in S^1$, $G(e^{\imath\theta})=S^1$. If $f:S^1\to S^1$ is defined as $f\left(e^{\imath\theta}\right)=e^{\imath\frac{\theta}{2}}$, then for $g=e^{-\imath\frac{\theta}{2}}\in S^1$, $f(e^{\imath\theta})=g.e^{\imath\theta}$. Therefore $e^{\imath\theta}\in Per_G(f)$ and hence $Per_G(f)=X$. Observe that here $Per\,f=\{e^{\imath 0}\}$.

Remarks 5.2.3. (i) Examples 5.2.2(b) and 5.3 3(c) show that a point may be a G- periodic point with respect to one group but need not G-periodic with

* Consider X, f, G as in Example 4 4 6

respect another group. Thus G- periodicity of a point depends upon the action of the group.

(ii) If the action of G on X is transitive then every point of X is a G- periodic point for any continuous onto maps f on X.

(iii) If x is a G-periodic point of a continuous onto map f defined defined on a metric G-space X then x is a G- non wandering point of f.

(iv) That G-non wandering point need not be a G-periodic point is justified by the following example.

Consider the map f on S^1 and group G of Example 4.4.6. Then from from Example 4.4.6 we have $\Omega_G(f)=\{e^{i0}, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3\pi}{2}}\}$ whereas $Per_G(f)=\{e^{i0}, e^{i\pi}\}$.

We first observe in the following result which relates G- periodic point of f and the periodic points of the induced map $\hat{f}$.

Proposition 5.2.4. *Let X be a metric G- space and $f: X \to X$ be a pseudoequivariant map. If $x \in X$ is a G- periodic point of f with period k, then every point of $G(x)$ is a G- periodic point of f with period k. Moreover, a point x is a G- periodic point of f if and only if $G(x)$ is a periodic point of $\hat{f}$.*

Proof. Let x be a G-periodic point of f with period $k \in N$ and let $g \in G$ such that $f^k(x)$ Now for any $t \in G$, f being pseudoequivariant, $f^k(tx)=t'f^k(x)=t'gx=mx$ where $m=t'g \in G$. Therefore tx is a G- periodic point of f with period k'.

Now if x is a G - periodic point of f, then there exists $k>0$ and $g\in G$ such that $f^k(x)=gx$. For this $k>0$ $\hat{f}^k(G(x))=G(f^k(x))=G(gx)=G(x)$, which implies $G(x)$ is a periodic point of $\hat{f}$.

Conversely, suppose $G(x)$ is a periodic point of $\hat{f}$ with period k. Then,

$$\begin{aligned}
&\hat{f}^k(G(x))= G(x)\\
&\Rightarrow G(f^k(x))=G(x)\\
&\Rightarrow g.f^k(x)=mx, \text{ for some } g,m\in G.\\
&\Rightarrow f^k(x)=lx\\
&\Rightarrow x \text{ is a } G\text{-periodic point of } f.
\end{aligned}$$

The following result gives the relation between $Per_G(f)$ and $\Omega_G(f)$.

Theorem 5.2.5. *Let X be a compact metric G - space, with G - compact and let $f:X\to X$ be an onto pseudoequivariant map. Suppose f is positively G-expansive map having G-shadowing property. Then the set of G-periodic points of f are dense in $\Omega_G(f)$.*

Proof. Let e be a G - expansive constant for f and ε be such that $0<\varepsilon<e$ and $x\in\Omega_G(f)$. We show that $U_\varepsilon(x)\cap Per_G(f)\neq\phi$. Choose η, $0<\eta<\varepsilon$ such that for each $x\in X$ and $g\in G$, $gU_\eta(x)\subset U_{\frac{\varepsilon}{2}}(gx)$. Since f has the G-shadowing property, there is a $\delta>0$ such that every $\delta-G$ pseudo orbit for f is η- traced by a point of X. Also, G-shadowing of f implies $\Omega_G(f)=CR_G(f)$ and $f_{|\Omega_G(f)}$ has the G-shadowing property. Since $x\in\Omega_G(f)=CR_G(f)$ there is a $\delta-G$ pseudo orbit $\{x=x_0,x_1,\ldots,x_k=x\}$.

Consider the $\delta - G$ pseudo orbit $\{y_i : i \geq 0\} = \{x\ , x_1, \ldots, x_k = x, x_1, \ldots\}$ of $f_{|\Omega_G(f)}$ i.e. for each $i \geq 0$, i.e. for each i, $y_i = y_{k+i}$. G-shadowing property of $f_{|\Omega_G(f)}$ implies $\{y_i : i \geq 0\}$ is η- traced by a point of $\Omega_G(f)$, say, y. Therefore for each $i \geq 0$, there exists $p_i \in G$ such that $d(f^i(y), p_i y_i) < \eta$. Also, $d(f^{k+i}(y), p_{k+i} y_{k+i}) < \eta$. But for each $i \geq 0$, $y_i = y_{k+i} = x_j$ for some j, $0 \leq j \leq k-1$. This implies there exist p_i, $p_{k+i} \in G$ such that $d(f^i(y), p_i x_j) < \eta$ and $d(f^{k+i}(y), p_{k+i} x_j) < \eta$. Now, $d(f^i(y), p_i x_j) < \eta$ implies

$$f^i(y) \in U_\eta(p_i x_j) \subset p_i U_{\frac{\varepsilon}{2}}(x_j)$$
$$\Rightarrow d(p_i^{-1} f^i(y), x_j) < \frac{\varepsilon}{2}$$

Similarly, $d(p_{i+k}^{-1} f^{k+i}(y), x_j) < \frac{\varepsilon}{2}$.

$$\Rightarrow d(p_i^{-1} f^i(y), p_{k+i}^{-1} f^i(f^k(y))) < \varepsilon \text{, for all } i \geq 0.$$

But f positively G - expansive Therefore, $G(y) = G(f^k(y))$

$$\Rightarrow f^k(y) = gy \text{, for some } g \in G$$
$$\Rightarrow y \text{ is a } G\text{ - periodic point of } f$$

Now for $i = 0$, there is $p_0 \in G$ such that $d(p_0 y, x) < \eta < \varepsilon$ which implies $p_0 y \in U_\varepsilon(x)$. Since f is a pseudoequivariant map, every point of $G(y)$ is a G-periodic point of f. Therefore, $p_0 y \in U_\varepsilon(x) \cap \Omega_G(f)$. Thus, G-periodic point of f are dense in $\Omega_G(f)$.

3. G-specification.

In this section we discuss another application of G-expansive maps having G-shadowing property. We first define the necessary terminologies.

Definition 5.3.1. Let $f: X \to X$ be a homeomorphism of a compact metric G-space. Then f is said to have *G-specification* if for any $\varepsilon > 0$ there exists $M = M(\varepsilon) > 0$ such that for any finite sequence of points $g_1x_1, g_2x_2, \ldots, g_kx_k \in X$, for some $g_1, g_2, \ldots, g_k \in G$ and for $2 \le j \le k$, choosing any sequence of integers $a_1 \le b_1 < a_2 \le b_2 < \ldots < a_k \le b_k$ such that $a_j - b_{j-1} \ge M (2 \le j \le k)$ and an integer p with $p \ge M(b_k - a_1)$ there exists a point $x \in X$ with $f^p(x) = gx$, for some $g \in G$ and satisfying $d(f^i(x), p_i f^i(x_j)) < \varepsilon$ some $p_i \in G$ and for $a_j \le i \le b_j$, $1 \le j \le k$.

In the following result we relate the G-specification with other dynamical properties.

Theorem 5.3.2 *Let (X,d) be a compact metric G-space with G compact and d an invariant metric. Suppose $f: X \to X$ is a G-expansive pseudoequivariant homeomorphism having the G*-shadowing property. If for non-empty open sets U, V *in X there is an $N > 0$ such that for all $n \ge N$ there exists $g_n \in G$ satisfying $U \cap g_n f^n(V) \ne \phi$, for some $p_i \in G$ then f has the G*-specification.

Proof. Let $e>0$ be a G-expansive constant for f and take ε such that $0<\varepsilon<\frac{e}{2}$. Since f has the G-shadowing property, there exists $\delta>0$ such that every δ-Gpseudo orbit for f is ε-traced by a point of X. Let $\wp=\{U_1,U_2,....,U_m\}$ be a finite open cover of X with $U_i \neq \varphi$ and $diam\, U_i<\frac{\delta}{2}$, for each i, $i \in \{1,2,....,m\}$. By hypothesis for each open sets U_i, U_j there is $M_{i,j}>0$ such that for all $n \geq M_{i,j}$ there is $g'_n \in G$ satisfying

$$U_j \cap g'_n f^n(U_i) \neq \phi \text{ -------(I)}$$

Let $M=\max\{M_{i,j}: 1 \leq i,j \leq m\}$ and $g_1x_1, g_2x_2,.....,g_kx_k \in X$, for some $g_1, g_2,....,g_k \in G$ and for $2 \leq j \leq k$, choosing any sequence of integers $a_1 \leq b_1 < a_2 \leq b_2 < ..... < a_k \leq b_k$ such that $a_j - b_{j-1} \geq M (2 \leq j \leq k)$ and an integer p with $p \geq M+(b_k-a_1)$. Define $a_{k+1}=b_{k+1}=p+a_1$, $x_{k+1}=f^{a_1-a_{k+1}}(g_1x_1)$. By $U(z)$ we mean an open ball U in $\wp$ containing z. Since $a_{j+1}-b_j \geq M$, by (I) there is $g'_{a_{j+1}-b_j} \in G$ such that

$$U(f^{a_{j+1}}(g_{j+1}x_{j+1})) \cap g'_{a_{j+1}-b_j} f^{a_{j+1}-b_j}(U(f^{b_j}(g_jx_j))) \neq \phi$$

This implies there is $y_j \in f^{a_{j+1}-b_j}(U(f^{b_j}(g_jx_j))) \neq \phi$ such that $f^{a_{j+1}-b_j}(y_j)=k'_{a_{j+1}-b_j}y'_j$. Construct a δ-Gpseudo orbit $\{z_i : i \in \mathbf{Z}\}$ for f in X as follows:

$$z_i = f^i(g_jx_j) \quad \text{if } a_j \leq i \leq b_j$$
$$z_i = f^{i-b_j}(y_j) \quad \text{if } b_j \leq j \leq a_{j+1}$$
$$z_{i+P} = z_i, \qquad \forall i \in \mathbf{Z}$$

Since f has the G-shadowing property, $\{z_i : i \in \mathbf{Z}\}$ is ε-traced by a point of X, say, x. Therefore for each $i \in \mathbf{Z}$, there is $l_i, l_{i+p} \in G$ such that

$$d(f^i(x), l_i z_i) < \varepsilon \text{ and } d(f^{i+p}(x), l_{i+p} z_{i+p}) < \varepsilon$$
$$\Rightarrow d(f^i(x), l_i z_i) < \varepsilon \text{ and } d(f^{i+p}(x), l_{i+p} z_i) < \varepsilon$$
$$\Rightarrow \text{for each } i \in \mathbf{Z}, \text{ there exist } l_i, l_{i+p} \in G$$

satisfying

$$d(l_{i+p}^{-1} f^{i+p}(x), l_i^{-1} f^i(x)) < 2\varepsilon < e$$

But f is a G-expansive homeomorphism. Therefore $G(f^p(x)) = G(x)$. This implies $f^p(x) = gx$, for some $g \in G$. Also for $a_j \le j$ or $b \le b_j$, $z_i = f^i(g_j x_j)$. Therefore, $d(f^i(x), l_i z_i) = d(f^i(x), l_i f^i(g_j x_j)) < \varepsilon$ and $f^p(x) = gx$. Therefore by definition f has the G-specification.

We now give a example of G-specification. We first recall the following terminologies from **[22]**.

Let $\Sigma_2 = \{0, 1\}^{\mathbf{N}}$, be space of all sequences of 0 and 1, with the metric $\tilde{d}(\tilde{x}, \tilde{y}) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$, where $\tilde{x} = (x_i)$, $\tilde{y} = (y_i) \in \Sigma_2$. Consider the natural shift map σ defined on Σ_2 by $\sigma(x_0 x_1 x_2 ......) = (x_1 x_2 \quad)$. Then it is observed that the periodic points of σ are dense in Σ_2 and also there is a point in Σ_2 whose σ-orbit is dense in Σ_2, say, $\tilde{y}$. Therefore for any open sets U and V in Σ_2 there is a periodic point in each of U and V and also there is a point from the σ-orbit of $\tilde{y}$ in U and V. Hence σ is a topologically mixing. Walters has proved in **[48]** that σ is an expansive homeomorphism having shadowing property.

Example 5.3.3. Consider two disjoint copies of Σ_2, say, Σ^1 and Σ^2 having one point in common, namely, $\widetilde{0}=(0,0,.....)$. Let $X=\Sigma^1\cup\Sigma^2$. We denote points of Σ^t by x^t, $t=1,2$. Define a metric d on X by $d(\widetilde{x},\widetilde{y})$ equal to zero if both $\widetilde{x}=\widetilde{y}$, is 2^{-l}, if $\widetilde{x},\widetilde{y}\in\Sigma^t$ for same t, where l is the smallest integer for which $x_l\neq y_l$, and it is one if $\widetilde{x}$ and $\widetilde{y}$ are in different Σ^t. Suppose $G=\mathrm{Z}_2$ act on X by the action $1x=x$; $-1x^1=x^2$ and $-1x^2=x^1$. If σ_t denotes the corresponding shift maps on Σ^t, then the map $f:X\to X$ defined by $f(x)=\begin{cases}\sigma_1(x), & \text{if } x\in\Sigma_1\\ \sigma_1(x), & \text{if } x\in\Sigma_2\end{cases}$ is a Z_2-expansive homeomorphism having Z_2-shadowing property since induced map is expansive as well as the induced map has the shadowing property. Also the hypothesis of Theorem 5.3.2 holds. Infact by the above discussion there is a point whose f-orbit is dense each of Σ^t. Therefore union is dense in X Hence by Theorem 5 3.2, f has Z_2-specification.

4. G - minimality.

In this section we define the notion of minimality for a continuous self map on a metric G - space. Recall the definition of a minimal map on a metric space A map f is said to be a *minimal* map if for each $x\in X$, $cl(O_f(x))=X$. Suppose the action of G on X is trivial. Then for each $x\in X$ and $g\in G$,

$cl(O_f(gx)) = X$. This implies $cl(\bigcup_{g\in G} O_f(gx)) = X$. This motivates our definition of G - minimal map.

Definition 5.4.1. Let X be a compact metric G-space and $f : X \to X$ be a continuous onto map. Then f is said to be G-*minimal* if for each $x \in X$ $cl(\bigcup_{g\in G} O_f(gx)) = X$.

Remark 5.4.2. (i) Under the trivial action of G on X, both the notions of G-minimality and minimality coincides.

(ii) If X is a compact metric G - space and continuous onto map $f : X \to X$ is minimal then it is G - minimal.

Examples 5.4.3. (a) Let $X = S^1$ and suppose $G = S^1$ acts on X by the usual action of complex multiplication. We denote a point of S^1 by its argument. Consider the map $f : S^1 \to S^1$ defined by $f(\theta) = 2\theta$. Also for $\theta \in S^1, G(\theta) = S^1, \theta \in [0,2\pi)$. Now, for each $\theta \in S^1$, choose $g \in G$ such that $g\theta = \theta'$, where θ' is an irrational number. Consider $O_f(\theta') = \{f^n(\theta') / n \geq 0\}$. Then θ' being an irrational number, $O_f(\theta')$ is dense in S^1. Therefore $X = cl(\bigcup_{g\in G} O_f(g\theta))$ implies f is an S'- minimal map. Since $0 \in S^1$ is a fixed point of f, f is not a minimal map.

Real Numbers $\{\beta_1,\beta_2,...,\beta_n\}$ are said to be *rationally independent* if $\{\beta_1,\beta_2,...,\beta_n,1\}$ are linearly independent over $\mathbf{Q}$. Recall that a rotation f on T^n defined by $f((\theta_1,\theta_2,...,\theta_n))=(\theta_1+\beta_1,\theta_2+\beta_2,...,\theta_n+\beta_n)$ is minimal iff $\{\beta_1,\beta_2,...,\beta_n\}$ are rationally independent **[46]**.

5.4.3. (b) Consider n- dimensional Torus $\mathrm{T}^n=S^1\times S^1...\times S^1$. Suppose $G=S^1$ acts on T^n by the action $\theta(\theta_1,\theta_2,...,\theta_n)=(\theta+\theta_1,\theta_2,...,\theta_n)$. Define a map $h:\mathrm{T}^n\to\mathrm{T}^n$ by $h(\theta_1,\theta_2,...,\theta_n)=(\theta_1,\theta_2+\beta_2,...,\theta_n+\beta_n)$, where $\langle\beta_2,\beta_3,...,\beta_n\rangle$ are rationally independent. Then, since $\theta_n(\theta_1,\theta_2,...,\theta_n)$ is dense in $\{\theta_1\}\times\mathrm{T}^{n-1}$ it follows that $\bigcup_{\theta\in S^1} O_h((\theta(\theta_1,\theta_2,...,\theta_n)))$ is dense in T^n. Therefore, h is G-minimal. h is not minimal as $O_h((0,0,0,...,0))$ is not dense in T^n.

Now if $G_1=\mathrm{T}^k, k<n$, acts on T^n by the action $(\eta_1,\eta_2,...,\eta_k)$ $(\theta_1,\theta_2,...,\theta_n)$ = $(\eta_1+\theta_1,\eta_2+\theta_2,...,\eta_k+\theta_k,\theta_{k+1},...,\theta_n)$ then by a similar argument $h_1:\mathrm{T}^n\to\mathrm{T}^n$ defined by $h_1((\theta_1,\theta_2,...,\theta_n))=(\theta_1,\theta_2,..,\theta_k,\theta_{k+1}+\beta_{k+1},\theta_{k+2}+\beta_{k+2},...\theta_n+\beta_n)$ where $\{\beta_{k+1},\beta_{k+2},...,\beta_n\}$ are rationally independent, is a T^n minimal homeomorphism but not a minimal homeomorphism.

In the following result we characterize G-minimal homeomorphisms through G invariant, f-invariant subsets of a compact metric G-space X.

Theorem 5.4.4. Let X be a compact metric G-space and $f: X \to X$ be a homeomorphism. Then f is G-minimal if and only if the only f-invariant G-invariant closed subset of X is either X or empty set.

Proof. Suppose G is a minimal homeomorphism and E is a closed f-invariant G-invariant subset of X. If $E = \phi$, nothing to prove. Suppose $E \neq \phi$ then we show that $E = X$. Since E is G - invariant, $x \in E$ and $g \in G$, $gx \in E$. Also, E is f-invariant. Therefore $f(E) = E$. This implies for any $x \in E$

$f^n(gx) \in E$, for each $g \in G$ and each $n \in \mathbf{Z}$.

$$\Rightarrow O_f(gx) \subset E, \text{ for } g \in G.$$
$$\Rightarrow \bigcup_{g \in G} O_f(gx) \subset E$$
$$\Rightarrow cl(\bigcup_{g \in G} O_f(gx)) \subset E$$

Therefore G - minimality of f implies $E = X$.

Conversely, suppose the only f-invariant G-invariant subset of X is either X or empty set. We show that f is a G-minimal homeomorphism. For $x \in X$, $\bigcup_{g \in G} O_f(gx)$ is an f-invariant G-invariant subset of X. Therefore, $cl(\bigcup_{g \in G} O_f(gx)) = X$. This implies f is G-minimal.

Further studying relation of G-minimality with G-shadowing, we observe first the following Lemmas.

Lemma 5.4.5. *Let $f: X \to X$ be a pseudoequivariant homeomorphism defined on a compact metric G - space with metric d, where G is compact. If*

f has the G - shadowing property then for given $\varepsilon > o$ and $x \in \Omega_G(f)$ there exists $y \in X$ and $k = k(x,\varepsilon)$ such that $cl\left(O_{\hat{f}^k}(G(y))\right) \subset U_\varepsilon(G(x))$, where $U_\varepsilon(G(x))$ is a ε-neighbourhood of $G(x)$ with respect to metric d_1 on X/G induced by d.

Proof. Let $\varepsilon > 0$ be given. Since π is a uniformly continuous map, there exists $\beta > 0$ such that for $x, y \in X$,

$$d(x,y) < \beta \Rightarrow d_1(\pi(x),\pi(y)) < \frac{\varepsilon}{3}.$$

Choose an η, $o < \eta < \frac{\beta}{2}$, such that for each $g \in G$ and each $y \in X$, $gU_\eta(y) = U_{\frac{\beta}{2}}(gy)$. Since f has the G-shadowing property, therefore there exists $\delta, 0 < \delta < \frac{\beta}{3}$, such that every $\delta - G$ pseudo orbit for f is η-traced by a point of X. For $x \in \Omega_G(f)$, consider the $\frac{\delta}{2}$ - neighbourhood of X, say U. Since x is a G-non wandering point of f, therefore there is an integer $k > 0$ and $g' \in G$ such that $g' f^k(U) \cap U \neq \phi$. Let $z \in g^1 f^k(U) \cap U$, then there exists $y \in U$ such that $z = g^1 f^k(U) \Omega U$ i.e. $z = f^k(gy) \in U$, for some $g \in G$. Construct a k-periodic $\delta - G$ pseudo orbit $\{y_i : i \in \mathbf{Z}\}$ for f as follows:

$y_{nk} = y$, $y_{nk+j} = f^j(gy), 1 \le j \le k-1$

i.e. $\{y_i : i \in \mathbf{Z}\} = \{\ldots f^{k-1}(gy), y, f(gy), \ldots, y, \ldots\}$. But f has the G-shadowing property. Therefore $\{y_i : i \in \mathbf{Z}\}$ is η - traced by a point of X, say, z. This implies for each $i \in \mathbf{Z}$, there exists $p_i \in G$ such that $d(f^i(z), p_i z_i) < \eta$.

In particular for $i = nk$, there exists $p_{nk} \in G$ such that

$$
\begin{aligned}
& d(f^{nk}(z), p_{nk}y) < \eta \\
& \Rightarrow y \in p_{nk}^{-1}U_\eta(f^{nk}(z)) \subset U_{\frac{\beta}{2}}(p_{nk}^{-1}f^{nk}(z)) \\
& \Rightarrow d\left(y, p_{n-k}^{-1}f^{nk}(z)\right) < \frac{\beta}{2} \\
& \Rightarrow \text{for each } n \in \mathbf{Z},\ d(p_{nk}^{-1}f^{nk}(z), x) \leq d(p_{nk}^{-1}f^{nk}(z), y) + d(x, y) \\
& \qquad\qquad\qquad\qquad\qquad\qquad < \frac{\beta}{2} + \frac{\delta}{2} < \beta. \\
& \Rightarrow d_1\left(\pi(p_{nk}^{-1}f^{nk}(z), \pi(x)\right) < \frac{\varepsilon}{3} \\
& \Rightarrow d_1\left(G(f^{nk}(z), G(x)\right) < \frac{\varepsilon}{3} \\
& \Rightarrow \hat{f}^{nk}\left(G(z)\right) \in U_{\frac{\varepsilon}{3}}\left(G(x)\right) \\
& \Rightarrow cl(O_{\hat{f}^k}(G(z))) \subset cl(U_{\frac{\varepsilon}{3}}(G(x))) \subset U_\varepsilon(g(x))
\end{aligned}
$$

Hence the proof.

In the following Lemma we observe that every point of a compact metric G-space X is a G-non wandering point of a map f if f is a G-minimal map.

Lemma 5.4.6. *Let X be a compact metric G - space, where G is compact and $f : X \to X$ is a pseudoequivariant G-minimal map. Then every point of X is a G - non wandering point of f.*

Proof. For a given $\varepsilon > 0$, let $U = U_\varepsilon(x), x \in X$. Since f is G-minimal map, for each $y \in X$, $cl(\bigcup_{g \in G} O_f(gy)) = X$. Therefore, there are $g_1, g_2 \in G$ and integers $m, k > 0$ with $m < k$, say, such that $f^m(g_1y), f^k(g_2y) \in U$

Now, $f^m(g_1y) \in U \Rightarrow g_1 y \in f^{-m}(U)$

$\Rightarrow f^k(g_2 y) \in f^k\left(g_2 g_1^{-1} f^{-m}(U)\right)$

$\Rightarrow f^k(g_2 y) \in g f^{k-m}(U)$, for some $g \in G$

$\Rightarrow g\ f^n(U) \cap U \neq \phi$

$\Rightarrow\ x \in \Omega_G(f)$

Since x in X is arbitrary, we obtain $X = \Omega_G(f)$

Theorem 5.4.7. *Let X be a compact connected metric G - space with metric d and having more than one point, where G is compact then a pseudo equivariant G-minimal homeomorphism does not posses the G-shadowing property.*

Proof. Since X is compact therefore X/G is also compact. Let $l = diam X/G$ and $\varepsilon = \frac{l}{3}$. Suppose f is a G - minimal homeomorphism by Lemma 5.4.6. every point of X is a G - non wandering point of f. Therefore for each $x \in X$, by Lemma 5.4.5. there exists $y \in X$ and $k > 0$ such that $cl(O_{f^k}(G(y)) \subset U_\varepsilon(G(x))$, where $U_\varepsilon(G(x))$ is the ε-neighbourhood of $G(x)$ with respect to metric d_1 on X/G. Since f is a homeomorphism.

$\bigcup_{j=0}^{k-1} cl(\bigcup_{g \in G} O_{f^k}(f^i(gy))) = cl(\bigcup_{j=0}^{k-1}(\bigcup_{g \in G} O_{f^k}(f^i(gy)))) = cl(\bigcup_{g \in G} O_f(gy))$ $= X$ as f is a G - minimal. We show that from connectedness and G-minimality we get, $cl(\bigcup_{g \in G} O_{f^k}(gy)) = X$. For suppose $k=3$. Let $A_j = cl(\bigcup_{g \in G} O_{f^3}(f^j(gy)))$, $j=0,1,2.$

Then $A_1 = f(A_0)$ & $A_2 = f^2(A_0)$. Connectedness of X implies A_0, A_1, A_2 are not pairwise disjoint. Let $z \in A_0 \cap A_1$ then $z, f(z) \in A_0$. We claim that $B_0 \bigcup f^2(B_0)=X$, where $B_0=cl(\bigcup_{g\in G} O_{f^3}(gz))$. Let $t \in X$ be such that $t \in A_0$.

Then there exists $\{n_j\}$ such that $f^{-3p_j}(g_{nj}y)\to t$.

Also $z\in A_0$

$$\Rightarrow \text{There exists } \{p_j\} \text{ such that } f^{3p_j}(k_{p_j}y)\to z.$$
$$\Rightarrow f^{-3p_j}(k'_{p_j}z)\to y.$$
$$\Rightarrow g_{n_j}f^{-3p_j}(k'_{p_j}z)\to g_{n_j}y. \text{ for each } n_j \text{.for some } k'_{pj}\in G$$
$$\Rightarrow f^{3n_j}\left(g_{n_j}f^{-3p_j}(k'_{p_j}z)\right)\to f^{3n_j}\left(g_{n_j}y\right)\to t$$
$$\Rightarrow f^{3n_j}\left(g'_{n_j}z\right)\to t$$
$$\Rightarrow t\in B_0$$

Similarly if $t\in f(A_0)$ or $f^2(A_0)$ then $t\in B_0$ or $t\in f^2(B_0)$. Hence, $f^2(B_0)\cup B_0=X$. Again $f^2(B_0)\cap B_0\neq\phi$. Let $w\in f^2(B_0)\cap B_0$. Then by similar argument we have $cl(\bigcup_{g\in G} O_{f^3}(gw))=X$. But $gw\in B_0=cl(\bigcup_{g\in G} O_{f^3}(gz))$. This implies $cl(\bigcup_{g\in G} O_{f^3}(gy))=X$. Thus, in general there is $y\in X$, $k>0$ such that

$cl(\bigcup_{g\in G} O_{f^k}(gy))=X$.

$$\Rightarrow \pi(cl(\bigcup_{g\in G} O_{f^k}(gy)))=\pi(X)=X/G.$$

But $\pi(Cl(\bigcup_{g\in G} O_{f^k}(gy)))\subset Cl(\bigcup_{g\in G} O_{f^k}(gy))$ and $\pi(\bigcup_{g\in G} O_{f^k}(gy))=O_{\hat{f}^k}(G(y))$.

Therefore, $X/G\subseteq ClO_{\hat{f}^k}(G(y))$

$$\Rightarrow cl(O_{\hat{f}^k}(G(y)))=X/G$$

But $cl(O_{\hat{f}^k}(G(y))) \subset U_\varepsilon(G(x))$, implies $X/G \subset U_\varepsilon(G(x))$. Therefore $l \leq 2\varepsilon$, which is not possible as $l = \frac{\varepsilon}{3}$. Hence f does not have the G- shadowing property.