Appendix B

The Difference equations for a set of differential equations for a second order system are derived as follows :

The difference equations (5.29) and (5.30) combined together give

$$\ddot{\bar{x}}_1 + \alpha_1 \dot{\bar{x}}_1 + \alpha_2 \bar{x}_1 = u \tag{B.1}$$

where

$$\alpha_1 = \bar{x}_3 \bar{x}_5 (1 + \bar{x}_4 \bar{x}_6) \tag{B.2}$$

and

$$\alpha_2 = (\bar{x}_3 + \bar{x}_5) \tag{B.3}$$

The equation (B.1) is written in a general form which can represent state variable equations of any second order system with α_1 and α_2 chosen suitably. The two roots of equation (B.1) are given by

$$q_1 = \frac{\alpha_1}{2} + \tfrac{1}{2}\sqrt{\alpha_2^2 - 4\alpha_1} \tag{B.4}$$

$$q_2 = \frac{\alpha_1}{2} - \tfrac{1}{2}\sqrt{\alpha_2^2 - 4\alpha_1} \tag{B.5}$$

The roots are

(a) Real ; if $(\alpha_2^2 - 4\alpha_1) > 0$,

(b) Complex ; if $(\alpha_2^2 - 4\alpha_1) < 0$ and

(c) Real and Equal ; if $(\alpha_2^2 - 4\alpha_1) = 0$.

The difference equations for the nominal trajectory for the three cases (a), (b) and (c) are given as follows :

(a) <u>Difference Equations for REAL Roots</u>

$$\bar{x}_1(i+1) = \frac{1}{q_1-q_2}\Big[(q_1 e^{-q_2 T} - q_2 e^{-q_1 T})\,\bar{x}_1(i) +$$

$$(e^{-q_2 T} - e^{-q_1 T})\,\bar{x}_2(i) + u(i)\left\{\frac{1-e^{-q_2 T}}{q_2} - \frac{1-e^{-q_1 T}}{q_1}\right\}$$

$$+ \frac{u(i+1)-u(i)}{T}\left\{\frac{q_2 T - 1 + e^{-q_2 T}}{q_2^2} - \frac{q_1 T - 1 + e^{-q_1 T}}{q_1^2}\right\}\Big] \quad (B.6)$$

and

$$\bar{x}_2(i+1) = \frac{1}{q_1-q_2}\Big[q_1 q_2 (e^{-q_1 T} - e^{-q_2 T})\,\bar{x}_1(i) + (q_1 e^{-q_2 T} - q_2 e^{-q_1 T})$$

$$+ u(i)(e^{-q_2 T} - e^{-q_1 T}) + \frac{u(i+1)-u(i)}{T}\left\{\frac{q_1 T - 1 + e^{-q_1 T}}{q_1} -\right.$$

$$\left.\frac{q_2 T - 1 + e^{-q_2 T}}{q_2}\right\}\Big] \quad (B.7)$$

where T is the sampling period. The difference equations for equations (5.31) to (5.34) are already given by equations (5.37) to (5.40).

(b) Difference Equations for COMPLEX Roots

Let equations (B.4) and (B.5) be written as

$$q_1 = r_1 + j\,r_2 \quad (B.8)$$

$$q_2 = r_1 - j\,r_2 \quad (B.9)$$

where

$$r_1 = \frac{\alpha_1}{2} \quad (B.10)$$

and

$$r_2 = \frac{1}{2}\sqrt{4\alpha_1 - \alpha_2^2} \quad (B.11)$$

The difference equations are given by

$$\bar{x}_1(i+1) = e^{-r_1T}\left\{\frac{r_1}{r_2}\sin r_2T + \cos r_2T\right\}\bar{x}_1(i) + \frac{\sin r_2T}{r_2}\bar{e}^{r_1T}\,\bar{x}_2(i)$$
$$+ \frac{u(i)}{r_1^2 + r_2^2}\left\{1 - \frac{r_1}{r_2}e^{-r_1T}\sin r_2T - e^{-r_2T}\cos r_2T\right\} +$$
$$\frac{u(i+1)-u(i)}{T(r_1^2+r_2^2)}\left\{T - \frac{2r_1}{r_1^2+r_2^2} + \frac{e^{-r_1T}\sin r_2T}{r_2} + \frac{2r_1e^{-r_1T}\cos r_2T}{r_1^2+r_2^2}\right\} \qquad (B.12)$$

and

$$\bar{x}_2(i+1) = -\left\{\left(r_2 + \frac{r_1^2}{r_2}\right)e^{-r_1T}.\sin r_2T\right\}\bar{x}_1(i) + e^{-r_1T}\left\{-\frac{r_1}{r_2}\sin r_2T + \right.$$
$$\left.\cos r_2T\right\}\bar{x}_2(i) + \frac{u(i)}{r_2}e^{-r_1T}\sin r_2T + \frac{u(i+1)-u(i)}{T(r_1^2+r_2^2)}\left\{1 - \right.$$
$$\left. e^{-r_1T}\cos r_2T - \frac{r_1}{r_2}e^{-r_1T}\sin r_2T\right\} \qquad (B.13)$$

The difference equations for $\bar{x}_3$, $\bar{x}_4$, $\bar{x}_5$ and $\bar{x}_6$ are already given by equations (5.37) to (5.40).

(c) Difference Equations for REAL & EQUAL Roots

The roots q_1 and q_2 are equal and given by

$$q_1 = q_2 = q = \frac{r_1}{2} \qquad (B.14)$$

Thus, the difference equations are

$$\bar{x}_1(i+1) = (1+qT)\,e^{-qT}\bar{x}_1(i) + T\,\bar{e}^{qT}\bar{x}_2(i) + \frac{u(i)}{q^2}\left\{1 - \bar{e}^{qT}(1+qT)\right\} +$$
$$\frac{u(i+1)-u(i)}{T\,q^2}\left\{T(1+\bar{e}^{qT}) - \frac{2}{q}(1-\bar{e}^{qT})\right\} \qquad (B.15)$$

$$\bar{x}_2(i+1) = -q^2T\,\bar{e}^{qT}\,\bar{x}_1(i) + (1-qT)\,\bar{e}^{qT}\,\bar{x}_2(i) + T\,\bar{e}^{qT}\,u(i) +$$
$$\left\{u(i+1) - u(i)\right\}\left\{1 - \bar{e}^{qT}(1+qT)\right\}/(T\,q^2) \qquad (B.16)$$

Other equations are the same as in case (b) .