CHAPTER 1

INTRODUCTION

An isometry h of the real line $\mathbb{R}$ is a self-homeomorhism of $\mathbb{R}$ for which there obviously does not exist a positive real number δ such that for x, y in $\mathbb{R}$, $x \neq y$,

$$|h^n(x) - h^n(y)| \ (= |x - y| \ ) > \delta$$

for some n in $\mathbb{Z}$; on the other hand, the left / right multiplication h_α by a fixed real number α, $\alpha \neq -1, 0, 1$, is a self-homeomorphism of $\mathbb{R}$ for which there does exist a positive real number δ (in fact any $\delta > 0$ works) such that for x, y in $\mathbb{R}$, $x \neq y$,

$$|h^n(x) - h^n(y)| \ (= \alpha^n|x - y| \ ) > \delta$$

for some n in $\mathbb{Z}$. The self-homeomorphisms of the later type drew attention of Utz [37] in 1950 who termed them *unstable* and carried out their study first time on a general metric space. The term *expansive homeomorphism* which got into use in the literature for an unstable homeomorphism of Utz seems rather natural and descriptive of the notion and was suggested by Gottschalk and Hedlund (refer [6, 13]). The definition of such a homeomorphism on a metric space given by Utz is as follows :

Definition 1.1. Given a metric space (X,d), a self-homeomorphism h of X is called *expansive* if there exists a positive real number δ such that whenever $x, y \in X$, $x \neq y$, one can find an integer n satisfying $d(h^n(x),h^n(y)) > \delta$. The number δ is then called an *expansive constant* for h.

Throughout, an expansive homeomorphism will mean an expansive self-homeomorphism.

Since the appearance of Utz's paper in 1950, extensive work has been done on expansive homeomorphisms. The work mainly concerns the study of properties of expansive homeomorphisms, their existence / non-existence on different metric spaces, their extension problems, their characterizations, their asymptotic properties, and so on.

The concept of expansive homeomorphism is defined and studied also in other contexts by various authors. For example, expansive homeomorphism on compact uniform spaces / surfaces are studied in [5, 15, 16]; refer [34] for pointwise expansive homeomorphism on a metric space, [40] for expansive maps, [12] for positively expansive maps and [23] for positively pseudo expansive maps; expansive automorphisms of topological groups are studied in [1 , 44] and of topological vector spaces are studied in [10]; for expansive transformation semigroups / groups refer [11, 30]; uniformly expansive homeomorphisms are studied in [36]; also refer [3] for expansive flows, [17] for expansive foliations, [26] for continuumwise expansive homeomorphism; refer also [24, 31]. However, it appears to us that the concept of expansive homeomorphism is yet not defined and studied in the settings of general topological spaces and G-spaces. Taking up this task, we study several examples and analyse the definition of expansive homeomorphism on a metric space and define the notion of expansiveness in these general settings. We then carry out the study of existence / non-existence of such homeomorphisms on different spaces, their properties, their characterizations, their extension problems and several other related problems. The present thesis is the outcome of interesting researches carried out by the author mainly along these lines. There are five chapters in the thesis and this chapter aims at providing introduction to the subject matter of the thesis through the recent developments regarding the concerned problems of expansive homeomorphisms.

Some of the interesting properties of the expansive homeomorphisms are concerning their compositions, restrictions, products, periodic and fixed points, etc. For example, considering multiplications $h_{1/2}$ and h_2 on the real line $\mathbb{R}$ respectively by fixed real numbers 1/2 and 2, and observing that their composition is the identity on $\mathbb{R}$, it is easily seen that the composition of two expansive homeomorphisms need not be expansive. In this regard consider the following example also.

Example 1.1. Let $X = \{ 1/n, 1 - 1/n \mid n \in \mathbb{N} \}$ with usual metric. Clearly, X is a compact metric space. Consider the mapping h defined on X by

$h(x) = x$ if $x = 0$ or 1,

$=$ point of X which is immediately to the right of x.

This map may be called the right shift operator on X and is an expansive homeomorphism with expansive constant δ, $0 < \delta < 1/6$. The left shift operator on X, defined similarly, is also expansive with the same expansive constant δ. But their composition being an identity is obviously not expansive. This shows that the composition of two expansive homeomorphisms need not be expansive — even if the underlying space is compact metric. However, the following result for compact metric spaces concerning the composition of an expansive homeomorphism with itself is proved by Utz [37].

Theorem 1.1. *Let* X *be a compact metric space and let* h *be an expansive homeomorphism on* X. *Then for each integer* m, $m \neq 0$, h^m *is expansive on* X.

Concerning the restrictions and product of expansive homeomorphisms, it is easy to see that the restriction of an expansive homeomorphism h on a metric space X to a subspace Y of X is expansive if $h(Y) = Y$; and, if h_ι are expansive homeomorphisms on metric spaces X_ι, $\iota = 1,2$, then so is the homeomorphism $h_1 \times h_2$ on $X_1 \times X_2$. The later property extends to any finite product but not to infinite product [2].

Next, observe that if δ is an expansive constant for an expansive homeomorphism h on a metric space X, then obviously so is any θ such that $0 < \theta \leq \delta$; and in general, the set of all expansive constants for h need not be bounded (for example every real number $\delta > 0$ is an expansive constant for the multipliation h_α of the real line **R**, $\alpha \neq -1, 0, 1$). However, if X is compact, then the set of all expansive constants for any expansive homeomorphism h on X is a bounded subset of real numbers and hence has a least upper bound. The question whether this least upper bound is an expansive constant for h was answered in negation by Bryant in [6]. His result follows.

Theorem 1.2. *If X is a compact metric space and θ is a least upper bound of the expansive constants for an expansive homeomorphism* h *on* X, *then θ is not an expansive constant for* h.

Observe that for any $\alpha \neq -1, 0, 1$, not only the set of expansive constants for the homeomorphism h_α on **R** is an unbounded set but for each integer $m \neq 0$, h_α^m is also expansive (recall Theorem 1.1; notice that **R** is not compact). Therefore, the following questions arise.

(i) Does there exist an expansive homeomorphism h on **R**, or more generally on $\mathbf{R}^n$, $n \in \mathbf{N}$, such that h^m is not expansive if $m \neq -1, 0, 1$?

and

(ii) Does there exist an expansive homeomorphism h on **R**, or more generally on $\mathbf{R}^n$, such that the set of expansive constants for h is bounded ?

For an affirmative answer to these questions, refer Bryant and Coleman [7].

Recall that a point x of a metric space X is said to be a periodic point of period k, $k \in \mathbf{N}$, with respect to a homeomorphism h of X if x is a fixed point of h^k. The following result of Utz [37] concerning periodic points of a compact metric space with respect to an expansive homeomorphism on it shows that an expansive homeomorphism on a compact metric space can have only finitely many fixed points. (In general it is not true, for example the identity homeomorphism on the set of integers $\mathbf{Z}$ with usual topology is expansive with expansive constant δ, $0 < \delta < 1$ and obviously all points of $\mathbf{Z}$ are its fixed points.)

Theorem 1.3. *If* X *is a compact metric space and* h *is an expansive homeomorphism on* X, *then for each positive integer* k, *the points of* X *of period* k *are finite in number. Thus the periodic points of* X *form a countable set.*

In the same paper, Utz observes that expansiveness is a strong contradiction of regularity(*) on a metric space which is dense-in-itself; and recalls that [37 p.773] a pointwise periodic (or, even an almost periodic) homeomorphism on a compact metric space has to be regular. It therefore follows from

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(*) A transformation f from a metric space (X,d) onto itself is said to be *regular* if given any real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that $x, y \in X$ with $d(x,y) < \delta$ implies $d(f^n(x),f^n(y)) < \varepsilon$ for all integer n.

Theorem 1.3 that there does not exist a pointwise periodic or even an almost periodic expansive homeomorphism on a compact metric space which is dense in itself.

Concerning the existence of an expansive homeomorphism on a compact continuum, Williams [38] gave an example of such a homeomorphism in 1955. In 1958, Gottschalk [14] asked whether the n-cell can carry an expansive homeomorphism or not. A partial answer to this was given by Bryant [6] who showed that there exists no expansive homeomorpism on a closed 1-cell (in fact on any finite interval). In 1960, Jakobsen and Utz [18] proved that there exists no expansive homeomorphism on a circle and from this obtained the same result for a simple closed curve and a closed 2-cell. However in [33], Reddy shows that an open cell of even positive dimension as well as an n-dimensional torus for $n \geq 2$ do carry expansive homeomorphisms. In connection with the existence / non-existence of expansive homeomorphisms on different spaces refer also [32, 35, 27, 19, 20, 21, 22, 25].

As a consequence of the facts that the real line R does carry expansive homeomorphisms but there does not exist an expansive homeomorphism on the open unit interval (0,1), one sees that possessing an expansive homeomorphism is not a topological property for metric spaces. In this connection, Bryant [6] proves the following theorem giving sufficiency condition for preserving expansiveness under a homeomorphism.

Theorem 1.4. *Let* h *be an expansive homeomorphism on a metric space* X *and let* g *be a homeomorphism from* X *onto a metric space* Y. *If* g^{-1} *is uniformly continuous, then* ghg^{-1} *is an expansive homeomorpism on* Y.

One can also see from the same facts that an expansive homeomorphism on a metric space need not remain expansive under an equivalent metric; however, for a compact metric space expansiveness of a homeomorphism is independent of the choice of a metric as far as metric generates the same topology [8].

An extension problem for expansive homeomorphisms concerns finding conditions under which a homeomorphism on a metric space which is expansive on a subset turns out to be expansive on the whole space. Here a homeomorphism h on a metric space (X,d) is said to be expansive on a subset A of X if there exists a positive real number δ such that for x, y $\in$ A, x $\neq$ y, one has $d(h^n(x),h^n(y)) > \delta$ for some integer n. Notice that we do not assume h(A) = A. The following is the first result along this line which is obtained by Bryant [6].

Theorem 1.5. *Let* h *be a homeomorphism on a metric space* X *and suppose* h *is expansive on a subset* A *of* X. *If* A *is such that* X − A *is finite, then* h *is expansive on* X.

An extension of this result due to Williams [39] follows the following definition.

Definition 1.2. Given a homeomorphism h on a metric spae X the set $O(x) = \{ h^n(x) \mid n \in \mathbb{Z} \}$ is called the h-*orbit* of x in X.

Theorem 1.6. *Let* X *be a compact metric space and let* h *be a homeomorphism on* X. *If* h *is expansive on* $X - \bigcup_{i=1}^{n} O(x_i)$ *for some* n *points* $x_1, \ldots, x_n$ *of* X, *then* h *is expansive on* X.

These results on extension involve some finiteness condition on the remainder. In the following, another type of result involving a concept of a basis is proved by Wine [42] for the homeomorphic extension of an expansive homeomorphism to be expansive. First we give the definition of a basis due to Wine.

Definition 1.3. Given a homeomorphism h on a metric space X, a set $\{ x_\alpha \in X \mid \alpha \in \mathscr{A}, \mathscr{A} \text{ is an index set} \}$ is called a *basis of* X *with respect to* h, if $\alpha, \beta \in \mathscr{A}$, $\alpha \neq \beta$, implies $O(x_\alpha) \cap O(x_\beta) = \varphi$ and $\cup \{ O(x_\alpha) \mid \alpha \in \mathscr{A} \} = X$.

Theorem 1.7. *Let* (Y,ρ) *be a metric space*, X *be its subspace*, h *be an expansive homeomorphism on* X *with expansive constant* δ *and* f *be a homeomorphic extension of* h *to* Y. *Then* f *is expansive with expansive constant* δ *if*

(i) $f|_{Y-X}$ *is expansive with expansive constant* δ, *and*

(ii) *there exists a basis* $\mathscr{B}$ *of* X *with respect to* h *such that* $\rho(x, Y-X) > \delta$ *for every* x *in* $\mathscr{B}$.

The proof of Wine's this theorem is based on the following characterization of an expansive homeomorphism of a metric space obtained by him in the same paper.

Theorem 1.8. *A homeomorphism* h *on a metric space* (X,d) *is expansive with expansive constant* δ *iff*

(i) h *δ-separates* h*-orbits, and*

(ii) *for any* h*-orbit* O(p) *and integer* n, *not a period of* O(p), *there exist integers* r *and* s *such that*

$$r - s = n \quad \textit{and} \quad d(h^r(p), h^s(p)) > \delta .$$

Notice that the following definition due to Wine [42] is used in this result.

Definition 1.4. A homeomorphism h on a metric space (X,d) is said to *δ-separate* h-orbits if given any basis $\mathcal{B} = \{ x_\alpha \mid \alpha \in \mathcal{A} \}$ of X with respect to h whenever x_α, $x_\beta \in \mathcal{B}$ with $x_\alpha \neq x_\beta$, there exist integers m and M depending on α and β such that $M - m \geq 2$ and and $d(h^i(x_\alpha), h^i(x_\beta)) > \delta$ for all i, $m < i < M$.

Regarding extension of an expansive homeomorphism, one may also refer Wine's further work in [43].

Another characterization of expansive homeomorphisms on a compact metric space is obtained by Keynes and Robertson [29] in terms of topological analogue of generators of measure preserving transformations — the concept defined by them in the same paper. The definition of this concept of topological generators is as follows.

Definition 1.5. Given a compact Hausdorff space X and a homeomorphism h on X, a finite open cover $\mathcal{U}$ of X is called a *generator* (respectively *weak generator*) for (X,h) if for each bisequence $\{U\}_{i\in\mathbb{Z}}$ of members of $\mathcal{U}$, $\bigcap_{i=-\infty}^{\infty} h^{-i}(\mathrm{Cl}U_i)$ (respectively $\bigcap_{i=-\infty}^{\infty} h^{-i}(U_i)$) contains at most one point.

Note. A generator is obviously a weak generator but a weak generator need not be a generator. However, Keynes and Robertson show that if (X,h) has a weak generator, where X is a compact Hausdorff space, then X has a generator. Moreover, they prove that if (X,h) has a generator, then X is metrizable. The Keynes-Robertson characterization of expansive homeomorphism on a compact metric space is as follows.

Theorem 1.9. *Let* X *be a compact metric space and and let* h *be a homeomorphism on* X*. Then* h *is expansive iff* (X,h) *has a generator.*

In Chapter 2, we formulate and study the notion of expansive homeomorphism on a general topological space. Given a topological space X and a homeomorphism h on X, the notion of A-expansiveness of h is introduced relative to a subset A of X×X. We then observe that when X is a metric space, for a specific choice of $A \subset X\times X$, A-expansiveness coincides with usual expansiveness on a metric space. As in the case of expansiveness, here too the finite products and restrictions of A-expansive homeomorphisms do turn to be A-expansive. We give several examples of A-expansive homeomorphisms for certain non-trivial choices of subsets A of the product space. We prove in Theorem 2.5 that an analogue of Theorem 1.1 concerning powers h^m, $m \neq 0$, of an A-expansive homeomorphism h also holds true. Moreover, while expansiveness is not a topological property, our Theorem 2.3, from which Bryants Theorem 1.4 follows as a corollary, shows that A-expansiveness is a topological property. As a consequence of this, the behaviour of A-expansive homeomorphisms regarding their existence turns out to be typical. In fact, when X is a finite interval I or is the unit circle S^1, we have several non-trivial choices of $A \subset X \times X$ for which an A-expansive homeomorphism can be constructed on X. This is interesting if one recalls that I or S^1 does not carry any expansive homeomorphism in the usual sense. Notice that while expansiveness involves an expansive constant δ, A-expansiveness involves A; and we show that analogue of Theorem 1.2 does not hold. In fact, we give a homeomorphism h on a compact metric space [0,1] which is not only A_i-expansive for all i, but also $\bigcup_{i=1}^{\infty} A_i$-expansive for a certain increasing sequence $\{A_i\}_{i=1}^{\infty}$ of non trivial regular closed subsets A_i of $[0,1]\times[0,1]$ containing the diagonal. It is also shown that the set of fixed points of an A-expansive homeomorphism may be uncountable and obtain in Theorem 2.6 conditions on A and the space X so that the set of fixed points of an A-expansive homeomorphism is finite.

Concerning the extension of an A-expansive homeomorphism, we prove in Theorem 2.7 that for paracompact Hausdorff spaces the analogue of Bryant's Theorem 1.5 holds true. A characterization of A-expansive homeomorphisms in terms of basis is obtained in Theorem 2.8 which generalizes Wine's Theorem 1.8 and then using this characterization in Theorem 2.9 an analogue of Theorem 1.7 is obtained for A-expansive homeomorphisms — giving another extension theorem.

Finally, in Chapter 2, we define the notion of generators and weak generators for a homeomorphism on a paracompact Hausdorff space and show that for such spaces the existence of a weak generator guarantees the existence of a generator. It is then proved in Theorem 2.11 that for paracompact Hausdorff spaces X, a homeomorphism h has a generator if and only if h is A-expansive for some neighbourhood A of the diagonal in $X \times X$. This result is the analogue of the Keyne-Robertson Theorem 1.9.

In Chapter 3, we define and study the notion of expansive homeomorphism on a metric G-space. Before elaborating our work, we give the necessary notations and terminologies.

Recall that a *topological group* is a triple $(G, \tau, \cdot)$, where $(G, \cdot)$ is a group and τ is a Hausdorff topology on G such that the map $\eta : G\times G \to G$ defined by $\eta(x,y) = x\cdot y^{-1}$ is continuous. Some of the standard examples of topological groups are : the additive groups $\mathbb{R}$ and $\mathbb{Z}$ with usual topology, the additive group $\mathbb{Z}_m$ of residue classes modulo m with discrete topology, the orthogonal group O(n) of all n×n real metrices having determinant 1 or -1 under multiplication and with the subspace topology of $\mathbb{R}^{n^2}$, the multiplicative group U(n) of n^{th} roots of unity with usual subspace topology of the complex plane, and so on.

By a *topological transformation group* or a G-*space* X, we mean [4] a triple (X,G,θ) consisting of a topological space X, a topological group G and an action θ of G on X i.e., a continuous map $\theta : G \times X \to X$ satisfying $\theta(e,x) = x$ and $\theta(g_1,\theta(g_2,x)) = \theta(g_1.g_2,x)$, where e is the identity of G, $x \in X$ and g_1, $g_2 \in G$. An action θ of G on X is called *trivial* if $\theta(g,x) = x$ for each g in G and x in X. By a *metric* G-*space* X, we mean a metric space X on which a topological group G acts. For g in G and x in X, we denote $\theta(g,x)$ by g.x (or simply by gx) and for $A \subseteq X$, let $g.A = \{ ga \mid a \in A \}$. A subset A of a G-space X is called G-*invariant* if $\theta(G \times A) \subseteq A$; and for x in X, the set $G(x) = \{ gx \mid g \in G \}$ is called the G-*orbit* of x in X. Notice that the relation '$\sim$' defined on X as $x \sim y$ iff $x = gy$ for some g in G, where $x, y \in X$ is an equivalence relation. Therefore these G-orbits form a partition of X. The quotient space X/G of X having G-orbits as its members is called the *orbit space* of X; and the quotient map $p : X \to X/G$, sending x to G(x), is called the *orbit map* which is clearly open and continuous [4; p.37]. Given G-spaces X and Y, a continuous map $f : X \to Y$ satisfying $f(g.x) = g.f(x)$ for all g in G and x in X is called an *equivariant* map. In case an equivariant map is a homeomorphism, then f^{-1} is also equivariant. An equivariant map $f : X \to Y$ naturally induces a map $f_G : X/G \to Y/G$ which is defined by $f_G(G(x)) = G(f(x))$. If (X,d) is a metric G-space with G compact, then X/G is also a metric space with metric ρ defined by $\rho[G(x),G(y)] = \text{Inf } \{ d(gx,ky) \mid g, k \in G \}$.

In [11], Eisenberg defined and studied the notion of expansive transformation group. This definition is as follows.

Definition 1.6. Let a uniform space X with uniformity $\mathcal{U}$ be a topological transformation group i.e., a G-space. Then X is called *expansive* if there exists α in $\mathcal{U}$ such that whenever $x, y \in X$ with $x \neq y$, one can find a g in G satisfying $(gx,gy) \notin \alpha$; α is then called an *expansive index* of X.

Observe that every metric space is a uniform space and hence according to this definition, an expansive metric G-space can be defined. However, it does not involve any kind of study of expansiveness of a homeomorphism of that G-space. Therefore, we introduce the notion of G-expansive homeomorphism on a metric G-space and continue our researches to study them in this Chapter. First observing that every metric space is a metric G-space under trivial action of G, we analyse the definition of expansiveness of a homeomorphism on a metric space in a specific way. Then through several examples we motivate the definition of G-expansiveness of a homeomorphism on a metric G-space. The definition is such that when the action of G is trivial, the G-expansiveness of a homeomorphism on a metric G-space turns out to be equivalent to the expansiveness in the usual sense. However, examples are provided to show that under a non-trivial action of G neither expansiveness of a homeomorphism implies nor is implied by its G-expansiveness. This leads us to determine some conditions under which expansiveness implies or is implied by G-expansiveness. In this process, we introduce the notion of pseudoequivariant maps between G-spaces. Every equivariant map is a pseudoequivariant but the fact that the converse is not true is justified by an example. We prove that if a homeomorphism h of a G-space X is pseudoequivariant, then for any intrger n, h^n is also pseudoequivariant. Like expansiveness, here also, under relavent conditions a finite product of G-expansive homeomorphisms as well as restriction of a G-expansive homeomorphism turn out to be G-expansive. A result similar to Theorem 1.1 concerning powers h^m, $m \neq 0$, of a G-expansive homeomorphism is shown to hold true in our Theorem 3.3. It is observed that unlike expansiveness, the set of fixed points of a G-expansive homeomorphism may be infinite even if the metric G-space under consideration is compact. A result of the type of Theorem 1.4 giving some conditions for G-expansiveness to be preserved under a homeomorphism is obtained and an extension theorem for G-expansive homeomorphism, along the line of Theorem 1.5, is proved in our Theorem 3.8. We also obtain in Theorem 3.7 an analogue for G-expansive homeomorphisms of the following result due to Bryant [6].

Theorem 1.10 *If* (X,d) *is a compact metric space and* h *is an expansive homeomorphism on* X *with expansive constant* δ, *then for each* θ, $0 < \theta \leq \delta$, *there exists a positive integer* $k(\theta)$ *such that* $d(x,y) > \theta$ implies $d(h^n(x),h^n(y)) > \delta$ for some n with $|n| \leq k(\theta)$.

Defining the notion of G-δ separate h-orbits for a homeomorphism on a metric G-space, we finally obtain in Chapter 3, a characterization of a G-expansive homeomorphism in terms of a basis in Theorem 3.9; and using this characterization a sufficient condition for a pseudoequivariant homeomorphic extension of a pseudoequivariant G-expansive homeomorphism to be G-expansive is obtained in Theorem 3.10. Our Theorem 3.9 and Theorem 3.10 extend Wine's Theorems 1.8 and 1.7 respectively in the sense that when the action of G on X is trivial, our results coincide with those of Wine's.

Next, observe that the notion of a generator for (X,h) defined in definition 1.5 by Keynes and Robertson for a homeomorphism h on a compact Hausdorff space X was also generalized by them in the same paper [29] to the notion of a generator for a G-space X with X a compact Hausdorff space and G a discrete group as follows.

Definition 1.7. Let X be a G-space with X a compact Hausdorff space and G a discrete group. Then a finite open cover $\mathcal{U}$ of X is called a *generator* for (X,G) if for every G-family $\{ A_g \mid g \in G \}$ of members in $\mathcal{U}$, $\cap\, g^{-1}(\mathrm{Cl}A_g)$ contains at most one point.

In terms of this definition, the following characterization is obtained in [29] for expansive transformation groups.

Theorem 1.11. *A compact Hausdorff uniform G-space X with G a discrete group and compatible uniformity of X is expansive iff there exists a generator for* (X,G).

Also, the notion of asymptotic points for a homeomorphism on a metric space was defined in [37] as follows.

Definition 1.8. Let (X,d) be a metric space and let h be a homeomrophism on X. Then two distinct points x, y in X are said to be positively asymptotic (respectively negatively asymptotic) to each other under h if $\mathrm{Lim}_{n\to\infty} d(h^n(x),h^n(y) = 0$ (respectively, $\mathrm{Lim}_{n\to-\infty} d(h^n(x),h^n(y) = 0$).

On metric spaces, the existence of asymptotic points under expansive homeomorphisms is studied by Utz [37], Bryant [5, 6] and others. Using the concept of generators as defined in Definition 1.5, Bryant and Walters [8] obtained the following results while studying asymptotic points further.

Theorem 1.12. *Let* $\mathcal{U}$ *be a generator for* (X,h), *where* X *is a compact metric space and* h *is a homeomorphism on* X. *Then for each non-negative integer* n, *there exists* $\varepsilon > 0$ *such that* $d(x,y) < \varepsilon$ *implies the existence of* $A_{-n}, A_{-n+1}, \ldots\ldots, A_0, A_1, \ldots\ldots A_n \in \mathcal{U}$ *with* $x, y \in \bigcap_{i=-n}^{n} h^{-i}(A_i)$. *Conversely, for each* $\varepsilon > 0$, *there is an* n *in* **N** *such that if* $x, y \in \bigcap_{i=-n}^{n} h^{-i}(A_i)$, *where* $A_{-n}, \ldots, A_0, \ldots, A_n \in \mathcal{U}$, *then* $d(x,y) < \varepsilon$.

Theorem 1.13. *Let* $\mathcal{U}$ *be generator for* (X,h), *where* X *is a compact metric space and* h *is a homeomorphism on* X. *Then two points* x *and* y *are positively asymptotic under* h *iff there is a natural number* N *such that for each* $i \geq N$, *there is an* $A_i \in \mathcal{U}$ *with* $x, y \in \bigcap_{i=N}^{\infty} h^{-i}(A_i)$.

Both these theorems are also true if A_i is replaced by $\mathrm{Cl}A_i$.

Observe that the notion of a generator for (X,h) as defined in Definition 1.5 involves a homeomorphism of X, while that for a G-space X, ((X,G) with G discrete) as defined in Definition 1.7 does not involve any homeomorphism of X — though X is a compact Hausdorff space in both the cases; and hence the notion of a generator for a G-space X (with G any group) involving a homeomorphism needs an attention. In Chapter 4 of the present thesis, we define such a notion of generators and weak generators for a homeomorphism on a compact Hausdorff G-space X with G any topological group and term them G-generators and weak

G-generators. We provide examples to show that under a non-trivial action of a group G on a compact Hausdorff space X, neither a G-generator need be a generator nor a generator need be a G-generator for a homeomorphism on X — though under a trivial action of G on X both of them coincide. It is proved that the existence of a weak generator implies that of a G-generator for a compact Hausdorff G-space. We then prove some properties of pseudoequivariant maps and use them to prove in Theorem 4.3 that the existence of a G-generator for a homeomorphism on a compact Hausdorff G-space X assures the metrizability of the orbit space X/G. Also, in Theorem 4.5 we obtain a characterization of a G-expansive homeomorphism on a compact metric G-space in terms of a G-generator — a result analogous to Theorem 1.9. We then define the notion of positively (negatively) G-asymptotic points for a homeomorphism on a metric G-space — the notion which under the trivial action of G on X, coincide with positively (negatively) asymptotic points. However, under a non-trivial action of G on X, while positively (negatively) asymptotic points are positively (negatively) G-asymptotic, the converse is not true. Studying G-asymptotic points in relation to G-generators for a homeomorphism on a compact metric G-space, we obtain analogues of Theorems 1.12 and 1.13 in the setting of G-spaces.

Finally, in Chapter 5 we consider the case of a topological G-space X which need not be a metric G-space and define the notion of expansiveness of a homeomorphism on this G-space X relative to a subset A of X×X. We term such a homeomorphism as GA-expansive homeomorphism.

We observe that in case X is a metric G-space, for a specific choice of a subset A of X×X, GA-expansiveness is equivalent to G-expansiveness and on G-space when action of G is trivial, GA-expansiveness is equivalent to A-expansiveness. Examples are provided to illustrate that in general neither GA-expansiveness implies nor is implied by the A-expansiveness — showing that the two concepts are independent of each other. As in the case of A-expansive and G-expansiveness, it turns out that the restriction and finite products of GA-expansive homeomorphisms are GA-expansive. The analogue of Theorem 1.1 concerning powers h^m, $m \neq 0$ of GA-expansive homeomorphism also holds true. In Theorem 5.4, we find a condition for GA-expansiveness to be preserved under a homeomorphism; and then prove in Theorem 5.5, a result concerning extension of GA-expansive homeomorphism. Defining the notion of GA-separate h-orbits for a homeomorphism of a G-space X, we obtain in Theorem 5.6 a characterization of GA-expansive homeomorphism in terms of basis and using this one more result (see Theorem 5.7) concerning extension of GA-expansive homeomorphisms is proved. Theorems 5.7 and 5.6 reduce to Wine's Theorems 1.7 and 1.8 respectively for a certain $A \subset X \times X$, when X is a metric G-space with a trivial action.