CHAPTER 2

EXPANSIVE HOMEOMORPHISMS ON TOPOLOGICAL SPACES

In this chapter we formulate and study the notion of expansiveness of a homeomorphism on a topological space X relative to a subset A of $X \times X$. Let X throughout denote a topological space and H(X) denote the collection of all homeomorphisms on X.

1. A-expansiveness : Definitions, examples and properties.

Recall that in case X is a metric space with metric d, an h in H(X) is defined to be expansive with expansive constant $\delta > 0$ if for x, y in X with $x \neq y$, there exists an integer n such that $d(h^n(x),h^n(y)) > \delta$, that is, $(h^n(x),h^n(y)) \notin A_\delta$, where

$$A_\delta = d^{-1}[0,\delta]$$
$$= \{ (x,y) \in X \times X \mid d(x,y) \leq \delta \}$$

which is a subset of $X \times X$. Therefore, if one regards an expansive homeomorphism in this usual sense to be A_δ-expansive, then it motivates the following definition of A-expansiveness of an h in H(X) when X is mearly a topological space and A is any subset of $X \times X$.

Definition 2.1. Let X be a topological space and let A be a subset of $X \times X$. Then a homeomorphism h on X is called *A-expansive* if for x, y in X with $x \neq y$, there exists an integer n such that $(h^n(x),h^n(y)) \notin A$.

Obviously, if h in $H(X)$ is A-expansive, then it is B-expansive for any subset B of A. Also, every h in $H(X)$ (respectively no h in $H(X)$) will be A-expansive if $A = \varphi$ (respectively $A = X \times X$), and hence we assume that A is a non-trivial subset of $X \times X$, that is, $A \neq \varphi$ and $A \neq X \times X$. In case A is an at most countable subset $\{ (a_j, b_j) \mid j \in J \}$ of $X \times X$, it is easy to see that an h in $H(X)$ is A-expansive iff for each index i in J, either

$$O(a_i) \cap \{ a_j \mid j \in J \}^c \neq \varphi$$

or

$$O(b_i) \cap \{ b_j \mid j \in J \}^c \neq \varphi,$$

where $O(x) = \{ h^n(x) \mid n \in \mathbb{Z} \}$ is the h-orbit of x in X. Since A-expansiveness depends on A, the notion provides a wider scope even on metric spaces; in fact, on metric spaces the behaviour of A-expansiveness is observed to be completely different at some places from the behaviour of expansiveness in the usual sense. The typical illustration being the fact that certain metric spaces which admit no expansive homeomorphism, i.e., A_δ-expansive homeomorphism, for any $\delta > 0$, do admit A-expansive homeomorphisms for certain non-trivial subsets A of the product space. For example, we know that the space $X = I = [0,1)$ (in fact, any finite interval) and the unit circle S^1 both with usual metric do not admit any expansive homeomorphism; however, there are many non-trivial choices of $A \subset I \times I$ and $A \subset S^1 \times S^1$ for which A-expansive homeomorphisms can be constructed on I and S^1 respectively. A typical choice of a subset A in $I \times I$ is given in the following example.

Example 2.1. Let $A = [1/5,1/2] \times [1/3,2/3] \subset I\times I$, $I = [0,1)$ with usual metric, and let $h : I \to I$ be defined by

$$
\begin{aligned}
h(x) &= x, && \text{if } x \in [0,1/8] \\
&= (19x - 2)/3, && \text{if } x \in [1/8,1/5] \\
&= (11x + 5)/12, && \text{if } x \in [1/5,1/2] \\
&= (x + 3)/4, && \text{if } x \in [1/2,1).
\end{aligned}
$$

Then it is easily seen that h is an A-expansive homeomorphism on I.

In fact, for any $A \subset I\times I$ such that either $I - p_1(A)$ or $I - p_2(A)$ contains a non trivial segment ($p_1(A)$ and $p_2(A)$ are respectively the first and the second projections of A on I), one can construct a homeomorphism on I which is A-expansive. Before giving other examples of A-expansive homeomorphisms, we obtain some simple properties of such homeomorphisms.

We have the following result concerning the restriction of A-expansive homeomorphisms.

Theorem 2.1. *Let* h *in* $H(X)$ *be A-expansive, where* $A \subset X\times X$ *and* $Y \subset X$ *be any subspace of* X *such that* $h(Y) = Y$*, then* $h|_Y$ *is B-expansive for a suitable subset* B *of* $Y\times Y$.

Proof. Let $x, y \in Y$ and $x \neq y$. Then $x, y \in X$, and therefore A-expansiveness of h on X implies the existence of an integer n such that $(h^n(x),h^n(y)) \notin A$, that is, $(h^n(x),h^n(y)) \notin [A \cap Y\times Y]$. The Theorem follows if we take $B = A \cap Y\times Y$.

As in the case of expansive homeomorphisms, we have the following result regarding product of A-expansive homeomorphisms.

Theorem 2.2. *Let* X *and* Y *be topological spaces,* f *in* $H(X)$ *be* A-*expansive and* g *in* $H(Y)$ *be* B-*expansive, where* $A \subset X\times X$ *and* $B \subset Y\times Y$. *Then* $f\times g : X\times Y \to X\times Y$ *is* C-*expansive, where* $C = h^{-1}(A\times B)$ *in which* $h : (X\times Y)^2 \to X^2\times Y^2$ *is such that* $h(x,y,x',y') = (x,x',y,y')$, *where* $x, x' \in X$ *and* $y, y' \in Y$.

Proof. Let (x,y) and (x',y') be any two distinct points of $X\times Y$. Therefore either $x \neq x'$ or $y \neq y'$. Assume $x \neq x'$. Then by A-expansiveness of f on X we find an integer n satisfying $(f^n(x),f^n(x')) \notin A$. It follows that

$$(f^n(x),f^n(x'),g^n(y),g^n(y')) \notin A \times B$$

and hence as h is bijective we get

$$\begin{aligned}(f^n(x),g^n(y),f^n(x'),g^n(y')) &= ((f\times g)^n(x,y),(f\times g)^n(x',y')) \\ &\notin h^{-1}(A\times B) = C.\end{aligned}$$

This proves the result.

Observe that if (X,d) is a metric space then for any positive real number δ, A_δ always contains the diagonal D in $X\times X$ and is a regular closed set (in fact, A_δ is a closed neighbourhood of D). Therefore study of A-expansive homeomorphisms on a topological space X where A is a regular closed subset containing the diagonal D in $X\times X$ may be useful. The set A in Example 2.1 does not contain the diagonal D in $I\times I$. It is therefore natural to inquire whether there exists an A-expansive homeomorphism on I for an A which is a regular closed set containing the diagonal in $I\times I$. We shall show that there does exist such an A-expansive homeomorphism on I. Before that let us prove the following theorem which shows that for topological spaces admitting an A-expansive homeomorphism is a topological property.

Theorem 2.3. *Let* X *and* Y *be topological spaces,* $g : X \to Y$ *be a homeomorphism and* $A \subset X\times X$. *Then an* h *in* H(X) *is* A-*expansive iff* $ghg^{-1} \in H(Y)$ *is* B-*expansive, where* $B = (g\times g)(A)$.

Proof. Suppose h in H(X) is A-expansive. Let $y_1, y_2 \in Y$ and $y_1 \neq y_2$. Then $g^{-1}(y_1)$ and $g^{-1}(y_2)$ are distinct points of X and thus from A-expansiveness of h on X we find an integer n for which

$$(h^n(g^{-1}(y_1)),h^n(g^{-1}(y_2))) \notin A.$$

But this gives

$$((ghg^{-1})^n(y_1),(ghg^{-1})^n(y_2)) \notin (g\times g)(A) = B.$$

Hence ghg^{-1} is B-expansive.

For the converse, suppose $x_1, x_2 \in X$ and $x_1 \neq x_2$. Then $g(x_1)$ and $g(x_2)$ are distinct points of Y and therefore by the B-expansiveness of ghg^{-1}, there exists an integer n satisfying

$$((ghg^{-1})^n(g(x_1)),(ghg^{-1})^n(g(x_2))) \notin B$$

which gives

$$((gh^ng^{-1})(g(x_1)),(gh^ng^{-1})(g(x_2))) \notin B,$$

that is

$$(h^n(x_1), h^n(x_2)) \notin (g\times g)^{-1}(B) = A.$$

Hence h is A-expansive.

Recall that the above theorem is in contrast to the known fact that for metric spaces admitting an expansive homeomorphism is not a topological property. As a corollary to the above theorem we get the following result due to Bryant (Theorem 1.4 of our Chapter 1) giving a sufficient condition for the expansiveness on metric spaces to be preserved under a homeomorphism.

Corollary 2.4. *If* (X,d) *and* (Y,ρ) *are metric spaces,* h *in* H(X) *is expansive and* g *is a homeomorphism of* X *onto* Y *such that* g^{-1} *is uniformly continuous, then* ghg^{-1} *is an expansive homeomorphism on* Y.

Proof. Suppose h is A-expansive, where $A = A_\delta = d^{-1}[0,\delta]$ for some $\delta > 0$. Then by Theorem 2.3 ghg^{-1} is B-expansive on Y, where $B = (g\times g)(A_\delta)$. On applying the uniform continuity of g^{-1} we obtain an $\varepsilon > 0$ such that $B \supset B_\varepsilon = \rho^{-1}[0,\varepsilon]$. In fact, g^{-1} being uniformly continuous, there exists an $\varepsilon > 0$ such that

$$\rho(y_1, y_2) \le \varepsilon \Rightarrow d(g^{-1}(y_1), g^{-1}(y_2)) < \delta;$$

or equivalently

$$d(x_1, x_2) \ge \delta \Rightarrow \rho(g(x_1), g(x_2)) > \varepsilon.$$

This shows

$$(g\times g)(X\times X - A_\delta) \subset Y\times Y - B_\varepsilon \quad \text{or} \quad B \supset B_\varepsilon.$$

Hence ghg^{-1} is expansive on Y.

As noted in the proof of corollary 2.4, A_δ-expansiveness of h does give B-expansiveness of ghg^{-1}, where $B = (g\times g)(A_\delta)$. However, the following example will show that in general ghg^{-1} may fail to be expansive in the usual sense as B may not contain $B_\varepsilon = \rho^{-1}[0,\varepsilon]$ for any $\varepsilon > 0$.

Example 2.2. Consider the subspaces $[0,\infty)$ and $[0,1)$ of the usual space of the real numbers **R**. Obviously $h(x) = 2x$ is an A_δ-expansive homeomorphism of $[0,\infty)$ for any $\delta > 0$ and $g(x) = x/(x+1)$ is a homeomorphism from $[0,\infty)$ to $[0,1)$. Hence by Theorem 2.3, ghg^{-1}: $[0,1) \to [0,1)$, sending u to $2u/(u+1)$, is B-expansive on $[0,1)$, where

$B = (g\times g)(A_\delta)$

$= \{ (u,v) \in [0,1)\times[0,1) \mid u = x/x+1, v = y/y+1;\ (x,y) \in A_\delta \}$.

Observe that

$A_\delta = \{ (x,y) \mid x \geq 0,\ y \geq 0 \text{ and } x - \delta \leq y \leq x + \delta \}$

lies in the first quadrant $[0,\infty) \times [0,\infty)$ of xy-plane between the lines $y = x - \delta$ and $y = x + \delta$; and hence B is a subset of uv-plane contained in $[0,1) \times [0,1)$ and enclosed by the u-axis $v = 0$, v-axis $u = 0$, the lower branch of the rectangular hyperbola

$$(u - (\delta - 1)/\delta).(v - (\delta + 1)/\delta) = -1/\delta^2$$

(image of $y = x - \delta$ under $g\times g$) and the upper branch of the ractangular hyperbola

$$(u - (\delta + 1)/\delta).(v - (\delta - 1)/\delta) = -1/\delta^2$$

(image of $y = x + \delta$ under $g\times g$). The set B is clearly a regular closed set containing the diagonal D in $[0,1)\times[0,1)$, but do not contain B_ε for any $\varepsilon > 0$, where

$$B_\varepsilon = \{ (u,v) \mid 0 \le u,v < 1 \text{ and } |u - v| \le \varepsilon \}.$$

In fact, suppose $B \supset B_\varepsilon$ for some $\varepsilon > 0$. Since $g(x) \to 1$ as $x \to \infty$, there is an N in **N** such that

$$1 - x/(x + 1) = 1/(x + 1)$$
$$\le \varepsilon/2\delta$$

for $x \ge N$. Consider any u, v with $0 < u,v < 1$, $\varepsilon/2 < |u - v| \le \varepsilon$ and $g^{-1}(u) = x > N + \delta$. Then $(u,v) \in B_\varepsilon \subset B$ and must be the image under $g\times g$ of some $(x,y) \in A_\delta$. But, this is impossible because then

$$|u - v| = |(x/(x + 1)) - (y/(y + 1))|$$
$$= |x - y|/|x + 1|.|y + 1|$$
$$\le \delta/|1 + x|$$
$$\le \delta.\varepsilon/2\delta = \varepsilon/2$$

as $x = g^{-1}(u) > N + \delta$.

The observation that B does not contain B_ε for any $\varepsilon > 0$ can also be concluded from the fact that $[0,1)$ possesses no expansive homeomorphism, i.e., B_ε-expansive homeomorphism for any $\varepsilon > 0$ (refer [6]). Observe that the homeomorphism $ghg^{-1} = \psi$ on $[0,1)$ given as above by $u \to 2u/(u + 1)$ is strictly increasing homeomorphism with $\mathrm{Fix}\psi = \{ u \in [0,1) \mid \psi(u) = u \} = \{0\}$. However, the Example 2.3 given below shows that not every strictly increasing homeomorphism on $[0,1)$ with $\{0\}$ as its set of fixed points is $B = (g\times g)(A_\delta)$-expansive for some $\delta > 0$.

Example 2.3. Let $A = [a_0,\infty)$, where $a_0 > 0$ and let $I_i = [a_i, a_{i+1}]$, where $a_i < a_{i+1}$ for $i = 0,1,2,\dots,$ be a countably infinite partitioning of A. Let f be a homeomorphism on $[0,\infty)$ such that

(1) $f[0,a_0] = [0,a_1]$

(2) $f(x) > x$ for $0 < x < a_0$

(3) $f(I_i) = I_{i+1}$ for $i = 0,1,2,\dots$.

For example, let $I_0 = [a_0,a_1] = [1,2]$ and define f on $[0,a_1]$ as $f(x) = 2x$. Then $I_1 = [a_1,a_2] = [2,4]$ and define f on I_1 as $f(x) = x + 2$. Now, for $n \geq 2$, let $I_n = f(I_{n-1})$ (note that $I_2 = [4,6]$) and define f on I_n as $f(x) = m_n x + b_n$, where the sequences $\{m_n\}$ and $\{b_n\}$ are defined as follows :

$m_1 = 1$ and for $n \geq 2$, m_n is so chosen that

$$m_n \cdot m_{n-1} \cdot \dots \cdot m_2 \cdot m_1 = 1, \text{ if } n \text{ is not prime}$$
$$= n, \text{ if } n \text{ is prime;}$$

$b_1 = 1$ and for $n \geq 2$

$$b_n = (m_{n-1} - m_n)a_n + b_{n-1}.$$

Then f is a homeomorphism on $[0,\infty)$ of the type described above such that it is expansive with any $\delta > 0$ as an expansive constant but f^k is not expansive when $k \neq -1, 0, 1$ [7].

We now define $\psi : [0,1) \to [0,1)$ by $\psi = ghg^{-1}$, where g is as in Example 2.2 and $h = f^k$, $k \neq -1, 0, 1$. It is easy to observe that ψ is a strictly increasing homeomorphism on $[0,1)$ with $\mathrm{Fix}\psi = \{0\}$. In view of Theorem 2.3, one then concludes that ψ is not $B = (g\times g)(A_\delta)$-expansive for any $\delta > 0$.

Note. Consider the usual closed unit interval $[0,1]$ and define the homeomorphism $f : [0,1] \to [0,1]$ by $f(u) = 2u/(u + 1)$. By taking g and A_δ to be same as in example 2.2, one can easily show that f is B_δ-expansive on $[0,1]$, where

$$B_\delta = \{ (1,1) \cup (g\times g)(A_\delta) \} \subset [0,1]\times[0,1].$$

Moreover, for any sequence $0 < \delta_1 < \delta_2 < \ldots\ldots$ of reals, we have $B_{\delta_1} \subset B_{\delta_2} \subset \ldots\ldots\subset B_{\delta_n} \subset B_{\delta_{n+1}} \subset \ldots\ldots$, and f is not only B_{δ_i}-expansive for each i but is also $\bigcup_{i=1}^{\infty} B_{\delta_i}$-expansive on $[0,1]$. This is interesting if one compares it with the known result due to Bryant (Theorem 1.2) which says that the least upper bound of expansive constants of an expansive homeomorphism on a compact metric space is not an expansive constant.

Similarly, it can be shown using Theorem 2.3 that any finite interval admits an A-expansive homeomorphism.

We now give an example of an A-expansive homeomorphism on the unit circle S^1.

Example 2.4. In the unit circle S^1, suppose C_k denote the arc $(e^{i2\pi k/n}, e^{i2\pi(k+1)/n})$, where $k = 0,1,\ldots\ldots,n-1$. Let I_k denote the closed unit interval $[0,1]$ for each k. Then obviously $f_k : I_k \to C_k$ defined by

$$f_k(s) = e^{i2\pi(s+k)/n},\ k = 0,1,\ldots,n-1,$$

is a homeomorphism for each k. Since on I_k, $g_k: I_k \to I_k$ defined by

$$g_k(x) = 2x/(x + 1),\ x \in I_k$$

is B_δ-expansive, where B_δ is as described in the Note following Example 2.3, therefore in view of Theorem 2.3 it follows that for a fixed $\delta > 0$,

$$f_k g_k f_k^{-1} = h_k \text{ is } (f_k \times f_k)(B_\delta)\text{-expansive on } C_k.$$

Define $h : S^1 \to S^1$ by

$$h|_{C_k} = h_k,\ k = 0,1,\ldots\ldots,n-1.$$

Then h is a homeomorphism on S^1 which is $\bigcup_{k=0}^{n-1} (f_k \times f_k)(B_\delta)$-expansive on S^1.

In general composition of two A-expansive homeomorphisms need not be A-expansive. For example recall that the homeomorphisms h_2 and $h_{1/2}$ on $\mathbb{R}$ sending x to 2x and x to x/2 respectively are expansive with any $\delta > 0$ as an expansive constant. However, their composition being the identity map is not A-expansive for any A in $\mathbb{R}\times\mathbb{R}$. We now prove a result about powers h^m, $m \neq 0$, of an A-expansive homeomorphism h (compare Theorem 1.1 of Chapter 1).

Theorem 2.5. *Let* X *be a paracompact Hausdorff space and let* $\mathcal{U}$ *be the uniformity on it consisting of all neighbourhoods of the diagonal. Suppose* h *is a homeomorphism on* X *such that* h^m, $m \neq 0$, *is uniformly continuous with respect to* $\mathcal{U}$. *Then* h *is* U*-expansive for some* U *in* $\mathcal{U}$ *implies each* h^m, $m \neq 0$, *is* V*-expansive for some* V *in* $\mathcal{U}$.

Proof. Consider any integer m differnt from 0 and let ι belong to

$\{ \pm1,\ldots\ldots\pm m \}$. Since h^i is a homeomorphism and $h^{-i} = (h^i)^{-1}$ is uniformly continuous, there exists $V_i \in \mathcal{U}$ such that

$$(h^i\times h^i)^{-1}(V_i) \subset U$$

or equivalently

$$(h^i\times h^i)(X\times X - U) \subset X\times X - V_i.$$

Set

$$V = \cap \{ V_i \mid i \in \{\pm1,\ldots\ldots,\pm m\} \}$$

and note that

$$(h^i\times h^i)(X\times X - U) \subset (X\times X - V).$$

Applying U-expansiveness of h to distinct points x, y in X one obtains an integer n satisfying $(h^n(x),h^n(y)) \in (X\times X - U)$ and hence from the above observation,

$$(h^i(h^n(x)),h^i(h^n(y))) \in (X\times X - V)$$

for each i in $\{\pm1,..,\pm m\}$. Let r be an integer such that

$$0 < |r - n/m| \le 1 \quad \text{or} \quad 0 < |rm - n| \le |m|.$$

Now putting $i = (rm - n)$ one obtains

$$((h^m)^r(x),(h^m)^r(y)) \in X\times X - V.$$

This proves the V-expansiveness of h^m.

2. A-expansiveness and fixed points.

We recall (Theorem 1.3) that the fixed point set of an expansive homeomorphism on a compact metric space is always a finite set. However this is not the situtation with A-expansive homeomorphisms. For example, if we replace I by [0,1] in Example 2.1 and then define h at 1 by h(1) = 1, then with the same A the homeomorphism h is A-expansive on the compact metric space $[0,1]$ with $\mathrm{Fix}h = \{ x \in [0,1] \mid h(x) = x \} = [0,1/8]$, an uncountable set. Of course here A does not contain the diagonal D in $[0,1] \times [0,1]$. On the other hand, if A is a neighbourhood of D, one can very easily conclude that there can not be any A-expansive homeomorphism on $[0,1]$. But one can ask what happens if A is a regular closed set containing the diagonal. For an answer to this we refer the note following Example 2.3 in which we have described an A-expansive homeomorphism h on $[0,1]$ with $\mathrm{Fix}h = \{0,1\}$ where A is a regular closed set containing D. It is therefore interesting to inquire whether the set of fixed points of every A-expansive homeomorphism on $[0,1]$, with A as a regular closed set containing the diagonal in $[0,1]\times[0,1]$, is finite. The following example shows that this is not true in general.

Example 2.5. Consider the subset $T = \{ 1 - 1/n \mid n \in \mathbf{N} \}$ of $[0,1]$ and the family $\mathscr{A} = \{ A_n = [1 - 1/n,\ 1 - 1/(n+1)] \mid n \in \mathbf{N} \}$. That $\mathscr{A}$ is a locally finite family of closed subsets of $[0,1]$ follows easily. For each n in $\mathbf{N}$ define $g_n : [0,1] \to A_n$ by

$$g_n(t) = t/[n(n+1) + (n-1)/n].$$

Also, define f and B_δ for a fixed $\delta > 0$, as it is done in the Note following the Example 2.3. Then $h_n = g_n f g_n^{-1}$ is a $(g_n \times g_n)(B_\delta)$-expansive homeomorphism on A_n. Now, define

$h : [0,1] \to [0,1]$ by

$$h|_{A_n} = h_n \text{ for each } n; \text{ and}$$

$$h(1) = 1.$$

Obviously, h is a homeomorphism on [0,1]. Set

$$A = \{(1,1)\} \cup [\cup \{ (g_n \times g_n)(B_\delta) \mid n \in \mathbf{N} \}]$$

and notice that h is A-expansive. Finally, observe that A is a regular closed set containing the diagonal in $[0,1]\times[0,1]$ and Fixh = T, an infinite set.

Remark. Given any finite set F in [0,1] containing {0,1}, one can give a constructive proof (as is done in Example 2.5) of the existence of an A-expansive homeomorphism h on [0,1] such that Fixh = F and A is a suitable regular closed set containing the diagonal in $[0,1]\times[0,1]$. It will be interesting to know whether there exists an A-expansive homeomorphism h on [0,1] for which Fixh is a given countable / uncountable set, where A is a suitable regular closed set containing the diagonal. We do not have a definite answer to this.

We now prove the following result which gives a class of spaces on which any A-expansive homeomorphism has a finite fixed point set, whenever A is a neighbourhood of the diagonal in the product space.

Theorem 2.6. *Let* X *be a first countable, countably compact Hausdorff space and let* h *be an* A*-expansive homeomorphism on* X, *where* A *is a neighbourhood of the diagonal in* $X\times X$. *Then* Fixh *is a finite set.*

Proof. If possible, assume that Fixh is an infinite set. Then it has a limit point, say x. As Fixh is a closed set, $x \in$ Fixh. Since A is a neighbourhood of the diagonal in $X \times X$, there exists a neighbourhood W_x of x such that

$$(x,x) \in W_x \times W_x \subset A.$$

Now x being a limit point of Fixh, there exists another fixed point y different from x such that $y \in W_x$. Clearly

$$(x,y) \in W_x \times W_x \subset A.$$

But this is not possible because h is A-expansive. Hence Fixh is a finite set.

3. Extension of A-expansive homeomorphisms.

Suppose h is a homeomorphism on a topological space X, $A \subset X \times X$, and $B \subset X$. Then we say h is A-expansive on B if for distinct points x, y in B, there exists an integer n such that $(h^n(x),h^n(y)) \notin A$. Note that here we do not require the invariancy $h(B) = B$. In this Section we consider an extension problem for A-expansive homeomorphisms which concerns finding conditions under which A-expansiveness on a subset turns out to be A'-expansive on the whole space X for some suitable A' of $X \times X$. (Refer Theorem 1.5 due to Bryant [6] .)

Theorem 2.7. *Let* X *be a paracompact Hausdorff space and let* $Y \subset X$ *be such that* $X - Y$ *is finite. Suppose* h *is a* U*-expansive homeomorphism on* Y *where* U *is a neighbourhood of the diagonal in* $X \times X$*. Then* h *is* B*-expansive on* X *for some* B.

Proof. Let $X - Y = \{ x_0, x_1, \ldots\ldots, x_n \}$. We first show that h is B-expansive on $Y \cup \{x_0\}$ for a suitable subset B of $X \times X$ and then the required result can be proved using induction for finitely many steps.

Since X is a paracompact Hausdorff space and U is a neighbourhood of the diagonal in $X \times X$, there exists a symmetric neighbourhood V of the diagonal such that $V \circ V \subset U$, where

$$V \circ V = \{ (x,y) \mid \text{there exists } z \in X \text{ such that } (x,z) \in V \text{ and } (z,y) \in V \},$$

refer [28, p.137]. Further, as h is U-expansive we assert that there exists at most one point p in Y such that

$$(h^n(p), h^n(x_0)) \in V$$

for all integers n. In fact, if there are two such points, say p and q in Y, then we obtain

$$(h^n(p), h^n(x_0)) \in V$$

and

$$(h^n(q), h^n(x_0)) \in V$$

for all integers n, and hence

$$(h^n(p), h^n(q)) \in V \circ V \subset U$$

for all integers n as V is symmetric. But this contradicts the U-expansiveness of h on Y. Having justified the assertion, we consider the following two cases : either such a point p in Y exists or there is no such point p in Y. Choose

$$\begin{aligned} B &= V - \{(p,x_0),(x_0,p)\}, \text{ if such a } p \text{ exists} \\ &= V, \text{ otherwise.} \end{aligned}$$

Now it is easy to observe that h is B-expansive on $Y \cup \{x_0\}$.

4. A characterization of A-expansiveness.

Recall that Wine [42] has obtained a characterization of an expansive homeomorphism on a metric space in terms of a basis (refer Theorem 1.8). In this Section we give a definition about separation of h-orbits in topological setting, obtain a characterization of A-expansive homeomorphisms and then use this characterization to obtain one more extension theorem for A-expansive homeomorphisms.

Definition 2.2. Let h be a homeomorphism on a topological space X and let $A \subset X \times X$. Then h is said to *A-separate* h-orbits if given any basis $\mathscr{B} = \{ x_\alpha | \alpha \in \mathscr{A} \}$ of X with respect to h, whenever $x_\alpha, x_\beta \in \mathscr{B}$ with $\alpha \neq \beta$, there exists an integer n satisfying $(h^n(x_\alpha), h^n(x_\beta)) \notin A$.

Theorem 2.8. *Let* X, A *and* h *be as in the above definition. Then* h *is* A*-expansive iff*

(i) h A*-separates* h*-orbits*;

(ii) *given* p *in* X *and an integer* n *such that* $h^n(p) \neq p$, *there exists an integer* r *satisfying*

$$(h^r(p), h^{r-n}(p)) \notin A.$$

Proof. First suppose h is A-expansive. Then (i) is obvious. For (ii), if $p \in X$ and an integer n are such that $h^n(p) \neq p$, then apply A-expansiveness of h to obtain an integer k such that

$$(h^k(h^n(p)), h^k(p)) \notin A.$$

Now setting $r = k + n$ one gets (ii).

Conversely, suppose (i) and (ii) hold. Let x and y be distinct points of X. In case h-orbits $O(x)$ and $O(y)$ are disjoint, then choose that basis of X with respect to h which has x and y as its members. Now use condition (i) of the hypothesis to conclude that h is A-expansive. On the other hand if $O(x) = O(y)$, then for some n, $x = h^n(y)$. Clearly $y \neq h^n(y)$. Then condition (ii) with p replaced by y gives an integer r satisfying

$$(h^r(y), h^{r-n}(y)) \notin A.$$

Substitute $y = h^{-n}(x)$ to obtain

$$(h^{r-n}(x),\ h^{r-n}(y)) \notin A.$$

But this proves h is A-expansive.

We now give a sufficient condition for a homeomorphic extension of an A-expansive homeomorphism on a subspace to be A-expansive on the whole space.

Theorem 2.9. *Let* $A \subset X \times X$ *and* $Y \subseteq X$. *Suppose* h *is an* A*-expansive homeomorphism on* Y. *Then a homeomorphic extension* f *of* h *to* X *is* A*-expansive if*

(i) $f|_{X-Y}$ *is* A*-expansive;*

(ii) *there exists a basis* $\mathscr{B}$ *of* Y *with respect to* h *such that* $(x,y) \notin A$ *for all* x *in* $\mathscr{B}$ *and* y *in* $X - Y$.

Proof. In order to prove that f is A-expansive, we show that f satisfies conditions (i) and (ii) of Theorem 2.8. Let $\mathscr{C} = \{ x_\alpha \mid \alpha \in \mathscr{A} \}$ be any basis of X with respect to f. Then for distinct elements x_α, x_β in $\mathscr{C}$ we have three possibilities :

(a) x_α, $x_\beta \in Y$,

(b) x_α, $x_\beta \in X - Y$, and

(c) $x_\alpha \in Y$ and $x_\beta \in X - Y$ or $x_\alpha \in X - Y$ and $x_\beta \in Y$.

In case (a) holds we use the A-expansiveness of $f|_Y = h$ and when (b) holds we use the A-expansiveness of $f|_{X-Y}$ and obtain an integer n satisfying $(f^n(x_\alpha), f^n(x_\beta)) \notin A$, i.e., f A-separates f-orbits. Now suppose (c) holds. Let $x_\alpha \in Y$ and $x_\beta \in X - Y$. Since $\mathcal{B}$ is a basis for Y with respect to h, there exists y in $\mathcal{B}$ such that $x_\alpha = h^n(y)$ for some integer n. Further, as f is a homeomorphic extension of h, we can therefore write $x_\alpha = f^n(y)$. But this gives

$$(f^{-n}(x_\alpha), f^{-n}(x_\beta)) \notin A,$$

meaning by f A-separates f-orbits. In case $x_\alpha \in X - Y$ and $x_\beta \in Y$, we argue along the same lines. Thus, we have shown that condition (i) of Theorem 2.8 is satisfied.

To show that f satisfies condition (ii) of Theorem 2.8, assume there is a p in X and an integer n such that $f^n(p) \neq p$. Now, either $p \in Y$ or $p \in X - Y$. But, we know that restrictions of f to Y and X -Y are A-expansive and therefore in both the cases we will find an integer r satisfying

$$(f^r(p), f^{r-n}(p)) \notin A.$$

This completes a proof of the Theorem.

5. Generators and A-expansiveness.

Recall that the notions of a generator and a weak generator (refer definition 1.5) for a homeomorphism on a compact Hausdorff space was defined by Keynes and Robertson in [29] and in the same paper they obtain a characterization of an expansive homeomorphism on a compact metric space in terms of generator (refer Theorem 1.9). Here we define the notions of a generator and a weak generator for homeomorphism on a paracompact Hausdorff space (not necessarily a compact metric space) and obtain a characterization of an A-expansive homeomorphism on a paracompact Hausdorff space in terms of generator, when A is a neighbourhood of the diagonal in the product space.

Definition 2.3. Let X be a paracompact Hausdorff space and let h be a homeomorphism on X. Then

(a) a locally finite open covering $\mathcal{U}$ of X is called a *generator* for (X,h) if for each bisequence $\{U_i\}_{i\in\mathbb{Z}}$ of members of $\mathcal{U}$, $\bigcap_{i=-\infty}^{\infty} h^{-i}(\mathrm{Cl}U_i)$ is at most one point.

(b) an open covering $\mathcal{A}$ of X is called a *weak generator* for (X,h) if for each bisequence $\{A_i\}_{i\in\mathbb{Z}}$ of members of $\mathcal{A}$, $\bigcap_{i=-\infty}^{\infty} h^{-i}(A_i)$ contains at most one point.

That a generator is a weak generator is immediate. For the converse, we have the following result.

Theorem 2.10. *Let* X *be a paracompact Hausdorff space and let* h *be a homeomorphism on* X*. Then* (X,h) *has a generator whenever it has a weak generator.*

Proof. Suppose $\mathcal{A}$ is a weak generator for (X,h). Then $\mathcal{A}$ being an open cover of X, for each x in X there is an A_x in $\mathscr{A}$ such that $x \in A_x$. Now, regularity of X guarantees the existence of an open set G_x such that $x \in G_x \subseteq \mathrm{Cl}G_x \subseteq A_x$. Obviously $\mathscr{G} = \{ G_x \mid x \in X \}$ is an open cover of X. Since X is paracompact Hausdorff, there exists a locally finite open refinement say, $\mathscr{V} = \{ V_\beta \mid \beta \in \mathscr{B} \}$ of $\mathscr{G}$. We claim that $\mathscr{V}$ is a generator for (X,h). To establish the claim, choose any bisequence $\{V_i\}_{i\in\mathbb{Z}}$ of members of $\mathscr{V}$. For each index i, there exists a G_i in $\mathscr{G}$ and an A_i in $\mathscr{A}$ such that $V_i \subseteq G_i \subseteq \mathrm{Cl}G_i \subseteq A_i$. Clearly $\{A_i\}_{i\in\mathbb{Z}}$ is bisequence of members of $\mathscr{A}$ and as $\mathscr{A}$ is a weak generator, $\bigcap_{i=-\infty}^{\infty} h^{-i}(A_i)$ contains at most one point and therefore the inclusion $\bigcap_{i=-\infty}^{\infty} h^{-i}(\mathrm{Cl}V_i) \subseteq \bigcap_{i=-\infty}^{\infty} h^{-i}(A_i)$ shows that $\bigcap_{i=-\infty}^{\infty} h^{-i}(\mathrm{Cl}V_i)$ can not contain more than one point. This establishes the claim.

Now we give a characterization of an A-expansive homeomorphism on a paracompact Hausdorff space in terms of generator defined in Definition 2.3.

Theorem 2.11. *Let* X *be a paracompact Hausdorff space and let* h *be a homeomorphiusm on* X. *Then* (X,h) *has a generator iff* h *is A-expansive, for some neighbourhood* A *of the diagonal in* X×X.

Proof. Suppose h is A-expansive, where A is a neighbourhood of the diagonal in X×X. Then for each x in X there exists a neighbourhood U_x of x in X such that $U_x \times U_x \subset A$. Let $\mathscr{U} = \{ U_x \mid x \in X \}$. We first observe that $\mathscr{U}$ is a weak generator for (X,h). Clearly $\mathscr{U}$ is an open cover of X. Now consider any bisequence $\{U_i\}_{i\in\mathbb{Z}}$ of members of $\mathcal{U}$ and suppose $x, y \in \bigcap_{i=-\infty}^{\infty} h^{-i}(U_i)$. Then $h^i(x) \in U_i$, $h^i(y) \in U_i$ for each integer i and hence

$$(h^i(x), h^i(y)) \in U_i \times U_i \subset A$$

for each i in $\mathbb{Z}$. Since h is A-expansive, it therefore follows that $x = y$. But this proves that $\bigcap_{i=-\infty}^{\infty} h^{-i}(U_i)$ contains at most point i.e., $\mathcal{U}$ is a weak generator for (X,h). Finally, Theorem 2.10 implies $\mathcal{U}$ generator for (X,h).

Conversely, suppose (X,h) possesses a generator, say, $\mathcal{A}$. Since X is paracompact and Hausdorff, $\mathcal{A}$ is an even cover, i.e., there exists a neighbourhood V of the diagonal in $X \times X$ such that for each x in X, $V[x] = \{ y \in X \mid (x,y) \in V \} \subset A$ for some A in $\mathcal{A}$ [28, p.156]. Now we complete the proof by showing that h is V-expansive. If contrary, assume that there is a pair of distinct points in X, say x, y, such that $(h^n(x), h^n(y)) \in V$ for all integers n. Then due to the property of V, for each integer n, there exists an A_n in $\mathcal{A}$ such that

$$h^n(y) \in V[h^n(x)] \subset A_n.$$

Also, since V contains the diagonal in $X \times X$, $h^n(x) \in V[h^n(x)]$ for all integers n. Therefore

$$x, y \in \bigcap_{n=-\infty}^{\infty} h^{-n}(A_n) \subset \bigcap_{n=-\infty}^{\infty} h^{-n}(\mathrm{Cl}A_n).$$

Now, applying the fact that $\mathcal{A}$ is a generator for (X,h), we get $x = y$. But this is a contradiction to the choice of the pair x, y. Hence h is V-expansive.