CHAPTER 3

EXPANSIVE HOMEOMORPHISMS ON G-SPACES

In this Chapter we propose to introduce and study the notion of G-expansive homeomorphism on a metric G-space. Observe that every metric space (X,d) is a uniform space with uniformity $\mathcal{U}$ consisting of the sets of the form $V_\delta = \{(x,y) \in X \times X \mid d(x,y) < \delta\}$, where δ is any positive real number; and hence, the Definition 1.6 of expansive transformation group is applicable. Consequently, the definition of an expansive metric G-space then will be as follows: A metric G-space X with metric d is called expansive if there exists $\delta > 0$ such that whenever $x, y \in X$, $x \neq y$, one can find a g in G satisfying $(gx,gy) \notin V_\delta$, i.e., $d(gx,gy) \geq \delta$. However, it may be seen that this concept of expansive metric G-space does not involve any kind of expansiveness of a homeomorphism on the underlying G-space. Therefore, it will be interesting to define and study the notion of G-expansive homeomorphism on a metric G-space.

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1. G-expansive homeomorphisms.

Let X throughout denote a metric space with metric d, H(X) denote the collection of all homeomorphisms on X and G denote a topological group. Obviously X is a metric G-space under the trivial action of G on X, and then every expansive h in H(X) with expansive constant $\delta > 0$ clearly satisfies the following condition :"For x,y in X with distinct G-orbits i.e., $G(x) \neq G(y)$, there exists an n in $\mathbf{Z}$ satifying $d(h^n(u),h^n(v)) > \delta$ for all u in G(x) and v in G(y). (If an h in H(X) satisfies the above condition for a fixed positive real number δ, then we say that h *δ-expands* each pair of distinct G-orbits.)" However, under a non-trivial action of a G on X, every expansive homeomorphism h on X need not satisfy the above condition. We consider the following examples.

Examples. 3.1(a). Consider the space $X = \{ 1/m, 1- 1/m \mid m \text{ in } \mathbf{N} \}$ with the usual metric defined through the absolute value. Then the h in H(X) which fixes 0 and 1 and sends $t \in X - \{0,1\}$ to the point of X which is next to the right of t, is expansive with expansive constant δ, where $0 < \delta < 1/6$ [6]. Let the topological group $G \equiv \mathbf{Z}_2 = \{-1, 1\}$ act on X with the action defined on X by $1t = t$ and $-1t = 1 - t$, $t \in X$. It is easily seen that for t, s in $X - \{1/2\}$ with $G(t) \neq G(s)$, there exists no integer n satisfying $|h^n(u) - h^n(v)| > \delta$ for all $u \in G(t)$, $v \in G(s)$, whatever $\delta > 0$ may be.

However, in the following examples we see that expansive h does δ-expand distinct G-orbits.

3.1(b). Consider the space $X = \{ f : \mathbf{Z} \to \{0,1\} \}$ with metric d defined by

$d(f,g) = 1/[1 + \max \{m \mid f(i) = g(i) \text{ for } |i| < m \}]$, if $f(0) = g(0)$;

$= 1$, if $f(0) \neq g(0)$.

Then h in H(X) which takes f in X to g where g is defined by $g(i) = f(i + 1)$ for all i in $\mathbf{Z}$, is expansive with each positive real number δ less than 1 as an expansive constant. Let the topological group $G \equiv \mathbf{Z}_2$ act on X with action defined by $1f = f$ and $-1f = 1 - f$. Let f_1, f_2 be in X with distinct G-orbits, i.e., $G(f_1) \neq G(f_2)$. Put $F_j = 1 - f_j$ for $j = 1, 2$ and $k = \min \{ |i| \mid f_1(i) \neq f_2(i) \}$. Then

$$d(h^k(f_1),h^k(f_2)) = d(h^k(F_1),h^k(F_2)) = 1$$

and

$$d(h^k(f_1),h^k(F_2)) = d(h^k(F_1),h^k(f_2)) = 1/2.$$

Threfore h δ-expands each pair of distinct G-orbits for each δ in $(0,1/2)$.

3.1(c). Consider the space $X = \{ \mp 1/m, \mp(1 - 1/m) \mid m \in N \}$ with usual metric. Then the h in H(X) which fixes -1, 0 and 1 and sends $t \in X - \{-1,0,1\}$ to the point of X which is next to the right of t if $0 < t < 1$; while next to the left of t if $-1 < t < 0$, is expansive with expansive constant δ, $0 < \delta < 1/6$. If we consider the action of the group $G \equiv \mathbf{Z}_2$ on X defined by $1t = t$ and $-1t = -t$, $t \in X$, then for any δ, $0 < \delta < 1/6$, h δ-expands each pair of distinct G-orbits.

3.1(d). For an n in Z, let $X_n = \{ n + 1/m, n + 1 - 1/m \mid m \in \mathbf{N} \}$ and consider $X = \cup \{ X_n \mid n \in \mathbf{Z} \}$ with usual metric. Then the h in H(X) which fixes all integers and sends (n + t) to (n + t'), where n is an integer, $t \in Y \equiv \{ 1/m, 1 - 1/m \mid m \in \mathbf{N} - \{1\} \}$ and t' is that element of Y which is next to the right of t, is expansive with expansive constant δ, $0 < \delta < 1/6$. Under the additive action of the group $G \equiv \mathbf{Z}$ on X, it can be observed that for any δ, $0 < \delta < 1/6$, h δ-expands each pair of distinct G-orbits.

3.1(e). Consider the space $X = \mathbf{R}^2$ with usual metric and h_α in $H(\mathbf{R}^2)$ defined by sending u in $\mathbf{R}^2$ to αu, where $\alpha \in \mathbf{R} - \{-1,0,1\}$. Then h_α is expansive with any $\delta > 0$ as an expansive constant. Let $G \equiv \mathbf{R}$ act on $\mathbf{R}^2$ by $x(y, z) = (x + y, z)$, $x, y, z \in \mathbf{R}$. Then for any $\delta > 0$, h δ-expands each pair of distinct G-orbits.

3.1(f). Consider the Euclidean n-space $\mathbf{R}^n$, $n \geq 1$ and h_α in $H(\mathbf{R}^n)$ defined by $h_\alpha(x) = \alpha x$, where $\alpha \in \mathbf{R} - \{-1,0,1\}$. Then h_α is expansive with any $\delta > 0$ as an expansive constant. Let the orthogonal group $G \equiv O(n)$ act on $\mathbf{R}^n$ with the usual action defined by the matrix multiplication. Note that the G-orbit of any point x in $\mathbf{R}^n$ is the (n - 1)-sphere with centre at origin and radius equal to the distance of x from origin. Let δ be any positive real number. Then for x, y in $\mathbf{R}^n$ with distinct G-orbits G(x) and G(y), there exists $u \in G(x)$, $v \in G(y)$ such that $d(u,v) = d(G(x),G(y))$ where d is the Euclidean metric on $\mathbf{R}^n$ and hence from the expansiveness of h_α, there exists an integer n, satisfying

$d(h_\alpha^n(u), h_\alpha^n(v)) > \delta$. It can be verified that $d(h_\alpha^n(p), h_\alpha^n(q)) > \delta$ for all p in G(x) and q in G(y). Thus h δ-expands each pair of distinct G-orbits.

3.1(g). For each positive integer n, we define the space

$$X^{(n)} = \{ e^{i2\pi(k+t)/n}, \text{ where } k = 0,1,\ldots\ldots,n-1 \text{ and } t \in Y \equiv \{ 1/m,\ 1 - 1/m;\ m \in \mathbf{N} \} \}$$

with usual metric d and consider h_n in $H(X^{(n)})$ defined by

$$h_n(e^{i2\pi(k+t)/n}) = e^{i2\pi(k+t)/n}, \text{ when } t = 0 \text{ or } t = 1;$$
$$= e^{i2\pi(k+t')/n}, \text{ when } t \in Y - \{0,1\}$$

wherein t' is that element of Y which is next to the right of t. Then h_n is expansive with expansive constant δ, where $0 < \delta < d(e^{i\pi/n}, e^{i2\pi/3n})$. Under the usual action on $X^{(n)}$ of the group $G \equiv U(n)$ consisting of the nth roots of unity it is easily seen that h_n δ-expands all pairs of distinct G-orbits, where $0 < \delta < d(e^{i\pi/n}, e^{i2\pi/3n})$.

The observations made in these examples lead us to the following definition.

Definition 3.1. Let X be a metric G-space and $h \in H(X)$. Then h is called G-*expansive* if there exists a $\delta > 0$ such that whenever $x, y \in X$ with $G(x) \neq G(y)$, there exists an integer n satisfying $d(h^n(u), h^n(v)) > \delta$ for all $u \in G(x)$ and $v \in G(y)$; δ is then called a G-*expansive constant* for h.

Note (a). Clearly the notion of G-expansive homeomorphism on a metric G-space as defined above is completly different from the notion of expansive transformation group defined by Eisenberg (Definition 1.6) in [11].

Note (b). Under the trivial action of G on a metric space X, a G-expansive h in H(X) is expansive. However, under a non-trivial action of a G on X, a G-expansive h in H(X) need not be expansive as can be seen from the following example.

Example 3.2. Consider $X = \cup \{ C_k \mid k = 1,..,n \}$ with usual metric, where $n \in \mathbf{N}$ and C_k is the circle in the Euclidean plane $\mathbf{R}^2$ with centre at origin and radius k. Under the usual action of $G \equiv O(2)$ on X defined by matrix multiplication, the identity map on X is G-expansive with G-expansive constant δ, $0 < \delta < 1$, but obviously it is not expansive on X.

Also, under a non-trivial action of a group G on a metric space X an expansive homeomorphism need not be G-expansive. This can be seen from Example 3.1(a). Thus the notions of expansiveness and that of G-expansiveness on a metric G-space are independent of each other. This naturally therefore raises a question :

When an expansive homeomorphism on a metric G-space is G-expansive and conversely, when a G-expansive homeomorphism on it is expansive ?

A look at the Examples 3.1(c), (e) and (f) wherein expansive homeomorphisms do turn out to be G-expansive reveals the following facts :

(F1) h satisfies $h(G(x)) = G(h(x))$, $x \in X$;

(F2) G is a subgroup of ISO(X), the group of isometries on X ;

(F3) For each pair of distinct G-orbits G(x) and G(y) in X, there exists a $g_o \in G$ such that

(a) $d(g_o x, y) = d(G(x), G(y))$ and

(b) for all g_o in G satisfying (a),

$$d(h^{i}(g_o x), h^{i}(y)) \le d(h^{i}(g g_o x), h^{i}(y)),\ g \in G,\ i \in \{-1,1\}.$$

It is then natural to inquire whether an expansive homeomorphism satisfying (F1), (F2) and (F3) is G-expansive. We show in what follows that this is indeed the case.

Here it may be noted that in Examples 3.1(c) and 3.1(f), if $x = 0$, then $G(x) = \{0\}$ and every g in G works as a g_o in (F3)(a) and all of them trivially satisfy (F3)(b), but if x, y are different from 0, then there exists a unique g_o. In Example 3.1(e) we get unique g_o. However, in Example 3.1(e) if we consider the action on $\mathbb{R}^2$ of the subgroup $\mathbb{Z}$ instead of R then g_o may not be unique; for instance, consider

$G(x,y)$ and $G((x + 1)/2, u)$; $x, y, u \in \mathbb{R}$ and $y \neq u$,

then both 0 and 1 work as g_o, of course, (F3)(b) holds for all such g_o.

In view of (F1), we call a continuous map f from a G-space X to a G-spacce Y to be *pseudoequivariant* if $f(G(x)) = G(f(x))$ for all $x \in X$. (Recall that f is called *equivariant* if for all x in X and g in G, f satisfies $f(gx) = gf(x)$.) Clearly every equivariant map is pseudoequivariant but the fact that the converse is not true can be seen by considering the map h_α in the Example 3.1(e). The following result concerning pseudoequivariant homeomorphism will be used for getting a sufficiency condition for an expansive homeomorphism to be G-expansive. We study some more properties of pseudoequivariant maps in Chapter 4.

Lemma 3.1. *If* h *in* H(X) *is pseudoequivariant, then*

$$h^n(G(x)) = G(h^n(x)),$$

for each x *in* X *and* n *in* $\mathbb{Z}$.

Proof. When n = 0, the result is obviously true. We prove the Lemma for positive integers by applying the induction principle. For n = 1, it is true by definition of pseudoequivariany. For n = 2, we have

$$\begin{aligned} h^2(G(x)) &= h(h(G(x))) \\ &= h(G(h(x))) \\ &= G(h(h(x))) = G(h^2(x)). \end{aligned}$$

Suppose the result is true for n = m. Then we have

$$\begin{aligned} h^{m+1}(G(x)) &= h(h^m(G(x))) \\ &= h(G(h^m(x))) \\ &= G(h(h^m(x))) = G(h^{m+1}(x)). \end{aligned}$$

Therefore, the Lemma follows for all positive integers.

Now we show that $h^{-1}(G(x)) = G(h^{-1}(x))$. Let $u \in h^{-1}(G(x))$.

Then $h(u) \in G(x)$. Therefore $h(u) = gx$ for some g in G, i.e.,

$$x = g^{-1}h(u) \in G(h(u)) = h(G(u)).$$

Thus $x = h(g'u)$ for some $g' \in G$, i.e.,

$$u = (g')^{-1}h^{-1}(x) \in G(h^{-1}(x)).$$

Hence $h^{-1}(G(x)) \subseteq G(h^{-1}(x))$. For the reverse inclusion, if $v \in G(h^{-1}(x))$, then $v = gh^{-1}(x)$ for some g in G. Therefore

$$g^{-1}v = h^{-1}(x) \quad \text{or} \quad x = h(g^{-1}v).$$

This implies

$$x \in h(G(v)) = G(h(v)).$$

Thus $x = kh(v)$ for some k in G, i.e.,

$$v = h^{-1}(k^{-1}x) \in h^{-1}(G(x))$$

and so we get the desired containment. Hence,

$$h^{-1}(G(x)) = G(h^{-1}(x)).$$

This proves that $f = h^{-1}$ is pseudoequivariant. Therefore as proved above, $G(f^m(x)) = f^m(G(x))$ for for all m in $\mathbf{N}$, and hence we get

$$G(h^{-m}(x)) = h^{-m}(G(x))$$

for all m in $\mathbf{N}$. This completes the proof of the Lemma.

Lemma 3.2. *Let* X *be a metric* G*-space and* $h \in H(X)$. *Suppose* (F1), (F2) *and* (F3) *hold. Then whenever* $G(x) \neq G(y)$, *we have*

$$d(h^m(gg_o x), h^m(y)) \geq d(h^m(g_o x), h^m(y)) \qquad (*)$$

for all m *in* $\mathbf{Z}$ *and* g *in* G, *where* g_o *is same as described in* (F3).

Proof. Since $G(x) \neq G(y)$, in view of (F3) for $i = 1$ in (F3)(b), there exists a g_o in G such that $d(g_o x, y) = d(G(x), G(y))$ and

$$d(h(g_o x), h(y)) \leq d(h(gg_o x), h(y)) \qquad (A)$$

for all g in G. We claim that pseudoequivariancy of h then gives

$$d(h^2(g_0x),h^2(y)) \leq d(h^2(gg_0x),h^2(y))$$

for all $g \in G$. For a proof of the claim we proceed as follows. As h is given to be pseudoequivariant so for $g' \in G$ we have $g'h(g_0x) = h(gg_0x)$ for some g in G and hence by (A),

$$d(g'h(g_0x),h(y)) = d(h(gg_0x),h(y)) \geq d(h(g_0x),h(y)).$$

This gives

$$\text{Inf } \{ d(g'h(g_0x),h(y)) \mid g' \in G \} \geq d(h(g_0x),h(y)).$$

On the other hand

$$\text{Inf } \{ d(g'h(g_0x),h(y)) \mid g' \in G \} \leq d(h(g_0x),h(y))$$

by considering g' to be the identity of G. Thus

$$d(G(h(g_0x),h(y)) = d(h(g_0x),h(y)).$$

Now by applying (F2) we get,

$$d(G(h(g_0x)),h(y)) = d(G(h(g_0x)),G(h(y)))$$

and as h is pseudoequivariant $h(g_0x) = g_1h(x)$ for some g_1 in G, and hence

$$d(G(h(x)),G(h(y))) = d(g_1h(x),h(y)).$$

Since $G(x) \cap G(y) = \varphi$ implies $h(G(x)) \cap h(G(y)) = \varphi$ in view of h being bijective, from the pseudoequivariancy of h we have $G(h(x)) \cap G(h(y)) = \varphi$. Therefore using (F3)(b), with $i = 1$, g_1 in the place of g_0 and x, y replaced by h(x) and h(y) respectively, we get

$$d(h(g_1h(x)),h(h(y))) \leq d(h(g'g_1h(x)),h(h(y)))$$

for all $g' \in G$. Here $g_1h(x) = h(g_0x)$ and hence

$$d(h^2(g_0x),h^2(y)) \leq d(g'h(g_0x),h^2(y)),$$

for all g' in G. Now for a g in G, from the pseudoequivariancy of h there exists a $g' \in G$ such that

$h^2(gg_o x) = h(g'h(g_o x))$

and hence we obtain

$$d(h^2(gg_o x), h^2(y)) = d(hg'h(g_o x)), h^2(y)).$$

This establishes the claim.

Finally, as proved in Lemma 3.1 the pseudoequivariancy of h implies $h^n(G(x)) = G(h^n(x))$ for all n in $\mathbb{Z}$ and x in X. Thus, it can be shown that (*) will hold true for $m = k + 1$ if one assumes (*) to be true for m = k. Hence (*) follows for all positive integers. Using (F3)(b) for $i = -1$, one can similarly show that (*) holdsfor all negative integers. The m = 0 case follows from (F3)(a). This completes the proof of the Lemma.

Theorem 3.1. *Under the hypothesis of Lemma* 3.2, h *is* G-*expansive with* G-*expansive constant* δ *whenever it is expansive with expansive constant* δ.

Proof. Consider x, y in X with $G(x) \neq G(y)$. Then from Lemma 3.2 there exists a g_o in G which satisfis (*). Since h is expansive with expansive constant δ, there exists an m in $\mathbb{Z}$ satisfying $d(h^m(g_o x), h^m(y)) > \delta$ and hence

$$\begin{aligned} d(h^m(gx), h^m(y)) &= d(h^m(gg_o^{-1}g_o x), h^m(y)) \\ &\geq d(h^m(g_o x), h^m(y)) > \delta. \qquad (\#) \end{aligned}$$

for each $g \in G$. Now for any g,k in G, using (B) and (F2) we have

$$\begin{aligned} d(h^m(gx), h^m(ky)) &= d(h^m(gx), k'h^m(y)) \\ &= d((k')^{-1}h^m(gx), h^m(y)) \\ &= d(h^m(k_1 x), h^m(y)) > \delta \end{aligned}$$

by (#). This proves that h is G-expansive with G-expansive constant δ.

Remark. It may be observed that in Theorem 3.1 the condition of pseudoequivariancy of h is not a necessary condition. This is seen by considering the usual additive action of the group $\mathbb{Z}$ on $\mathbb{R}$ and the $\mathbb{Z}$-expansive homeomorphism $h_\alpha : \mathbb{R} \to \mathbb{R}$, defined by $h_\alpha(x) = \alpha x$, $\alpha \in \mathbb{R} - \{ -1, 0, 1 \}$, having any positive real number δ as G-expansive constant. Similarly, since property (F3)(b) is not true for Examples 3.1(d) and 3.1(g), clearly the condition (F3)(b) too is not necessary. For instance, in Example 3.1(d), if we take $x = -2/3$, $y = -1/7$ and $g = -1$, then $g_o = 1$ and therefore

$$d(h(gg_o x),h(y)) = 3/8$$

while

$$d(h(g_o x),h(y)) = 5/8.$$

Regarding the converse of the above theorem we have the following result.

Theorem 3.2. *Let* X *be a metric* G*-space and let* h in H(X) *be* G*-expansive with* G*-expansive constant* δ. *Then* h *is expansive with expansive constant* δ *if* h *is expansive on* G(x) *for each* x *in* X *with expansive constant* δ.

Proof. Let x, y be in X, $x \neq y$. Then either x, y lie in same G-orbit or they lie in distinct G-orbits. In case they lie in same G-orbit, the result follows by the hypothesis. Otherwise, the result follows by the G-expansiveness of h.

It may be noted here that the condition of expansiveness of h on G(x) for each x in X in Theorem 3.2 is necessary for if we consider the identity map on X in Example 3.2, then it is G-expansive but it is not expansive on any G-orbit in X.

2. Properties of G-expansive homeomorphisms.

Here we prove some properties of G-expansive homeomorphisms. First is an analogue of the Theorem 1.1 proved by Utz [37] for expansive homeomorphisms on metric spaces.

Theorem 3.3. *Let* X *be a compact metric* G-*space and let* $f \in H(X)$. *Then* h *is* G-*expansive iff* f^m, $m \neq 0$, *is* G-*expansive.*

Proof. First we suppose that f^m, $m \neq 0$, is G-expansive with G-expansive constant δ. Fix some integer m in $\mathbb{Z}$. Let $x, y \in X$ with $G(x) \neq G(y)$. Then by G-expansiveness of f^m, there exists an n in $\mathbb{Z}$ satisfying

$$d((f^m)^n(gx),(f^m)^n(ky)) > \delta$$

for all g, k in G. Substituting m.n = t, we get

$$d(f^t(gx),f^t(ky)) > \delta$$

for all g, k in G. Hence f is G-expansive with G-expansive constant δ.

Conversely, suppose f is G-expansive with δ as G-expansive constant. Since X is compact, any h in H(X) will satisfy the following condition :

Given any $\varepsilon > 0$, there exists an $\eta > 0$ such that

$d(x,y) \geq \varepsilon \Rightarrow d(h(x),h(y)) > \eta$.

Let $\phi = f^m$, where m is an integer different from 0. Consider f^i, $i = \pm1, \ldots, \pm m$. Then corresponding to $\delta > 0$, there exists η_i, for every $i \in \{\pm1, \ldots, \pm m\}$ such that

$$d(x,y) \geq \delta \Rightarrow d(f^i(x), f^i(y)) > \eta_i$$

for all x, y ∈ X. Let x, y ∈ X with distinct G-orbits. Since f is G-expansive with G-expansive constant δ, there exists an integer n satisfying $d(f^n(gx), f^n(ky)) > \delta$, for each g, K ∈ G. Thus,

$$d(f^i(f^n(gx)), f^i(f^n(ky))) > \min \{\eta_i\},$$

$i \in \{\pm1, \ldots, \pm m\}$, and g, k ∈ G. Since one can find an integer r such that $0 < |r - n/m| \leq 1$ or $0 < |rm - n| \leq |m|$, therefore $(rm-n) \in \{\pm1, \ldots, \pm m\}$ and hence

$$d(f^{rm-n}(f^n(gx)), f^{rm-n}(f^n(ky))) > \min \{\eta_i\},$$

or equivalently

$$d((f^m)^r(gx), (f^m)^r(ky)) > \min \{\eta_i\},$$

where $i \in \{ \pm1, \ldots, \pm m \}$ and g, k ∈ G. Hence $\phi = f^m$ is G-expansive with G-expansive constant α, where $\alpha = \min \{ \eta_i | \, i \in \{\pm1, \ldots \pm m\} \}$.

The following result concerns the restriction of G-expansive homeomorphism.

Theorem 3.4. *Let* X *be a metric* G*-space,* h *in* H(X) *be* G*-expansive and* A *be a* G*-nvariant subspace of* X *such that* h(A) = A. *Then* $h|_A$ *is* G*-expansive on* A.

Proof. Let δ be G-expansive constant for h on X. Choose x, y in A with distinct G-orbits. Since h is given to be G-expansive on X, there exists an integer n satisfying $d(h^n(gx), h^n(ky)) > \delta$ for all g, k in G; then A being G-invariant, $G(x) \subseteq A$ and $G(y) \subseteq A$ and hence h is G-expansive on A with G-expansive constant δ.

Next, we prove a result regarding product of two G-expansive homeomorphisms.

Theroem 3.5. *Let* (X,d) *and* (Y,ρ) *be two metric* G-*spaces, and let* h *in* H(X), f *in* H(Y) *be* G-*expansive homeomorphisms. Then* $h \times f \in H(X \times Y)$ *and is* G-*expansive when the product space* $X \times Y$ *is given the diagonal action of* G *i.e.,* $g(x,y) = (gx,gy)$, *where* $g \in G$ *and* $(x,y) \in X \times Y$.

Proof. Suppose D denotes the product metric on $X \times Y$. Let h in H(X) be G-expansive with G-expansive constant δ and let f in H(Y) be G-expansive with G-expansive constant ε. Obviously $h \times f \in H(X \times Y)$. Suppose $(x,y), (u,v) \in X \times Y$ with $G(x,y) \neq G(u,v)$, i.e., $(x,y) \neq g(u,v)$ for any g in G. Then either $x \neq gu$ for any g in G or $y \neq gv$ for any g in G. In case $x \neq gu$, by G-expansiveness of h there exists an integer n satisfying $d(h^n(gx),h^n(ku)) > \delta$ for each g, $k \in G$ and therefore we obtain

$$\begin{aligned} D((h\times f)^n(g(x,y)),(h\times f)^n(k(u,v))) &= D((h^n(gx),f^n(gy)),(h^n(ku),f^n(kv))) \\ &= [d(h^n(gx),h^n(ku))^2 + \rho(f^n(gy),f^n(kv))^2]^{1/2} \\ &> \delta \geq \min\{\delta,\varepsilon\}, \end{aligned}$$

for each g, k in G. In case $y \neq kv$ for any $k \in G$, we apply the G-expansiveness of f to obtain an integer m satisfying

$$D((h\times f)^m(g(x,y)),(h\times f)^m(k(u,v))) > \varepsilon \geq \min\{\delta,\varepsilon\}$$

for each g, k in G. This proves that $(h \times f)$ is G-expansive with G-expansive constant α, where $\alpha = \min\{\delta, \varepsilon\}$.

Using the same method it can easily be proved that if $h_i \in H(X_i)$, $i = 1, \ldots, n$; $n \in \mathbb{N}$, are G-expansive, then $\prod h_i$ is also G-expansive on $\prod X_i$ under the diagonal action of G on $\prod X_i$.

The following result is an analogue of Bryant's result (refer Theorem 1.4) proved in [6] for expansive homeomorphism on metric spaces.

Theorem 3.6. *If a pseudoequivariant homeomorphism* Ψ *from a metric G-space* X *to a metric G-space* Y *is such that* Ψ^{-1} *is uniformly continuous, then* $\Psi h \Psi^{-1}$ *is G-expansive on* Y *whenever* h *is G-expansive on* X.

Proof. Let d and ρ denote the metrics of X and Y respectively. Sine Ψ^{-1} is uniformly continuous so for a given $\varepsilon > 0$ there exists $\alpha > 0$ such that

$$\rho(u,v) \le \alpha \Rightarrow d(\Psi^{-1}(u), \Psi^{-1}(v)) < \varepsilon;$$

where u, $v \in Y$ or equivalently

$$d(\Psi^{-1}(u), \Psi^{-1}(v)) \ge \varepsilon \Rightarrow \rho(u,v) > \alpha$$

which means for p, q in X one has

$$d(p,q) \ge \varepsilon \Rightarrow \rho(\Psi(p), \Psi(q)) > \alpha.$$

Now, let h in H(X) be G-expansive with G-expansive constant δ and let u, v be in Y with distinct G-orbits. Then

$$G(u) \cap G(v) = \varphi \Rightarrow \Psi^{-1}(G(u)) \cap \Psi^{-1}(G(v)) = \varphi.$$

Using pseudoequivariancy of Ψ, we have from Lemma 3.1

$$G(\Psi^{-1}(u)) \cap G(\Psi^{-1}(v)) = \varphi.$$

That is $\Psi^{-1}(u)$ and $\Psi^{-1}(v)$ have distinct G-orbits. Thus by G-expansiveness of h, there exists an integer n satisfying

$$d(h^n(g\Psi^{-1}(u)),h^n(k\Psi^{-1}(v))) > \delta$$

for each g, k in G. Corresponding to δ there exists a $\beta > 0$ satisfying

$$\rho(\Psi h^n(g\Psi^{-1}(u)),\Psi h^n(k.\Psi^{-1}(v))) > \beta$$

for each g, k in G. Another use of pseudoequivariancy of Ψ gives

$$\rho(\Psi h^n\Psi^{-1}(g'u),\Psi h^n\Psi^{-1}(k'v)) > \beta$$

or equivalently

$$\rho((\Psi h\Psi^{-1})^n(g'u),(\Psi h\Psi^{-1})^n(k'v)) > \beta$$

for each g', k' in G. Hence $\Psi h\Psi^{-1}$ is G-expansive with G-expansive constant β.

We next prove a result which is analogue of Theorem 1.10 obtained by Bryant [6] for expansive homeomorphisms on metric spaces.

Theorem 3.7. *Let* X *be a compact metric G-space with* G *compact and let* h *in* H(X) *be G-expansive with G-expansive constant* δ. *Then there exists a* $k(\lambda)$ *in* **N** *for each* λ, $0 < \lambda \le \delta$, *such that* $Inf \{ d(gx,ky) \mid g, k \in G \} > \lambda$ *implies* $d(h^n(gx),h^n(ky)) > \delta$, *for all* g, $k \in G$ *and for some integer* n *satisfying* $|n| \le k(\lambda)$.

Proof. Suppose the result is not true. Then there exists a λ such that $0 < \lambda \le \delta$ and for each i in **N**, there exist x_i, y_i with

$$\text{Inf} \{ d(gx_i, ky_i) \mid g, k \in G \} > \lambda \qquad (\blacksquare)$$

and

$$d(h^n(gx_i), h^n(ky_i)) \leq \delta$$

for some g, $k \in G$ and each integer n such that $|n| \leq i$. Since X is compact we can assume that sequences $\{x_i\}$ and $\{y_i\}$ converge respectively to some elements x and y of X.

Now using ($\blacksquare$), one can easily conclude that x and y have distinct G-orbits. Next, choose an integer m. Since

$$d(h^m(gx_i), h^m(ky_i)) \leq \delta$$

for each $i \geq |m|$ and some g, $k \in G$, we have

$$d(h^m(gx), h^m(ky)) \leq \delta$$

for some g, $k \in G$. But, as choice of m was arbitrary, we get a contradiction to the fact that h is G-expansive with G-expansive constant δ. This completes the proof of the Theorem.

3. Fixed points, extension and characterization of G-expansive homeomorphisms.

Regarding the fixed points of an expansive homeomorphism, recall that an expansive homeomorphism on a compact metric space can have only finetely many fixed points. However, this need not be true for a G-expansive homeomorphism. For example the G-expansive homeomorphism of Example 3.2 on the compact metric G-space X has uncountably many fixed points.

We now deal with an extension problem of G-expansive homeomorphisms. If X is a metric G-space, $A \subseteq X$ is G-invariant and

$h \in H(X)$, then by G-expansiveness of h on A we mean that there exists a positive real number δ such that whenever $x, y \in A$ with with distinct G-orbits, there exists an n in **Z** satisfying $d(h^n(u),h^n(v)) > \delta$ for all $u \in G(x), v \in G(y)$. Obviously a G-expansive homeomorphism h on a metric G-space is G-expansive on every G-invariant subspace A of X. Regarding the extension of G-expansive homeomorphisms, in the following we prove a result which gives condition under which an h in H(X) is G-expansive on X whenever it is G-expansive on a G-invariant subspace of X.

Theorem 3.8. *Let* X *be a metric* G-*space and* A *be a* G-*invariant subspace of* X *such that* X − A *is a union of finitely many distinct* G-*orbits. Then an* h *in* H(X) *which is* G-*expansive on* A *is* G-*expansive on* X.

Proof. Let δ be a G-expansive constant of h on A and let

$$X - A = \cup \{ G(x_i) \mid i = 1,\ldots,n \},$$

where $G(x_i) \neq G(x_j)$ for $i \neq j$. We need to prove that h is G-expansive on $A \cup G(x_1)$ because then the result will follow by induction. To see that h is G-expansive on $A \cup G(x_1)$, first observe that there can not exist two points p, q in A with distinct G-orbits such that for given m in **Z**, one can get g, k, t in G satisfying

$$d(h^m(gp),h^m(tx_1)) \leq \delta/2$$

and

$$d(h^m(kq),h^m(tx_1)) \leq \delta/2.$$

For otherwise, using the triangle inequality of the metric d, we will arrive at a contradiction to the hypothesis that h is G-expansive on A with G-expansive constant δ. It follows that there exists at most one point p in A such that given m in $\mathbb{Z}$, one can find g, k in G satisfying $d(h^m(gp),h^m(kx_1)) \le \delta/2$.

Now, in case such a p exists in A, choose c such that

$$0 < c < \operatorname{Inf} \{ d(gx_1,kp) \mid g, k \in G \},$$

otherwise take $c = \delta/2$ and observe that h is G-expansive on $A \cup G(x_1)$ with c as a G-expansive constant.

Since in general expansiveness neither implies nor is implied by G-expansiveness, the following characterization of G-expansiveness will be interesting. First we give a necessary definition.

Throughout X denotes a metric G-space with metric d.

Definition 3.2. Given $\delta > 0$, an h in H(X) is said to G-δ *separate* h-orbits if given any basis $\mathcal{B} = \{ x_\alpha \mid \alpha \in \mathcal{A} \}$ of (X,h), whenever $G(x_\alpha) \neq G(x_\beta)$ there exists an integer r satisfying

$$d(h^r(gx_\alpha),h^r(kx_\beta)) > \delta$$

for all g, k in G.

Theorem 3.9. *Let* h *in* H(X) *be pseudoequivariant. Then* h *is* G-*expansive with* G-*expansive constant* δ *iff*

(i) h-*orbits are* G-δ *separated by* h *and*

(ii) *for* p *in* X *and* n *in* $\mathbb{Z}$ *such that* $h^n(p) \notin G(p)$, *there exists an* *integer r satisfying*

$$d(h^{n+r}(gp),h^{r}(kp)) > \delta$$

for all g, k *in* G.

Proof. From Lemma 3.1, the pseudoequivariancy of h gives

$$h^{q}(G(x)) = G(h^{q}(x)) \qquad (B)$$

where $x \in X$ and $q \in \mathbb{Z}$. Suppose h in H(X) is G-expansive with G-expansive constant δ. Let $\mathcal{B} = \{ x_\alpha \mid \alpha \in \mathcal{A} \}$ be any basis of X with respect to h. Then whenever $G(x_\alpha) \neq G(x_\beta)$, from G-expansiveness of h there exists an integer r satisfying $d(h^r(gx_\alpha),h^r(kx_\beta)) > \delta$ for each g, k in G. Thus, h G-δ separates h-orbits. Also, if p in X and n in $\mathbb{Z}$ are such that $h^n(p) \notin G(p)$, then we get $G(h^n(p)) \neq G(p)$. Therefore from G-expansiveness of h there exists an integer r satisfying $d(h^r(gh^n(p)),h^r(kp)) > \delta$, for each g, k in G. Using (B), we obtain $gh^n(p) = h^n(g'p)$ for some g' in G. But this gives

$$d(h^{n+r}(g'p),h^{r}(kp)) > \delta$$

for all g',k in G. This completes the proof of (ii).

Conversely, suppose both the conditions (i) and (ii) hold. Then we show that h is G-expansive with G-expansive constant δ. For this let x, y be in X with distinct G-orbits. Then either x and y are in the same h-orbit or they lie in different h-orbits. We consider both the two cases separately.

<u>Case</u> 1. Let x, y be in distinct h-orbits say, $O(x_\alpha)$ and $O(x_\beta)$ respetively. Let $x = h^n(x_\alpha)$ and $y = h^m(x_\beta)$. We put $x_\gamma = h^{m-n}(x_\beta)$, and write $y = h^n(x_\gamma)$. Now choose that basis $\mathcal{B}$ of (X,h) which has x_α and x_γ as its members. Since $G(x) \neq G(y)$, i.e.,

$G(h^n(x_\alpha)) \neq G(h^n(x_\gamma))$, using pseudoequivariancy of h, we have

$$h^n(G(x_\alpha)) \cap h^n(G(x_\gamma)) = \varphi$$

and hence

$$G(x_\alpha) \neq G(x_\gamma).$$

Now using (i) we get an integer r satisfying

$$d(h^r(gx_\alpha), h^r(kx_\gamma)) > \delta$$

for all g, k in G, i.e.,

$$d(h^r(gh^{-n}(x)), h^r(kh^{-n}(y))) > \delta$$

for all g, k in G. Now we again use pseudoequivariance of h to obtain

$$d(h^{r-n}(g'x), h^{r-n}(k'y)) > \delta$$

for all g', k' in G. Thus we get an integer t = (r - n) such that for all g', k' in G

$$d(h^t(g'x), h^t(k'y)) > \delta.$$

It follows in this case that h is G-expansive with G-expansive δ.

<u>Case</u> 2. Let x,y be in the same h-orbit say, $O(x_\alpha)$. Then $x = h^n(x_\alpha)$ and $y = h^m(x_\alpha)$ for some integers n and m. Since $G(x) \neq G(y)$, i.e.,

$$G(h^n(x_\alpha)) \neq G(h^n(h^{m-n}(x_\alpha))),$$

using pseudoequivariancy of h we get

$$h^n(G(x_\alpha)) \cap h^n(G(h^{m-n}(x_\alpha))) = \varphi$$

and hence $G(x_\alpha) \neq G(h^{m-n}(x_\alpha))$. Thus $h^{m-n}(x_\alpha) \notin G(x_\alpha)$ and hence by (ii) there exists an integer r satisfying

$$d(h^r(gx_\alpha), h^{m-n+r}(kx_\alpha)) > \delta,$$

or equivalently

$$d[h^r(gh^{-n}(x)), h^{m-n+r}(kh^{-m}(y))] > \delta$$

for all g, k in G. Now using the pseudoequivariancy of h we get

$$d(h^{r-n}(g'x), h^{r-n}(k'y)) > \delta$$

for all $g', k' \in G$, i.e., there exists an integer $q = (r - n)$ such that for all g', k' in G

$$d(h^q(g'x), h^q(k'y)) > \delta.$$

Hence, in this case also, h is G-expansive with G-expansive constant δ. This completes a proof of the Theorem.

Using the above result, we prove the following result which gives a sufficient condition for a homeomorphic extension of a G-expansive homeomorphism to be G-expansive.

Theorem 3.10. *Let* X *be a* G*-invariant subspace of a metric* G*-space* Y *and let* h *in* H(X) *be pseudoequivariant* G*-expansive with* G*-expansive constant* δ*. Then a pseudoequivariant extension* f *of* h, $f \in H(Y)$, *is* G*-expansive with* G*-expansive constant* δ *if*

(i) f *is* G*-expansive on* Y − X *with* G*-expansive constant* δ; *and*

(ii) *there exists a basis* $\mathcal{B}$ *of* (X,h) *such that* $d(g.x, (Y - X)) > \delta$ *for each* g *in* G *and* x *in* $\mathcal{B}$.

Proof. To prove that $f \in H(Y)$ is G-expansive with G-expansive constant δ, we show that f satisfies conditions (i) and (ii) of Theorem 3.9. Let $\mathcal{E} = \{ y_\alpha \mid \alpha \in \mathcal{A}, \mathcal{A}$ is an index set $\}$ be a basis of Y with respect to f. Let y_α, $y_\beta \in \mathcal{E}$ with distinct G-orbits. Then there are three possibilities :

(a) y_α, $y_\beta \in X$;

(b) y_α, $y_\beta \in Y - X$, and

(c) $y_\alpha \in X$ and $y_\beta \in Y - X$ or $y_\alpha \in Y - X$ and $y_\beta \in X$.

In the situation (a) and (b) the Theorem follows by using the fact that $f|_X$ and $f|_{Y-X}$ are G-expansive with G-expansive constant δ and also the fact that X is a G-invariant subspace of Y.

Now, suppose we are in the situation (c). Let $y_\alpha \in X$ and $y_\beta \in Y - X$. As $\mathcal{B}$ is a basis for (X,h), there exists a x in $\mathcal{B}$ such that $y_\alpha \in O(x)$, i.e., for some integer n, $y_\alpha = h^n(x)$. From the hypothesis we get

$$d(g'f^{-n}(y_\alpha), f^{-n}(ky_\beta)) > \delta$$

for each g' and k in G. Since f is pseudoequivariant, Lemma 3.1 gives

$$d(f^{-n}(gy_\alpha), f^{-n}(ky_\beta)) > \delta$$

for each g, k in G. Thus, f-orbits are G-δ separated by f. For condition (ii), let $p \in Y$ and $n \in \mathbb{Z}$ be such that $h^n(p) \notin G(p)$. Then either $p \in X$ or $p \in Y - X$. In case $p \in X$, X being G-invariant $G(p) \subseteq X$ and therefore by G-expansiveness of $f|_X = h$, there exists an integer r satisfying

$$d(f^{n+r}(gp), f^r(kp)) > \delta$$

for each g, k in G. Similarly if $p \in Y - X$, G-expansiveness of f on Y - X gives the required condition. Hence f in H(Y) is G-expansive with G-expansive constant δ.

Remark . If the action of G on X is trivial, then Theorems 3.9 and 3.10 reduce to respectively Theorems 1.8 and 1.9 due to Wine [42] stated in Chapter 1.

The following example shows that the sufficiency condition concerning basis in Theorem 3.10 is not necessary.

Example 3.3. Let the space X and the homeomorphism h in H(X) of Example 3.1(c) be here denoted by W and ϕ respecctively. On $Y = W \times W$ let $G = \mathbb{Z}_2$ act by

$$1.(s,t) = (s,t) \quad \text{and} \quad -1.(s,t) = (-s, -t).$$

Then

$$X = [\ (W - \{-1,0,1\}) \times (W - \{-1,0,1\})\ ]$$

is a G-invariant subspace of Y and the function $h = \phi \times \phi$ on X is a pseudoequivariant G-expansive homeomorphism on X with G-expansive constant δ, where $0 < \delta < 1/6$. Also, the function $f = \phi \times \phi$ on Y is in H(Y) and is obviously a pseudoequivariant extension of h to Y such that f is G-expansive on Y - X as well as on Y with the same G-expansive constant δ. But, for the h-orbit O(1/m,1 - 1/m) of any point (1/m,1 - 1/m) with $1/m < \delta$ one has $d(gt, Y - X) \le \delta$ for any g in G and t in O(1/m,1 - 1/m). Since any basis of X with respect to h contains a point of such h-orbit O(1/m, 1 - 1/m), it follows that the condition concerning the basis in Theorem 3.10 is not necessary.