CHAPTER 5

A-EXPANSIVENESS ON G-SPACES

Motivated by the concept of expansiveness of a homeomorphism on a metric space, in Chapter 2 we defined the notion of A-expansiveness of a homeomorphism on a topological space X relative to a subset A of X×X; while in Chapter 3 we defined the notion of G-expansiveness of a homeomorphism on a metric G-space wherein G is any topological group acting on the metric space. It is therefore natural to consider the general case of a G-space X, that is, a topological space X on which a topological group G acts; and to define and study the notion of expansiveness of a homeomorphism h in this setting. We take up this task in the present chapter. In fact, we define the notion of expansiveness of a homeomorphism h on a G-space X relative to a subset A of X×X and terming it GA-expansive homeomorphism we carry out their study obtaining some interesting results. Naturally, in case of a metric G-space, for a specific choice of A, the concept of GA-expansive homeomorphism coincides with that of G-expansive homeomorphism.

Let H(X) throughout denote the collection of all homeomorphisms on the topological space X.

1. GA-expansiveness.

The considerations of the following examples help us to motivate the concept of GA-expansiveness.

Examples 5.1(a). Let $X = [0,1]$ with usual metric. Choose either $A = [b,1]\times[c,d]$ where $b \in (1/2,1)$ and $c, d \in X$ or $A = [0,a]\times [c,d]$ where $a \in (0,1/2)$ and $c, d \in X$. Define $h : X \to X$ by $h(x) = 1 - x$. Then h is A-expansive :

Let $x,y \in X$ be such that $x \neq y$. If $(x,y) \notin A$, then for $n = 0$,

$$(h^n(x),h^n(y)) \notin A$$

and if $(x,y) \in A$, then

$$(h(x),h(y)) \notin A.$$

Next, let the topological group $G \equiv \mathbb{Z}_2 = \{-1,1\}$ act on X with the action $1t = t$ and $-1t = 1 - t$, where $t \in X$. Then it can be easily seen that there exist x, y in $X - \{1/2\}$ with distinct G-orbits such that for no n in $\mathbb{Z}$

$$[\ h^n(G(x)) \times h^n(G(y))\ ] \cap A = \varphi.$$

5.1(b). Let X be as in Example 5.1(a), $A = [1/5,1/2] \times [1/3,2/3] \subset X\times X$ and $h:X \to X$ be defined by

$$\begin{aligned} h(x) &= 3x, && \text{if } x \in [0,1/5]; \\ &= (11x+5)/12, && \text{if } x \in [1/5,1/2], \text{ and} \\ &= (x+3)/4, && \text{if } x \in [1/2,1]. \end{aligned}$$

It can be easily seen that h is an A-expansive homeomorphism on X. Let $G \equiv \mathbb{Z}_2$ act on X as defined in Example 5.1.(a). Then it can be observed that whenever $x, y \in X$ with distinct G-orbits, there exists an n in $\mathbb{Z}$ satisfying $[\ h^n(G(x)) \times h^n(G(y))\ ] \cap A = \varphi$.

In Example 5.1(b), given any $A = [a,b] \times [c,d] \subset X\times X$, where $a, b, c, d \notin \{0,1\}$, one can construct a suitable h satisfying the same property, namely given any pair of distinct G-orbits G(x) and G(y), there exists an n in $\mathbb{Z}$ such that

$$[\ h^n(G(x)) \times h^n(G(y))\ ] \cap A = \varphi.$$

In fact one may define h in such a way that $h([a,b]) \cap [c,d] = \varphi$. But here we observe that A does not contain the diagonal in X×X. However, we do have similar situation even if A is a regular closed set containing the diagonal as can be seen in the following example.

5.1 (c). Let X = [0,1] with the usual metric and consider the subset A^δ of X×X given by

$A^\delta = \{\ (\ x/(x+1),\ y/(y+1)\ ) \mid x,\ y \geq 0 \text{ with } |x - y| \leq \delta\ \} \cup \{(1,1)\}$,

where $\delta > 0$ is a fixed real number. Define $h : X \to X$ by

$$h(x) = \beta.x\ /\ [\ (\beta - 1).x + 1\ ],$$

$x \in X$, where β is a fixed positive real number and $\beta \neq 1$. Then as observed in the Note following Example 2.3 of Chapter 2, h is A^δ-expansive on X. Let $G \equiv \mathbb{Z}_2$ act on X as in Example 5.1(a). Then it can be seen that whenever $x, y \in X$ with $G(x) \neq G(y)$, there exists an r in $\mathbb{Z}$ satisfying $[\ h^r(G(x)) \times h^r(G(y))\ ] \cap A = \varphi$. For example take $\beta = 2$. Then Fixh = {0,1} and for any $x \in X - \{0,1\}$,

$$h^n(x) \to 1 \text{ and } h^{-n}(x) \to 0 \text{ as } n \to \infty.$$

Thus there exist integers ι, m, n, k such that

$$(h^\iota(x),0) \notin A^\delta;\ (0,h^m(y)) \notin A^\delta;$$

$$(h^n(1 - x),0) \notin A^\delta \text{ and } (0,h^k(1 - y)) \notin A^\delta.$$

If $r = \max\{\iota, m, n, k\}$, then it follows that

$$[\ h^r(G(x)) \times h^r(G(y))\ ] \cap A^\delta = \varphi.$$

5.1 (d). Consider the unit circle S^1 and the usual action of the multiplicative group $G \equiv U(n)$ of nth roots of unity on S^1. Let C_k denote the arc $(e^{i2\pi k/n}, e^{i2\pi(k+1)/n})$ of S^1, $k = 0,1,\ldots,n-1$ and f_k denote the homeomorphism from $I_k = [0,1]$ to C_k given by

$$f_k(s) = e^{i2\pi(s+k)/n}$$

where $s \in [0,1]$ and $k = 0,..,n-1$. Since the homeomorphism g_k on I_k defined by $g_k(x) = \beta x/[(\beta-1)x+1]$, for a fixed β, $\beta > 0$ and $\beta \neq 1$ is A^{δ}-expansive, where A^{δ} is as described in Example 5.1(c), it follows from Theorem 2.3 of Chapter 2 that $f_k g_k f_k^{-1} \equiv h_k$ is $[(f_k \times f_k)(A^{\delta})]$-expansive on C_k. Define $h : S^1 \to S^1$ by $h|_{C_k} = h_k$, where $k = 0,1,\ldots\ldots,n-1$. Obviously h is in $H(S^1)$. In fact h is an $\bigcup_{k=0}^{n-1}((f_k \times f_k)(A^{\delta}))$-expansive homeomorphism on S^1 and the subset $B_n = \bigcup_{k=0}^{n-1}[(f_k \times f_k)(A^{\delta})]$ of $S^1 \times S^1$ is a regular closed set which contains the diagonal in $S^1 \times S^1$. In this example also one can verify that for distinct G-orbits $G(x)$ and $G(y)$, there exists an n in $\mathbb{Z}$ satisfying $[\, h^n(G(x)) \times h^n(G(y)) \,] \cap B_n = \varphi$.

The observations made in the above examples lead us to the following definition of GA-expansiveness.

Definition 5.1. Let X be a topological space on which a topological group G acts, $A \subset X \times X$ and $h \in H(X)$. Then h is called *GA-expansive* if whenever $x, y \in X$ with $G(x) \neq G(y)$, there exists an integer n satisfying $[\, h^n(G(x)) \times h^n(G(y)) \,] \cap A = \varphi$.

Observe that a metric space can always be regarded as a metric G-space by considering the trivial action of any group G on it; and hence by choosing $A = A_\delta \equiv d^{-1}[0,\delta]$ for some $\delta > 0$ when X is a metric space with metric d in this definition, one sees that GA-expansiveness of h in H(X) is equivalent to expansiveness of h with expansive constant δ. However, if X is any G-space with action of G on X trivial, then the GA-expansiveness of h in H(X) is equivalent to A-expansiveness of h. Also, in case X is a metric G-space and $A = A_\delta$ for some $\delta > 0$, then GA-expansiveness of h in H(X) is equivalent to G-expansiveness of h with G-expansive constant δ.

Example 5.1 (a) shows that an A-expansive homeomorphism need not be GA-expansive and on the other hand Example 3.2 of Chapter 3 shows that a GA-expansive homeomorphism is not necessarily an A-expansive homeomorphism.

2. Properties of GA-expansive homeomorphisms.

We study some properties of GA-expansive homeomorphisms. To begin with, the following result regarding the restriction of a GA-expansive homeomorphism follows from the definition.

Theorem 5.1. *Let* X *be a* G-*space*, Y *be a* G-*invariant subspace of* X, $h \in H(X)$ *be* GA-*expansive where* $A \subset X\times X$, *and* $h(Y) = Y$. *Then* $h|_Y$ *is* GB-*expansive, where* B *is trace of* A *in* $Y\times Y$.

Proof. Suppose x and y are two points in Y with distinct G-orbits. Then GA-expansiveness of h on X gives an integer n satisfying $[\, h^n(G(x)) \times h^n(G(y))\,] \cap A = \varphi$. Now the Theorem follows if we take $B = A \cap Y\times Y$.

Next, we have a result regarding product of two GA-expansive homeomorphisms.

Theorem 5.2. *Let* X, Y *be* G-*spaces*, $A \subset X\times X$, $B \subset Y\times Y$, $h \in H(X)$ *be* GA-*expansive and* $f \in H(Y)$ *be* GB-*expansive. Then* $\psi = h\times f$ *is* $G(q^{-1}(A\times B))$-*expansive on* $W = X\times Y$, *where* $q : W\times W \to (X\times X)\times(Y\times Y)$ *is defined by* $q(x,y,x',y') = (x,x',y,y')$, $x,x' \in X$, $y,y' \in Y$ *and* W *is considered to be a* G-*space under the diagonal action of* G.

Proof. Let $(x,y), (x',y') \in W$ be such that $G(x,y) \neq G(x',y')$. Since action of G on W is diagonal action, i.e., $g(x,y) = (gx,gy)$, $g \in G$, $(x,y) \in W$, the following cases arise : (*i*) $G(x) \neq G(x')$ or (*ii*) $G(y) \neq G(y')$. In case (*i*) since $G(x) \neq G(x')$, from GA-expansiveness of h there exists an n in $\mathbb{Z}$ satisfying $[h^n(G(x))\times h^n(G(x'))] \cap A = \varphi$ which implies

$$[\, h^n(G(x))\times h^n(G(x'))\times f^n(G(y))\times f(^nG(y'))] \cap (A\times B) = \varphi.$$

Further, as q is a homeomorphism we obtain

$$q^{-1}[\, h^n(G(x))\times h^n(G(x'))\times f^n(G(y))\times(f^nG(y'))\,] \cap q^{-1}(A\times B) = \varphi$$

which implies

$$[\, (h\times f)^n(G(x)\times G(y)) \times (h\times f)^n(G(x')\times G(y'))\,] \cap q^{-1}(A\times B) = \varphi.$$

Since $G(x,y) \subseteq G(x)\times G(y)$ and $G(x',y') \subseteq G(x')\times G(y')$, we therefore obtain

$$[\, (h\times f)^n(G(x,y)) \times (h\times f)^n(G(x',y'))\,] \cap q^{-1}(A\times B) = \varphi$$

and hence $h\times f$ is $G(q^{-1}(A\times B))$-expansive on W. Similarly Case(*ii*) follows from GB-expansiveness of f on Y.

The above result extends to any finite product of GA-expansive homeomorphisms and can be proved in a similar way by using induction principle. Next we obtain a result regarding integral powers of a GA-expansive homeomorphism.

Theorem 5.3. *Let X be a paracompact Hausdorff G-space, $\mathcal{U}$ be the uniformity on it consisting of all the neighbourhoods of the diagonal in X×X and h ∈ H(X) be such that h^m, m ≠ 0 is uniformly continuous with respect to $\mathcal{U}$. Then h is GA-expansive for some A ∈ $\mathcal{U}$ iff h^m, m ≠ 0, is GB-expansive for a suitable B ∈ $\mathcal{U}$.*

Proof. Consider any integer m different from 0. Suppose $i \in \{\pm 1, \ldots, \pm m\}$. Since for each i, h^{-i} is uniformly continuous, there exists a B_i in $\mathcal{U}$ for each i such that

$$(h^{-i} \times h^{-i})(B_i) \subseteq A$$

or equivalently

$$(h^{i} \times h^{i})(X \times X - A) \subseteq X \times X - B,$$

where

$$B = \cap \{ B_i \mid i \in \{\pm 1, \ldots, \pm m\} \}.$$

Let x, y ∈ X with G(x) ≠ G(y). Then from the GA-expansiveness of h there exists an n in $\mathbb{Z}$ satisfying $[h^n(G(x)) \times h^n(G(y))] \cap A = \varphi$. But this gives

$$[h^i(h^n(G(x))) \times h^i(h^n(G(y)))] \cap B = \varphi \qquad (*)$$

for each $i \in \{\pm 1, \ldots, \pm m\}$. Let r be in $\mathbb{Z}$ such that $0 < |r - n/m| \le 1$, i.e., $0 < |rm - n| \le |m|$. Then putting $i = rm - n$ in (*) we get

$$[(h^m)^r(G(x)) \times (h^m)^r(G(y))] \cap B = \varphi$$

Thus h^m is GB-expansive, where B ∈ $\mathcal{U}$.

Conversely, let h in H(X) be such that h^m is GA-expansive for some m in $\mathbb{Z}-\{0\}$. Then, for x, y in X with $G(x) \neq G(y)$, the GA-expansivness of h^m implies that there exists an n in Z satisfying

$$[\,(h^m)^n(G(x)) \times (h^m)^n(G(y))\,] \cap A = \varphi.$$

Now put $r = m.n$ to see that h is also GA-expansive.

The following result shows that admitting a GA-expansive homeomorphism is a topological property for G-spaces under some condition.

Theorem 5.4. *Let* X *and* Y *be G-spaces*, $A \subset X\times X$ *and* $f : X \to Y$ *be a pseudoequivariant homeomorphism. Then an* h *in* H(X) *is* GA-*expansive iff* fhf^{-1} *is a* $G((f\times f)(A))$*-expansive homeomorphism of* Y.

Proof. Let $y, y' \in Y$ with $G(y) \neq G(y')$. Since f is a homeomorphism, there exist x, x' in X such that $f(x) = y$, $f(x') = y'$; and therefore

$$G(f(x)) \neq G(f(x')).$$

Now, pseudoequivariancy of f implies

$$f(G(x)) \cap f(G(x')) = \varphi$$

and therefore, f being bijective, we get

$$G(x) \neq G(x').$$

Now, GA-expansiveness of h implies the existence of an integer n satisfying

$$[\,h^n(G(x)) \times h^n(G(y))\,] \cap A = \varphi.$$

Again using bijectivity of f, it follows that

$$[\,fh^n(G(f^{-1}(y))) \times fh^n(G(f^{-1}(y')))\,] \cap (f\times f)(A) = \varphi.$$

As f is pseudoequivariant, from Lemma 3.1 of Chapter 3 it follows that f^{-1} is also pseudoequivariant. Hence

$$[\ (fh^nf^{-1})(G(y)) \times (fh^nf^{-1})(G(y'))\ ] \cap (f\times f)(A) = \varphi$$

or equivalently

$$[\ (fhf^{-1})^n(G(y)) \times (fhf^{-1})^n(G(y'))\ ] \cap (f\times f)(A) = \varphi.$$

This proves that fhf^{-1} is $G((f\times f)(A))$-expansive on Y.

Conversely, suppose $x, y \in X$ with distinct G-orbits, i.e., $G(x) \neq G(y)$. Then bijectivity of f gives $f(G(x)) \cap f(G(y)) = \varphi$. Since f is pseudoequivariant, we have $G(f(x)) \cap G(f(y)) = \varphi$, i.e., f(x) and f(y) also has distinct G-orbits. Further, since fhf^{-1} is $G((f\times f)(A))$-expansive on Y it follows that there exists an integer n satisfying

$$[\ (fhf^{-1})^n(G(f(x))) \times (fhf^{-1})^n(G(f(y)))\ ] \cap (f\times f)(A) = \varphi$$

that is

$$[\ (fh^nf^{-1})(G(f(x))) \times (fh^nf^{-1})(G(f(y)))\ ] \cap (f\times f)(A) = \varphi.$$

Another application of pseudoequivariancy of f then gives

$$[\ fh^n(G(x)) \times fh^n(G(y))\ ] \cap (f\times f)(A) = \varphi.$$

Finally, apply bijectivity of f to obtain

$$[\ h^n(G(x)) \times h^n(G(y))\ ] \cap A = \varphi.$$

Hence h is GA-expansive on X.

3. Extension and characterization of GA-expansive homeomorphisms.

Next result is regarding extension of GA-expansive homeomorphisms. If X is a G-space and S is a G-invariant subspace of X, then by GA-expansiveness of an h in H(X) on S we mean there exists a subset A of X×X such that whenever $x, y \in S$ with

$G(x) \neq G(y)$, an integer n will exist satisfying

$$[h^n(G(x)) \times h^n(G(y))] \cap A = \varphi.$$

Theorem 5.5. *Let* X *be a paracompact Hausdorff* G*-space,* $S \subseteq X$ *be such that* S *is* G*-invariant and* X − S *is finite. If* h *in* H(X) *is* GU*-expansive on* S*, where* U *is a neighbourhood of the diagonal in* X×X*, then* h *is* GV*-expansive on* X *for a suitable neighbourhood* V *of the diagonal in* X×X.

Proof. Let $X - S = \{x_0, x_1, \ldots, x_n\}$. We first show h is GV-expansive on $S \cup \{x_0\}$. Since X is a paracompact Hausdorff space and U is a neighbourhood of the diagonal in X×X, there exists a symmetric neighbourhood V' of the diagonal in X×X such that $V' \circ V' \subset U$, where

$$V' \circ V' = \{ (x,y) \in X \times X \mid \text{there exists } z \text{ in } X \text{ satisfying } (x,z) \in V' \text{ and } (z,y) \in V' \}$$

Since V' contains the diagonal, $V' \subset V' \circ V' \subset U$.

First note that h being GU-expansive on S, there does not exist two points p_1, p_2 in S such that $G(p_1) \neq G(p_2)$ and for some g_1, k_1, g_2 in G

$$(h^n(g_1 p_1), h^n(k_1 x_0)) \in V' \quad \text{and} \quad (h^n(g_2 p_2), h^n(k_1 x_0)) \in V'$$

for each integer n, i.e., there exists at most one point p in S such that for some g, k_1 in G,

$$(h^n(gp), h^n(k_1 x_0)) \in V'$$

for each integer n. In case no such p exists in S then h is GV-expansive on $S \cup \{x_0\}$, where V = V'. On the other hand if such a point p exists, then by taking

$$V = V' - \{ [(G(p) \times G(x_0)) \cup (G(x_0) \times G(p))] \cap V' \},$$

one can easily verify that h is GV-expansive on $S \cup \{x_o\}$.

Finally, the required result is proved using induction on the number of elements in X - S.

Recall that at the end of Section 1 of the present Chapter, we have observed that the notion of A-expansiveness and the notion of GA-expansivenss are independent of each other. In view of this the following characterization of GA-expansive homeomorphism is interesting. We first give a definition.

Definition 5.2. Let X be a G-space, $A \subset X \times X$ and $h \in H(X)$. Then h is said to GA-*separate* h-orbits if given any basis $\mathcal{B} = \{ x_\alpha \mid \alpha \in \mathcal{A} \}$ of X with respect to h, whenever $G(x_\alpha) \neq G(x_\beta)$, there exists an integer n satisfying $[h^n(G(x_\alpha)) \times h^n(G(x_\beta))] \cap A = \varphi$.

Theorem 5.6. *Let* X *be a* G-*space and* $A \subset X \times X$. *Suppose* h *in* H(X) *is pseudoequivariant. Then* h *is* GA-*expansive iff*

(*a*) h GA-*separates* h-*orbits*

(*b*) *given* p *in* X *and* n *in* **Z** *such that* $h^n(p) \notin G(p)$, *there exists an integer* r *satisfying*

$$[h^r(G(p)) \times h^{r-n}(G(p))] \cap A = \varphi.$$

Proof. Suppose h is a GA-expansive homeomorphism. Then we show that (*a*) and (*b*) are true. For (*a*), let $\mathcal{B} = \{ x_\alpha \mid \alpha \in \mathcal{A} \}$ be any basis of X with respect to h. Consider x_α and $x_\beta \in \mathcal{B}$ with distinct G-orbits. Then by GA-expansiveness of h, there exists an n in **Z** satisfying $[h^n(G(x_\alpha)) \times h^n(G(x_\beta))] \cap A = \varphi$. This proves (*a*). For

(b), we recall that (Lemma 3.1) as h is pseudoequivariant, so we have

$$h^n(G(x)) = G(h^n(x)) \qquad (*)$$

for each x in X and n in $\mathbb{Z}$. Now, suppose there is a $p \in X$ and an n in $\mathbb{Z}$ such that $h^n(p) \notin G(p)$. Then we obtain an r in $\mathbb{Z}$ for which (b) holds. As h is GA-expansive, so we find an integer m satisfying

$$[h^m G(h^n(p)) \times h^m G(p)] \cap A = \varphi.$$

Using (*) we get

$$[h^{m+n}(G(p)) \times h^m(G(p))] \cap A = \varphi.$$

On substituting m + n = r, we finally obtain

$$[h^r(G(p)) \times h^{r-n}(G(p))] \cap A = \varphi.$$

Conversely, suppose (a) and (b) hold. Then we prove that h is GA-expansive. Let $x, y \in X$ with $G(x) \neq G(y)$. Then two cases arise: Either x and y have disjoint h-orbits or they intersect. In case $O(x) \cap O(y) = \varphi$, we choose that basis of X with respect to h which has x and y as its members and then apply (a) to obtain an integer r satisfying $[h^r(G(x)) \times h^r(G(y))] \cap A = \varphi$. This proves that h is GA-expansive in this case. In the other case when the h-orbits of x and y intersect, there exists an integer n for which $x = h^n(y)$. Since x and y are having distinct G-orbits, we get $G(y) \neq G(h^n(y))$ which implies $h^n(y) \notin G(y)$. Now we apply (b) and obtain an integer r satisfying

$$[\ h^r(G(y)) \times h^{r-n}(G(y))\ ] \cap A = \varphi$$

which implies

$$[\ h^r(G(h^{-n}(x))) \times h^{r-n}(G(y))\ ] \cap A = \varphi.$$

Once again we make use of (*) and obtain

$$[\ h^{r-n}(G(x)) \times h^{r-n}(G(y))\ ] \cap A = \varphi.$$

This establishes the GA-expansiveness of h in this case.

The above characterization of GA-expansive homeomorphisms gives the following sufficient condition for the homeomorphic extension of a GB-expansive homeomorphism on a G-invariant subspace Y of a G-space X to be GB-expansive on the whole space.

Theorem 5.7. *Let* Y *be a* G-*invariant subspace of a* G-*space* X *and let* h *in* H(Y) *be pseudoequivariant* GB-*expansive, where* $B \subset X \times X$. *Then a pseudoequivariant homeomorphic extension* f *of* h *to* X *is* GB-*expansive on* X *if*

(*i*) f *is* GB-*expansive on* X - Y *and*

(*ii*) *there exists a basis* $\mathscr{B}$ *of* Y *with respect to* h *such that*

$$[\ G(y) \times (X - Y)\ ] \cap B = \varphi,$$

for each y *in* $\mathscr{B}$.

Proof. For proving the GB-expansiveness of f in H(X), we show that conditions (*a*) and (*b*) of Theorem 5.6 are satisfied by f.

For (*a*), choose any basis $\mathscr{B}' = \{ x_\alpha \mid \alpha \in \mathscr{A} \}$ of X with respect to f and consider any two members, say x_α and x_β, in $\mathscr{B}'$ with distinct G-orbits. We have following cases :

(*i*) x_α, $x_\beta \in Y$;

(*ii*) x_α, $x_\beta \in X - Y$ and

(*iii*) $x_\alpha \in Y$ and $x_\beta \in X - Y$ or $x_\alpha \in X - Y$ and $x_\beta \in Y$.

In cases (*i*) and (*ii*), we appply the fact that $f|_X = h$ and

$f|_{X-Y}$ are GB-expansive homeomorphisms and get the desired result (here we use the fact that a point lies in a G-invariant set iff the entire G-orbit of that point lies in that set).

Next we consider case (iii). Let us assume $x_\alpha \in Y$ and $x_\beta \in X-Y$. Then $x_\alpha \in O(y)$ for some $y \in \mathcal{B}$, i.e., $x_\alpha = h^n(y)$ for some integer n and therefore by condition (ii) of the hypothesis

$$[\ G(h^{-n}(x_\alpha)) \times (X - Y)\ ] \cap B = \varphi.$$

Now X - Y being G-invariant subspace of X, $G(x_\beta) \subseteq X - Y$. Also $f^{-n}(G(x_\beta)) \subseteq X - Y$. Therefore using pseudoequivariancy of $f|_X = h$, we obtain

$$[\ f^{-n}(G(x_\alpha)) \times f^{-n}(G(x_\beta))\ ] \cap B = \varphi.$$

Hence f-orbits are GB-separated by f, i.e., f satisfies condition (a) of Theorem 5.6. For condition (b), let p in X and integer n be such that $f^n(p) \notin G(p)$. Again, either $p \in Y$ or $p \in X-Y$. If $p \in Y$, then Y being G-invariant one gets $G(p) \subset Y$. Also, as $f|_X = h$ is GB-expansive Theorem 5.6 is applicable to the map f on Y and hence there will exists an integer r which satisfies

$$[\ f^r(G(p)) \times f^{r-n}(G(p))\ ] \cap B = \varphi.$$

For the case when $p \in X-Y$ we use the fact that X-Y is G-invariant and $f|_{X-Y}$ is GB-expansive and argue exactly as we did when $p \in Y$ to obtain an $n \in \mathbb{Z}$ such that

$$[f^r(G(p)) \times f^{r-n}(G(p))] \cap B = \varphi.$$

Thus we obtain that f is GB-expansive on whole of X.

It may be noted here that Theorems 5.6 and 5.7 reduce to respectively Theorems 1.8 and 1.7 stated in Chapter 1 due to Wine [42].