CHAPTER - 6

CERTAIN RESULTS INVOLVING THE POLYNOMIAL $g_n^c(x,r,s;q)$

6.1 INTRODUCTION

In view of the general nature of the class of polynomials $\{g_n^c(x,r,s)\}$ introduced by Rekha Panda [1], it is natural that its q-analogue $\{g_n^c(x,r,s;q)\}$ introduced in chapter-4 would yield the q-analogues of a large number of polynomials included in $g_n^c(x,r,s)$. A systematic and detailed study of the class of polynomils $\{g_n^c(x,r,s;q)\}$ is , therefore, very much desirable and needed. Motivated by such a need we present in this chapter certain miscellaneous results involving the polynomial $g_n^c(x,r,s;q)$.

In what follows we shall make use of the following known results in this chapter.

(6.1.1) $$(x;q)_n \equiv [x]_n = \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \left[{n \atop k}\right] x^k$$

(6.1.2) $$[a]_{2k} = (a;q^2)_k\ (aq;q^2)_k$$

(6.1.3) $$(a;q^2)_k = (\sqrt{a};q)_k\ (-\sqrt{a};q)_k\ ,\ \sqrt{a} = q^{a/2}$$

(6.1.4) $$\int_0^1 t^{\alpha-1}\ [tq]_{\beta-1}\ d_qt = \frac{(1-q)\ [q]_\infty\ [q^{\alpha+\beta}]_\infty}{[q^\alpha]_\infty\ [q^\beta]_\infty}$$ (Hahn[1],[2])

(6.1.5) $$\int_0^1 t^{\alpha-1}\ E_q(tq)\ d_qt = (1-q)\ \frac{[q]_\infty}{[q^\alpha]_\infty}$$

(6.1.6) $$\int_0^\infty t^{\alpha-1}\ e_q(-t)\ d_qt = (1-q)\ q^{-\alpha(\alpha-1)/2}\ \frac{[q]_\infty}{[q^\alpha]_\infty}$$ (Hahn [2]).

$$(6.1.7)\quad x^n (1-q)^n \mathcal{D}_q^n f(x) = (-1)^n q^{-n(n-1)/2} \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \cdot \left[{n \atop k} \right] f(xq^{n-k}),$$

where the q-derivative operator $\mathcal{D}_q$ is defined by

$$\mathcal{D}_q f(x) = \frac{f(x)-f(qx)}{x(1-q)}, \quad q \text{ is fixed.}$$

$$(6.1.8)\quad f\left(\frac{t}{[x-|y]}\right) = \sum_{n=0}^{\infty} a_n \frac{t^n}{[x-|y]_n},$$

$$(6.1.9)\quad f(t[x-|y]) = \sum_{n=0}^{\infty} a_n [x-|y]_n t^n \qquad \text{(Khan [1])},$$

where $f(x) = \sum_{n=0}^{\infty} a_n x^n$, and (Jackson [5])

$$[x-|y]_n = (x-y)(x-yq)(x-yq^2)\ldots.(x-yq^{n-1}).$$

$$(6.1.10)\quad F_N(z[1-t]_{(s)}) = \sum_{k=0}^{N} c_k z^k \frac{[t]_{sk}}{[q]_{N-k}},$$

in which $F_N(z) = \sum_{k=0}^{N} c_k \dfrac{z^k}{[q]_{N-k}}$;

$$(6.1.11)\quad \sum_{k=0}^{[n/s]} \sum_{j=0}^{k} A(n,k,j) = \sum_{j=0}^{[n/s]} \sum_{k=0}^{[n/s]-j} A(n,k+n,j),$$

$$(6.1.12)\quad \sum_{n=0}^{N} \sum_{k=0}^{[n/s]} A(n,k) = \sum_{k=0}^{[N/s]} \sum_{n=0}^{N-sk} A(n+sk,k).$$

It may be seen first that in view of the definition of $[x-|y]_n$

and, the relations (6.1.1) and (6.1.11),

$$g_n^c([x-|y],r,s;q) = \sum_{j=0}^{[n/s]} (-1)^j\, q^{((rs-s^2+1)j^2+(r-s-1)j-2(r-s)nj)/2} \cdot$$

$$\cdot \frac{y^j}{[q]_j} \sum_{k=0}^{[n/s]-j} q_1^{sk(sk-2n+2sj+1)/2} \frac{[cq^{rk+rj}]_{n-sj-sk}}{(q_1;q_1)_{n-sj-sk}} \cdot$$

$$\cdot \frac{[q]_{k+j}}{[q]_k}\, \delta_{k+j}\, x^k ,$$

wherein $q_1 = q^{(r-s)/s}$. This polynomial representation when particularized by means of the substitutions $s=1$, $r=2$, $c=1+\alpha+\beta$, and

$$\delta_k = \frac{[\alpha\beta q]_{2k}\,(-1)^k}{[\alpha q]_k\,[q]_k}$$

yields the corresponding expression involving little q-Jacobi polynomial as given below.

$$\frac{[\alpha\beta q]_n}{[q]_n}\, p_n([x-|y];\alpha,\beta;q) = \sum_{j=0}^{n} q^{j^2-nj} \frac{[\alpha\beta q]_{n+j}\quad y^j}{[\alpha q]_j\,[q]_{n-j}\,[q]_j} \cdot$$

$$\cdot \sum_{k=0}^{n-j} \frac{[q^{-n+j}]_k\,[\alpha\beta q^{n+j+1}]_k}{[\alpha q^{j+1}]_k\,[q]_k}\, x^k q^k .$$

Since, the inner series in this last expression also represents the same 'little polynomial' of degree $(n-j)$, one obtains

$$(6.2.1)\quad p_n([x-|y];\alpha,\beta;q) = \sum_{j=0}^{n} q^{j^2-nj} \frac{[\alpha\beta q^{n+1}]_j}{[\alpha q]_j}\, y^j \,.$$

$$\cdot\, p_{n-j}(x;\alpha+j,\beta+j;q).$$

Next, we prove the summation formula

$$(6.2.2)\quad \sum_{n=0}^{N} g_n^c(xq^{sn-sN-s},2s,s;q)\; q^n = g_N^{c+1}(x,2s,s;q),$$

by making use of the easily verifiable (by induction) result

$$(6.2.3)\quad 1 + \sum_{n=1}^{N} \frac{[a]_n}{[q]_n}\; q^n = \frac{[aq]_N}{[q]_N}\; .$$

In fact, in the light of the relation (6.1.12), it is not difficult to see that

$$\sum_{n=0}^{N} g_n^c\,(xq^{sn-sN-s},2s,s;q)\; q^n$$

$$= \sum_{n=0}^{N}\sum_{k=0}^{[n/s]} q^{sk(sk-2N+1)/2}\, \frac{[cq^{2sk}]_{n-sk}}{[q]_{n-sk}}\, \delta_k\, x^k\, q^{n-sk}$$

$$= \sum_{k=0}^{[N/s]} q^{(s^2k-2sN+s)k/2}\, \delta_k\, x^k \sum_{n=0}^{N-sk} \frac{[cq^{2sk}]_n}{[q]_n}\quad q^n.$$

The inner series occuring in (6.2.4) above may be further simplified with the aid of the result (6.2.3), in the form

$$\frac{[cq^{2sk+1}]_{N-sk}}{[q]_{N-sk}}\quad ,$$

and thus, one obtains

$$\sum_{n=0}^{N} g_n^c(xq^{sn-sN-s},2s,s;q)\; q^n = \sum_{k=0}^{[N/s]} q^{sk(sk-2N+1)/2}\; \frac{[cq^{2sk+1}]_{N-sk}}{[q]_{N-sk}}\, \delta_k x^k$$

$$= g_N^{c+1}(x,2s,s;q),$$

and hence the proof of (6.2.2).

A worth mentioning special case of the result (6.2.2) corresponding to the polynomial $p_n(x;\alpha,\beta;q)$ may be deduced by taking s=1, c $=\alpha+\beta$, and

$$\delta_k = \frac{[\alpha\beta q]_{2k}\ (-1)^k}{[\alpha q]_k \quad [q]_k} .$$

In this case, one obtains after some simplification,

$$\sum_{n=0}^{N} \frac{[\alpha\beta]_n}{[q]_n} \sum_{k=0}^{n} \frac{[q^{-n}]_k\ [\alpha\beta q^n]_k\ [\alpha\beta q]_{2k}}{[\alpha\beta]_{2k}\ [\alpha q]_k\ [q]_k}\ (xq)^k\ q^n$$

$$= g_N^{\alpha+\beta+1}(x,2,1;q)$$

$$= \frac{[\alpha\beta q]_N}{[q]_N}\ p_N(x;\alpha,\beta;q),$$

which with an appeal to the formula (6.1.2) and (6.1.3), leads to the summation formula :

$$(6.2.5)\quad p_N(x;\alpha,\beta;q) = \sum_{n=0}^{N} \frac{[\alpha\beta]_n [q]_N}{[\alpha\beta q]_N\ [q]_n} .$$

$$.\ {}_4\phi_3\left[\begin{matrix} q^{-n},\ \alpha\beta q^n,\ q\sqrt{\alpha\beta},\ -q\sqrt{\alpha\beta};\ q,\ xq \\ \alpha q,\ \sqrt{\alpha\beta},\ -\sqrt{\alpha\beta}; \end{matrix}\right] q^n .$$

6.3 BASIC DIFFERENCE OPERATIONS

Let

$$(6.3.1)\quad \lambda_{n,c}^{\alpha,\beta}(x,r,s,(a_p),(b_i);q) = \sum_{k=0}^{[n/s]} q_1^{sk(sk-2n+1)/2} \cdot \frac{[cq^{rk}]_{n-sk}\,[(a_p)]_{\alpha k}\,x^k}{[(b_i)]_{\beta k}\,[q]_k\,(q_1;q_1)_{n-sk}}$$

$$(q_1 = q^{(r-s)/s}).$$

The basic polynomial in (6.3.1), henceforth abbreviated as $\lambda_{n,c}^{\alpha,\beta}(x;q)$, is a special case of the general class $\{g_n^c(x,r,s;q)\}$ when

$$\delta_k = \frac{[(a_p)]_{\alpha k}}{[(b_i)]_{\beta k}\,[q]_k}, \quad \text{with } [(a_p)]_n = [a_1]_n\,[a_2]_n \cdots [a_p]_n.$$

Now, in view of the definition of the q-derivative operator $\mathcal{D}_q$ given by (6.1.7), it follows that

$$(6.3.2)\quad x(1-q)\,\mathcal{D}_q\{\lambda_{n,c}^{\alpha,\beta}(x;q)\} = \lambda_{n,c}^{\alpha,\beta}(x;q) - \lambda_{n,c}^{\alpha,\beta}(xq;q).$$

But since,

$$x(1-q)\,\mathcal{D}_q\{\lambda_{n,c}^{\alpha,\beta}(x;q)\} = \sum_{k=0}^{[n/s]} q_1^{sk(sk-2n+1)/2}\,\frac{[cq^{rk}]_{n-sk}\,[(a_p)]_{\alpha k}\,\mathcal{D}_q x^k}{[(b_i)]_{\beta k}(q_1;q_1)_{n-sk}\,[q]_k}$$

$$= q_1^{s(s-2n+1)/2}\,\frac{[(a_p)]_\alpha}{[(b_i)]_\beta} \sum_{k=0}^{[n/s]} q_1^{sk(sk-2n+2s+1)/2} \cdot \frac{[cq^{rk+r}]_{n-s-sk}\,[(a_p+\alpha)]_{\alpha k}\,x^k}{[(b_i+\beta)]_{\beta k}\,(q_1;q_1)_{n-s-sk}\,[q]_k}$$

$$= q_1^{s(s-2n+1)/2} \frac{[(a_p)]_\alpha}{[(b_i)]_\beta} \lambda^{\alpha,\beta}_{n-s,c+r}(xq_1^{s^2};q),$$

the expression in (6.3.2) assumes the form :

$$(6.3.3)\quad q_1^{s(s-2n+1)/2} \frac{[(a_p)]_\alpha}{[(b_i)]_\beta}(x)\ \lambda^{\alpha,\beta}_{n-s,c+r}(xq_1^{s},r,s,(a_p+\alpha),(b_i+\beta);q)$$

$$= \lambda^{\alpha,\beta}_{n,c}(x;q) - \lambda^{\alpha,\beta}_{n,c}(xq;q)\ .$$

The relation (6.3.3) may be put in a more general form by applying the operator $\mathcal{D}_q$ (m-1) - times. In fact, one finds in this case that

$$(6.3.4)\quad (1-q)^m\ x^m\ \mathcal{D}_q^m\ \{\lambda^{\alpha,\beta}_{n,c}(x;q)\} = q_1^{ms(ms-2n+2)/2} \frac{[(a_p)]_{m\alpha}}{[(b_i)]_{m\beta}}\ .$$

$$.\ \lambda^{\alpha,\beta}_{n-ms,c+mr}(xq_1^{ms^2},r,s,(a_p+m\alpha),(b_i+m\beta);q)$$

which with an appeal to the formula (6.1.7), becomes

$$(6.3.5)\quad q_1^{ms(ms-2n+1)/2} \frac{[(a_p)]_{m\alpha}}{[(b_i)]_{m\beta}}\ x^m\ \lambda^{\alpha,\beta}_{n-ms,c+mr}(xq_1^{ms^2},r,s,(a_p+m\alpha),$$

$$(b_i+m\beta);q)$$

$$= (-1)^m\ q^{-m(m-1)/2} \sum_{k=0}^{m} (-1)^k\ q^{k(k-1)/2} \begin{bmatrix} m \\ k \end{bmatrix} \lambda^{\alpha,\beta}_{n,c}(xq^{m-k};q).$$

6.4 BASIC INTEGRAL FORMULAE

In this section, the basic integral formulae (6.1.4) to (6.1.6) will be used to derive the following basic integral representations of the polynomial $g_n^c(x,r,s;q)$ for $r\geq s$.

$$(6.4.1)\quad (1-q)\,[q]_\infty\, g_n^c(x,r,s;q) = \int_0^1 t^{c+n-1}(1-t)^{-1}\, G_{n,c}^{r,s}(xt^{r-s}[1-t]_{(s)})\,d_q t,$$

$$(6.4.2)\quad (1-q)\,[q]_\infty\, g_n^c(x,r,s;q) = \int_0^1 t^{c+n-1}\, E_q(tq)\, F_{n,c}^{r,s}(xt^{r-s})\, d_q t,$$

and

$$(6.4.3)\quad (1-q)\,[q]_\infty\, g_n^c(x,r,s;q) = \int_0^\infty t^{c+n-1}\, e_q(-t)\,.\, H_{n,c}^{r,s}(x(t/q^c)^{r-s})\, d_q t,$$

where

$$(6.4.4)\quad G_{n,c}^{r,s}(xt^{r-s}[1-t]_{(s)}) = \sum_{k=0}^{[n/s]} q_1^{sk(sk-2n+1)/2}\, \frac{[cq^{rk}]_n\,[q^{sk}]_\infty}{(q_1;q_1)_{n-sk}}\,.\,\delta_k\, x^k\, t^{rk-sk}\,(t;q)_{sk}\,.$$

$$(6.4.5)\quad F_{n,c}^{r,s}(xt^{r-s}) = \sum_{k=0}^{[n/s]} q_1^{sk(sk-2n+1)/2}\, \frac{[cq^{rk}]_\infty\,\delta_k}{(q_1;q_1)_{n-sk}}\, x^k\, t^{rk-sk}$$

and

$$(6.4.6)\quad H_{n,c}^{r,s}(x(t/q^c)^{r-s}) = q^{-(c+n)(c+n+1)/2} \sum_{k=0}^{[n/s]} q^{sck-rck}\, t^{rk-sk}\,.\, q^{((2rs^2-2s^2-r^2)k+(r-s)k-4(r-s)n)/2}\,\delta_k\, \frac{[cq^{rk}]_\infty\, x^k}{(q_1;q_1)_{n-sk}}$$

where, as before, $q_1=q^{(r-s)/s}$.

The proof of the formula (6.4.1) as sketched below, uses the notation cited in (6.1.10).

Let 'I' denote the right hand member of (6.4.1), then in view of (6.4.4), one gets

$$I = \sum_{k=0}^{[n/s]} q_1^{sk(sk-2n+1)/2} \frac{[cq^{rk}]_n \, [q^{sk}]_\infty}{(q_1;q_1)_{n-sk}} \, \delta_k \, x^k \, .$$

$$. \int_0^1 t^{c+n+rk-sk-1} \, [tq]_{sk-1} \, d_q t .$$

In this, the q-integral on right hand side may be evaluated with the help of the formula (6.1.4).

Thus,

$$I = (1-q)[q]_\infty \sum_{k=0}^{[n/s]} q_1^{sk(sk-2n+1)/2} \frac{[cq^{rk}]_n \, [q^{sk}]_\infty \, [q^{c+n+rk}]_\infty \, \delta_k \, x^k}{(q_1;q_1)_{n-sk} \, [q^{c+n+rk-sk}]_\infty \, [q^{sk}]_\infty}$$

$$= (1-q)[q]_\infty \sum_{k=0}^{[n/s]} q_1^{sk(sk-2n+1)/2} \frac{[cq^{rk}]_{n-sk}}{(q_1;q_1)_{n-sk}} \, \delta_k \, x^k$$

$$= (1-q) \, [q]_\infty \, g_n^c(x,r,s;q),$$

which proves (6.4.1).

In a similar manner, the results stated in (6.4.2) and (6.4.3) may be proved by employing the formulae (6.1.5) and (6.1.6) respectively.

The particular instances of these results (i.e. (6.4.1) to (6.4.3)) corresponding to the reducibilities of $g_n^c(x,r,s;q)$ to the different polynomials, may be obtained by specializing the various parameters involved therein.

For example, the formula (6.4.2) when specialized by setting $s=1$, $r=2$, $c=1+\alpha+\beta$, and

$$\delta_k = \frac{[\alpha\beta q]_{2k}\ (-1)^k}{[\alpha q]_k\ \ [q]_k} ,$$

gets reduced to the form :

$$(6.4.7)\ \ p_n(x;\alpha,\beta;q) = \frac{[\alpha\beta q^{n+1}]_\infty}{(1-q)\ [q^{n+1}]_\infty\ [\alpha q]_n} \int_0^1 t^{\alpha+\beta+n}\ E_q(tq)\ .\ {}_qL_n^{(\alpha)}(xt)\ \ d_qt$$

in which

$${}_qL_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{[q^{-n}]_k\ [\alpha q]_n}{[q]_k\ [\alpha q]_k}\ (xq)^k$$

is another basic analogue of the Laguerre polynomial $L_n^{(\alpha)}(x)$ (Khan [1]).